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GENERALIZATION OF THE GENERAL DIOPHANTINE APPROXIMATION THEOREM OF KRONECKER

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1. Introduction

Let G be the abelian group

$$G = \mathsf{Z}^r = \{(h_1,\dots,h_r) : h_1,\dots,h_r \in \mathsf{Z}\}$$

We consider the additive characters

$$\chi_1(h_1,\dots,h_r) \equiv \beta_{1,1}h_1 + \dots + \beta_{1,r}h_r \pmod 1$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\chi_p(h_1,\dots,h_r) \equiv \beta_{p,1}h_1 + \dots + \beta_{p,r}h_r \pmod 1$$

where $\beta_{i,j} \in \mathsf{R}$. The necessary and sufficient condition that the p inequalities

$$|\chi_1(h_1,\dots,h_r) - a_1| \leqq \varepsilon \pmod 1$$
$$\dots\dots\dots\dots\dots\dots\dots$$
$$|\chi_p(h_1,\dots,h_r) - a_p| \leqq \varepsilon \pmod 1$$

where $a_1,\dots,a_p \in \mathsf{R}$, for every $\varepsilon > 0$ have at least one solution $(h_1,\dots,h_r)$ is that for arbitrary integers $n_1,\dots,n_p$ with

$$n_1\chi_1(h_1,\dots,h_r) + \dots + n_p\chi_p(h_1,\dots,h_r) \equiv 0 \text{ for } (h_1,\dots,h_r) \in \mathsf{Z}^r$$

or equivalently

$$n_1\beta_{1,1} + \dots + n_p\beta_{p,1} \equiv 0 \pmod 1$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$n_1\beta_{1,r} + \dots + n_p\beta_{p,}{}^{r} \equiv 0 \pmod 1$$

also

$$n_1a_1 + \dots + n_pa_p \equiv 0 \pmod 1$$

This is Kronecker's general diophantine approximation theorem from [5].

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We shall generalize this result in two directions. In both cases we replace the group Z^r by an arbitrary topological abelian group G, and the characters $\chi_1,\dots,\chi_p$ by arbitrary continuous additive characters of G.

2. The first direction.

The first direction we take consists in proving the

THEOREM 1. *A necessary and sufficient condition that the set C of solutions x of the inequalities*

$$(1)\qquad \begin{array}{c} |\chi_1(x) - a_1| \leqq \varepsilon \pmod 1 \\ \dots\dots\dots\dots\dots\dots\dots \\ |\chi_p(x) - a_p| \leqq \varepsilon \pmod 1 \end{array}$$

where $a_1,\dots,a_p \in \mathrm{R}$, for every $\varepsilon > 0$ be not empty, is that for arbitrary integers $n_1,\dots n_p$ with

$$(2)\qquad n_1\chi_1(x) + \dots + n_p\chi_p(x) \equiv 0 \pmod 1 \text{ for } x \in G$$

also

$$(3)\qquad n_1a_1 + \dots + n_pa_p \equiv 0 \pmod 1$$

(A simpler proof than mine can be given by help of theorem 1.8.3 in [6] but my method leads to the deeper-lying Theorem 5.)

It is obvious that the condition is necessary. In order to prove the sufficiency, we prove somewhat more. Let $0 < \varepsilon < \frac{1}{2}$ and let (2) $\Rightarrow$ (3). Then we prove that C for ε small is W-almost periodic and that its mean value MC is positive. Then of course C is not empty. We do it by proving that MC, when (2) $\Rightarrow$ (3), and ε is small, does not depend on $a_1,\dots,a_p$ (and consider $a_1 = \dots = a_p = 0$).

A generalization of this fact is used in the second direction we take, which consists in finding formulas for the mean value of C and more general sets.

For orientation about W-almost periodic (W-ap) functions on a topological group G, about the Bohr compactification BG of G, and about the W_1-ap functions on BG we refer to [2], [3], [4].

As stated above, G is a topological abelian group. Let χ: $G \to \mathrm{R}/\mathrm{Z} = H$ be a continuous additive character of G. We consider H as the interval $0 \leqq y \leqq 1$ with identification of 0 and 1, modulo 1 addition, and usual circle topology.

We shall need the following

THEOREM 2. *Let the values of $\chi(x)$ be everywhere dense on H and let g: $H \to \mathrm{C}$ be a Riemann-integrable function on H with the Fourier series*

$$(4)\qquad g(y) \sim \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n y}$$

Then $g(\chi(x))$ is a W-ap function on G with the Fourier series

$$g(\chi(x)) \sim \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n\chi(x)} \tag{5}$$

where $e^{2\pi i n\chi(x)}$

for each integer n is a certain continuous multiplicative character: $G \to \{z\colon z \in \mathbf{C}, |z| = 1\}$.

PROOF. Let

$$A = (\alpha_1, \ldots, \alpha_Q; x'_1, \ldots, x'_Q)$$

where $\alpha_q > 0$ and $\sum_{q=1}^{Q} \alpha_q = 1$ and $x'_q \in G$.

We know that the Riemann-integral

$$\int_0^1 \left| g(y) - \sum_{n=-N}^{N} c_n e^{2\pi i n y} \right| dy \to 0 \text{ for } N \to \infty$$

We shall show that

$$0 \leqq \bar{M} \left| g(\chi(x)) - \sum_{n=-N}^{N} c_n e^{2\pi i n\chi(x)} \right| = \tag{6}$$

$$\inf_A \sup_x \sum_{q=1}^{Q} \alpha_q \left| g(\chi(x + x'_q)) - \sum_{n=-N}^{N} c_n e^{2\pi i n\chi(x + x'_q)} \right| \leqq$$

$$\int_0^1 \left| g(y) - \sum_{n=-N}^{N} c_n e^{2\pi i n y} \right| dy \to 0 \text{ for } N \to \infty$$

For a Riemann-integrable function $f(y)$ on H, viz.

$$f(y) = \left| g(y) - \sum_{n=-N}^{N} c_n e^{2\pi i n y} \right|$$

we know that

$$\sum_{q=1}^{Q} f(y + \eta_q)((y + y_q) - (y + y_{q-1})) \to \int_0^1 f(y) dy$$

uniformly in y for $|D| \to 0$ where D: $y_0 = 0 < y_1 < y_2 < \ldots < y_Q = 1$ and $|D| = \max_q (y_q - y_{q-1})$ and $y_{q-1} \leqq \eta_q \leqq y_q$. Thus to $\varepsilon > 0$ we can find $\delta > 0$ such that

$$\sum_{q=1}^{Q} \left| g(y+\eta_q) - \sum_{n=-N}^{N} c_n e^{2\pi i n(y+\eta_q)} \right| (y_q - y_{q-1}) \leqq \int_0^1 f(y)dy + \varepsilon$$

for $y \in H$ and $|D| < \delta$. We take $y = \chi(x)$ and let $y_0 = 0 = \chi(x_0) < y_1 = \chi(x_1) < y_2 = \chi(x_2) < \ldots < y_Q = \chi(x_Q) = 1$ where $x_0 = x_Q = 0$, and $\chi(x_{q-1}) \leqq \eta_q = \chi(\xi_q) \leqq \chi(x_q)$ and $|D| < \delta$. Here we have used that the values of $\chi(x)$ are everywhere dense on H. Then we get

$$\sup_x \sum_{q=1}^{Q} (\chi(x_q) - \chi(x_{q-1})) \left| g(\chi(x+\xi_q)) - \sum_{n=-N}^{N} c_n e^{2\pi i n \chi(x+\xi_q)} \right|$$

$$\leqq \int_0^1 \left| g(y) - \sum_{n=-N}^{N} c_n e^{2\pi i n y} \right| dy + \varepsilon$$

Since $\chi(x_q) - \chi(x_{q-1}) > 0$ and $\sum_{q=1}^{Q} (\chi(x_q) - \chi(x_{q-1})) = 1$

we see that

$$\bar{M} \left| g(\chi(x)) - \sum_{n=-N}^{N} c_n e^{2\pi i n \chi(x)} \right| =$$

$$\inf_A \sup_x \sum_{q=1}^{Q} \alpha_q \left| g(\chi(x+x_q')) - \sum_{n=-N}^{N} c_n e^{2\pi i n \chi(x+x_q')} \right| \leqq$$

$$\int_0^1 \left| g(y) - \sum_{n=-N}^{N} c_n e^{2\pi i n y} \right| dy + \varepsilon$$

for every $\varepsilon > 0$ and hence (6) follows.

(6) shows that

$$\bar{M} \left| g(\chi(x)) - \sum_{n=-N}^{N} c_n e^{2\pi i n \chi(x)} \right| \to 0 \text{ for } N \to \infty$$

and hence $g(\chi(x))$ is W-ap with the Fourier series

$$g(\chi(x)) \sim \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n \chi(x)}$$

when we note that the multiplicative characters

$$e^{2\pi i n \chi(x)}, \quad n \in \mathbf{Z}$$

are continuous on G since $e^{2\pi i n y}$ is continuous on H and $y = \chi(x)$ is continuous on G, and they are different, because otherwise, for a certain $n \neq 0$

$$n\chi(x) \equiv 0 (\text{mod } 1) \text{ for } x \in G$$

which contradicts that the values of $\chi(x)$ are everywhere dense on H.

A special consequence of (4) and (5) is that

$$M_g(\chi(x)) = c_0 = \int_0^1 g(y)dy$$

which for $G = \mathbf{Z}$ gives a result of Weyl, viz. that for an irrational number α the sequence $\alpha n, n = 1, 2, \ldots$ satisfies

$$\frac{1}{N}\sum_{n=1}^{N} g(\alpha n (\text{mod } 1)) \to \int_0^1 g(y)dy \text{ for } N \to \infty$$

and thus, in particular, is equi-distributed modulo 1.

Next we assume that each of the first $t, 0 \leqq t \leqq p$, of the characters in (1) do not have values that are everywhere dense on H, while each of the rest do, and intend to prove the following

THEOREM 3. *Let* (2) $\Rightarrow$ (3) *and let* $0 < \varepsilon < \frac{1}{2}$. *Then the set* C_1 *of solutions of the* p *relations*

$$(7) \qquad \begin{gathered} \chi_1(x) \equiv a_1 (\text{mod } 1) \\ \cdots\cdots\cdots\cdots \\ \chi_t(x) \equiv a_t (\text{mod } 1) \\ |\chi_{t+1}(x) - a_{t+1}| \leqq \varepsilon (\text{mod } 1) \\ \cdots\cdots\cdots\cdots \\ |\chi_p(x) - a_p| \leqq \varepsilon (\text{mod } 1) \end{gathered}$$

is W-ap *and* $MC_1 > 0$. *Further,* MC_1 *does not depend on* $a_1, \ldots, a_p$. *For* ε *small, say* $0 < \varepsilon < \varepsilon_1 < \frac{1}{2}$, $C_1 = C$, *so that Theorem* 1 *follows.*

PROOF. Let T_j for $j = 1, \ldots, t$ be the smallest positive integer with

$$T_j \chi_j(x) \equiv 0 (\text{mod } 1) \text{ for } x \in G$$

Let $g_j(y)$ be the characteristic function of the one-point set $\{a_j (\text{mod } 1)\}$ on H for $j = 1, \ldots, t$ and of the set $a_j - \varepsilon \leqq y \leqq a_j + \varepsilon (\text{mod } 1)$ on H for $j = t + 1, \ldots, p$. Then the characteristic function of C_1 is

$$(8) \qquad C_1(x) = g_1(\chi_1(x)) \ldots g_p(\chi_p(x)).$$

Since (2) $\Rightarrow$ (3) we have $T_j a_j \equiv 0 \pmod 1$ for $j = 1, \ldots, t$, abbreviated $Ta \equiv 0 \pmod 1$, and hence for ε small, say $0 < \varepsilon < \varepsilon_1 < \frac{1}{2}$, the sets C and C_1 are identical. Similarly we write g for g_j and χ for χ_j when $j = 1, \ldots, t$.

Now

$$\frac{1}{T} \sum_{n=0}^{T-1} e^{2\pi i \frac{S}{T} n} = \begin{cases} 1 & \text{for } S = 0 \\ 0 & \text{for } S = 1, \ldots, T-1 \end{cases}$$

and if

$$a \equiv \frac{S_0}{T} \pmod 1$$

we have

$$\frac{1}{T} \sum_{n=0}^{T-1} e^{2\pi i \frac{S - S_0}{T} n} = \begin{cases} 1 & \text{for } S \equiv S_0 \pmod T \\ 0 & \text{otherwise} \end{cases}$$

$$= \frac{1}{T} \sum_{n=0}^{T-1} e^{-2\pi i n a} e^{2\pi i \frac{S}{T} n} = \frac{1}{T} \sum_{n=0}^{T-1} e^{-2\pi i n a} e^{2\pi i n \chi(x)}$$

$$= \begin{cases} 1 & \text{for } \chi(x) \equiv a \pmod 1 \\ 0 & \text{otherwise} \end{cases} = g(\chi(x))$$

a trigonometric polynomial, where the T continuous multiplicative characters

$$e^{2\pi i n \chi(x)}, \quad n = 0, \ldots, T-1$$

are different.

$g_{t+1}(\chi_{t+1}(x)), \ldots, g_p(\chi_p(x))$ are W-ap on account of Theorem 2, that also gives us their Fourier series. For $j = t+1, \ldots, p$, if

$$g_j(y) \sim \sum_{n=-\infty}^{\infty} c_{j,n} e^{2\pi i n y}$$

we have $c_{j,0} = \displaystyle\int_{a_j - \varepsilon}^{a_j + \varepsilon} 1\, dy = 2\varepsilon$, and for $n \neq 0$

$$c_{j,n} = \int_{a_j - \varepsilon}^{a_j + \varepsilon} e^{-2\pi i n y}\, dy = e^{-2\pi i n a_j} \frac{\sin 2\pi n \varepsilon}{\pi n}$$

We let it mean 2ε if $n = 0$.

The multiplication theorem for W-ap functions on topological abelian groups is proved as for the usual almost periodic functions in [1], so the Fourier series of the W-ap function $C_1(x)$ in (8) is obtained by formal multiplication of its factors. In particular, since (2) $\Rightarrow$ (3),

$$MC_1 = \frac{1}{T_1 \dots T_t} \sum \frac{\sin 2\pi n_{t+1}\varepsilon}{\pi n_{t+1}} \dots \frac{\sin 2\pi n_p \varepsilon}{\pi n_p}$$

$$n_1\chi_1(x) + \dots + n_p\chi_p(x) \equiv 0 \,(\mathrm{mod}\ 1) \text{ for } x \in G$$

$$0 \leqq n_1 < T_1, \dots, 0 \leqq n_t < T_t$$

where the series, as for the usual almost periodic functions, is absolutely convergent.

We see that MC_1, when (2) $\Rightarrow$ (3), is independent of $a_1, \dots, a_p$.

For $a_1 = \dots = a_p = 0$, trivially (2) $\Rightarrow$ (3), and the set C_0 of solutions of (7) in this case has $MC_0 = MC_1$. We shall show that $MC_0 > 0$. We divide the cube H^{p-t} in q^{p-t} closed and congruent cubes, where the positive integer q satisfies $\frac{1}{q} \leqq \varepsilon$. We consider the points

$$(\chi_1(x), \dots, \chi_p(x)) \in \left\{0, \frac{1}{T_1}, \dots, \frac{T_1 - 1}{T_1}\right\} \times \dots$$
$$\times \left\{0, \frac{1}{T_t}, \dots, \frac{T_t - 1}{T_t}\right\} \times H^{p-t} \text{ for } x \in G.$$

There are at most $T_1 \dots T_t q^{p-t}$ possibilities for the simultaneous choice (i.e. for a given x) of the values of $\chi_1(x), \dots, \chi_t(x)$ and a small cube in which $(\chi_{t+1}(x), \dots, \chi_p(x))$ is lying. Let $x_1, \dots, x_j, \dots, x_s$ give all the possibilities where $s \leqq T_1 \dots T_t q^{p-t}$. For an arbitrary $x \in G$ we can therefore find an x_j such that

$$\chi_1(x - x_j) \equiv 0 \,(\mathrm{mod}\ 1)$$
$$\dots\dots\dots\dots\dots\dots\dots$$
$$\chi_t(x - x_j) \equiv 0 \,(\mathrm{mod}\ 1)$$
$$|\chi_{t+1}(x - x_j)| \leqq \varepsilon \,(\mathrm{mod}\ 1)$$
$$\dots\dots\dots\dots\dots\dots\dots$$
$$|\chi_p(x - x_j)| \leqq \varepsilon \,(\mathrm{mod}\ 1)$$

Thus

$$MC_1 = MC_0 = \underline{M}C_0 \geqq \inf_x \frac{1}{s} \sum_{j=1}^{s} C_0(x - x_j)$$
$$\geqq \frac{1}{s} \geqq \frac{1}{T_1 \dots T_t} \left(\frac{1}{q}\right)^{p-t} > 0.$$

We remark that Theorem 1, in contrast to Theorem 2 and 3, is contained in its discrete form (we give G the discrete topology). We could also say that the characters do not have to be continuous in the original topology. Let us use the discrete form to the discrete abelian group G_1 of the continuous additive characters χ of the topological abelian group G. Since

$$x(\chi) = \chi(x), x \in G, \chi \in G_1$$

is an additive character of G_1, we get the

THEOREM 4. *Let $x_1, \ldots, x_p \in G$ and $a_1, \ldots, a_p \in \mathrm{R}$. A necessary and sufficient condition that the p inequalities*

$$|\chi(x_1) - a_1| \leqq \varepsilon \pmod 1$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$|\chi(x_p) - a_p| \leqq \varepsilon \pmod 1$$

have a solution $\chi \in G_1$ is that for arbitrary integers $n_1, \ldots, n_p$ with

$$\chi(n_1 x_1 + \ldots + n_p x_p) \equiv 0 \pmod 1 \text{ for } \chi \in G_1 \tag{9}$$

also

$$n_1 a_1 + \ldots + n_p a_p \equiv 0 \pmod 1$$

If, in particular, G is maximally almost periodic, (9) *means*

$$n_1 x_1 + \ldots + n_p x_p = 0$$

3. The second direction.

In the following we deal with continuous additive characters χ of the topological abelian group G, i.e. χ: $G \to \mathrm{R}/\mathrm{Z}$, but we prefer to let χ be multi-valued with values in R, only determined modulo 1. We build on a generalized form of Theorem 3, but since its proof goes in nearly the same way, we omit it. By H we understand the interval $0 \leqq y < 1$.

Let exactly $t, 0 \leqq t < p$, of the continuous additive characters $\chi_1(x), \ldots, \chi_p(x)$ be non-dense on R. We know the structure of the closed subgroups of R^p and

$$K = \overline{\{(\chi_1(x), \ldots, \chi_p(x)) : x \in G\}} = \{(\chi_1(x), \ldots, \chi_p(x)) : x \in BG\}$$

is such a subgroup, and it contains the subgroup Z^p.

Except when $K = \mathrm{R}^p$, by eventually rearranging the characters, we get

$$K = R \oplus F$$

where R is a vector space of the form

$$y_{s+1} = l(y_1, \ldots, y_s), \ldots, y_p = m(y_1, \ldots, y_s)$$

with linear forms $l,\ldots,m$ which have rational coefficients, and F is a discrete infinite subgroup of the $y_{s+1},\ldots,y_p$-space for which $F/\mathbf{Z}^{p-b}$ is finite, with, say, N elements.

This follows from Theorem 1 by which when

$J = \{(n_1,\ldots,n_p): n_1,\ldots,n_p \in \mathbf{Z}, n_1\chi_1(x) + \ldots + n_p\chi_p(x) \equiv 0 \pmod 1 \text{ for } x \in G\}$ then

$K = \{(a_1,\ldots,a_p): a_1,\ldots,a_p \in \mathbf{R}, n_1 a_1 + \ldots + n_p a_p \equiv 0 \pmod 1 \text{ for } (n_1,\ldots,n_p) \in J\}$

and since the dimension of J is $< p$, the largest vector space contained in K has dimension $s \geqq 1$, i.e. it is not the 0-space. The rationality of the coefficients of $l,\ldots,m$ stems from the fact that J is a subgroup of $\mathbf{Z}^p$.

When $K \neq \mathbf{R}^p$ we denote by P the set of points $(y_1,\ldots,y_p)$ given by

$$y_1,\ldots,y_s \in H$$

$$l(y_1,\ldots,y_s) \leqq y_{s+1} < l(y_1,\ldots,y_s) + 1$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$m(y_1,\ldots,y_s) \leqq y_p < m(y_1,\ldots,y_s) + 1$$

When $K = \mathbf{R}^p$ we let $P = H^p$. Obviously it is no restriction on $x \in BG$ to demand that

$$(\chi_1(x),\ldots,\chi_p(x)) \in P$$

because the values of the characters are only determined modulo 1.

We shall prove the following

THEOREM 5. *Let E be a subset of P, and if $K = \mathbf{R}^p$, let E be Jordan-measurable. If $K \neq \mathbf{R}^p$, let the projection on the $y_1,\ldots,y_s$-space*

$$\operatorname{proj} E \cap (R + f_j) \text{ for } j = 1,\ldots,N$$

be Jordan-measurable, where

$$\{f_1,\ldots,f_N\} = F \cap H^{p-s}$$

In both cases the set

$$B = \{x : x \in G, (\chi_1(x),\ldots,\chi_p(x)) \in E\}$$

is W-ap. If $K = \mathbf{R}^p$ then

$$MB = mE$$

If $K \neq \mathbf{R}^p$ then

$$MB = \frac{1}{N}\sum_{j=1}^{N} m \operatorname{proj} E \cap (R + f_j)$$

(We shall express the theorem by the phrase:

$$(\chi_1(x), \ldots, \chi_p(x)), x \in G$$

is W-ap equi-distributed from G into its closure K modulo 1.)

PROOF. Let

$$f_1 = (y_{s+1}^1, \ldots, y_p^1), \ldots, f_N = (y_{s+1}^N, \ldots, y_p^N)$$

where for instance $f_1 = (0, \ldots, 0)$. It is understood that they lie in the $y_{s+1}, \ldots, y_p$-space. Obviously Nf_j has integral coordinates.

For a positive integer q we divide H^s in q^s congruent cubes of the type

$$a_1 - \frac{1}{2q} \leqq y_1 < a_1 + \frac{1}{2q}, \ldots, a_s - \frac{1}{2q} \leqq y_s < a_s + \frac{1}{2q}$$

In each cube we choose the center $(a_1, \ldots, a_s) \in H^s$ and consider the points

$$(a_1, \ldots, a_s, l(a_1, \ldots, a_s) + y_{s+1}^j, \ldots, m(a_1, \ldots, a_s) + y_p^j) =$$

$$(a_1, \ldots, a_s, a_{s+1}^j, \ldots, a_p^j) \in P \cap (R + f_j) \subset P \cap K$$

We note that for two different j and k at least one of the numbers

$$|a_{s+1}^j - a_{s+1}^k|, \ldots, |a_p^j - a_p^k| \text{ is } \geqq \frac{1}{N}$$

We also note that for a fixed j and a fixed $Y = (y_{s+1}, \ldots, y_p) \in F \setminus H^{p-s}$ at least one of the numbers $|y_{s+1}^j - y_{s+1}|, \ldots, |y_p^j - y_p|$ is $\geqq \dfrac{1}{N}$.

Let $\varepsilon_0 = \min\left(\dfrac{1}{N}, \varepsilon_1\right)$. In generalization of Theorem 3 the Nq^s sets

$$C = C_j = \Big\{x : x \in BG, a_1 - \frac{1}{2q} \leqq \chi_1(x) < a_1 + \frac{1}{2q}, \ldots, a_s - \frac{1}{2q} \leqq \chi_s(x)$$

$$< a_s + \frac{1}{2q}, a_{s+1}^j - \frac{\varepsilon_0}{3} \leqq \chi_{s+1}(x) \leqq a_{s+1}^j + \frac{\varepsilon_0}{3}, \ldots, a_p^j - \frac{\varepsilon_0}{3} \leqq$$

$$\chi_p(x) \leqq a_p^j + \frac{\varepsilon_0}{3} \pmod 1\Big\} \text{ for } q \text{ large}$$

are W_1-ap and since

$$n_1 a_1 + \ldots + n_s a_s + n_{s+1} a_{s+1}^j + \ldots + n_p a_p^j \equiv 0 \pmod 1$$

for $(n_1, \ldots, n_p) \in J$, this generalization tells us that the Nq^s sets C all have the same W_1-mean value M_1^q.

Since $l, \ldots, m$ are uniformly continuous on H^s the Nq^s sets C will, for q sufficiently large have a disjoint union which is equal to

$$\left\{x : x \in BG, (\chi_1(x), \ldots, \chi_p(x)) \in P \cap \bigcup_{j=1}^{N} (R + f_j) = P \cap K\right\} = BG$$

Thus

$$(*) \qquad Nq^s M_1^q = M_1 BG = 1$$

Now

$$E \cap K = \bigcup_{j=1}^{N} E \cap (R + f_j) \text{ (disjoint union)}$$

where

$$\operatorname{proj} E \cap (R + f_j), \ j = 1, \ldots, N$$

is Jordan-measurable. Let

$$\bigcup_{(a_1,\ldots,a_s)}' C_j = B_j' \text{ and } \bigcup_{(a_1,\ldots,a_s)}'' C_j = B_j''$$

where $\bigcup'$ denotes union over those centers $(a_1, \ldots, a_s)$ whose A_j' cubes

$$a_1 - \frac{1}{2q} \leqq y_1 < a_1 + \frac{1}{2q}, \ldots, a_s - \frac{1}{2q} \leqq y_s < a_s + \frac{1}{2q}$$

are contained in

$$\operatorname{proj} E \cap (R + f_j)$$

while $\bigcup''$ denotes union over those centers $(a_1, \ldots, a_s)$ whose A_j'' cubes have points in common with this projection.

Since the projection is Jordan-measurable, both

$$\frac{A_j'}{q^s} \text{ and } \frac{A_j''}{q^s} \to m \operatorname{proj} E \cap (R + f_j) \text{ for } q \to \infty$$

The set

$$B_0 = \{x : x \in BG, (\chi_1(x), \ldots, \chi_p(x)) \in E\}$$

satisfies

$$B_0 \supset B_1' \cup \ldots \cup B_N' \qquad \text{(disjoint union)}$$

and

$$B_0 \subset B_1'' \cup \ldots \cup B_N'' \qquad \text{(disjoint union)}$$

Here B'_j *and* B''_j are W_1-ap and by (*)

$$M_1 B'_j = A'_j M_1^q = \frac{A'_j}{Nq^s}$$

and

$$M_1 B''_j = A''_j M_1^q = \frac{A''_j}{Nq^s}$$

Hence B_0 is W_1-ap and

$$M_1 B_0 = \frac{1}{N} \sum_{j=1}^{N} m \operatorname{proj} E \cap (R + f_j)$$

From this follows that the "restriction" of B_0 to G, viz.

$$B = \{x : x \in G, (\chi_1(x), \ldots, \chi_p(x)) \in E\}$$

is W-ap on G and

$$MB = M_1 B_0 = \frac{1}{N} \sum_{j=1}^{N} m \operatorname{proj} E \cap (R + f_j)$$

The case $K = \mathsf{R}^p$ can be treated in a similar, but simpler, way.

For E lying in a fundamental domain modulo 1 of the form

$$c_1 \leqq y_1 < c_1 + 1, \ldots, c_p \leqq y_p < c_p + 1$$

instead of P, we assume, for $K \neq \mathsf{R}^p$, that

$$\operatorname{proj} E \cap (R + f)$$

is Jordan-measurable for $f \in F$. Then

$$B = \{x : x \in G, (\chi_1(x), \ldots, \chi_p(x)) \in E\}$$

is W-ap with

$$MB = \frac{1}{N} \sum_{f \in F} m \operatorname{proj} E \cap (R + f)$$

where only a finite number of terms are different from zero. For $K = \mathsf{R}^p$ we assume that E is Jordan-measurable. Then B is W-ap and

$$MB = mE$$

If $t = p$, and the set E is contained in a fundamental domain modulo 1, then the set B is almost periodic, and

$$MB = \frac{|E \cap K|}{|H^p \cap K|}$$

where $|\ |$ denotes "number of elements in". This follows from the proof of Theorem 3..

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