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ON SIMPLE GERMS WITH NON-ISOLATED SINGULARITIES

A. ZAHARIA

§1. Introduction.

Let $\mathcal{O} = \mathcal{O}_n$ denote the local ring of germs of analytic functions $f: (\mathbf{C}^n, 0) \to \mathbf{C}$ and m its maximal ideal. For an analytic germ $f \in \mathcal{O}$ we denote by J_f its Jacobi ideal, namely $J_f = \left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}\right)$. For an ideal $I \subset \mathcal{O}$ we consider as in [8], [9]:

– the primitive ideal $\int I$, defined by $\int I = \{f \in \mathcal{O} \mid (f) + J_f \subset I\}$; we have $I^2 \subset \int I \subset I$;

– the group $\mathcal{D}_I$ of local analytic isomorphisms $h: (\mathbf{C}^n, 0) \to (\mathbf{C}^n, 0)$ such that $h^*(I) = I$; it is a subgroup of the group of all germs of local analytic isomorphisms of $(\mathbf{C}^n, 0)$.

$\mathcal{D}_I$ acts on $\int I$ and we shall consider the $\mathcal{R}_I$ (right-equivalence) relation on $\int I$. In the next section we prove the following.

THEOREM 1. *Let $I \subset \mathcal{O}$ be a radical ideal defining a germ of a quasihomogeneous complete intersection in $(\mathbf{C}^n, 0)$ with isolated singularity. Suppose that there exist $\mathcal{R}_I$-simple germs in $\int I$. Then in some coordinates $(z_1, \ldots, z_n)$ of $(\mathbf{C}^n, 0)$ we have either*

a) *there exists $k \in \{1, \ldots, n\}$ such that $I = (z_1, \ldots, z_k)$, or*

b) *there exists $k \in \{1, \ldots, n\}$ and a quasihomogeneous isolated singularity $g = g(z_1, \ldots, z_k) \in \mathcal{O}_k$ such that $I = (g, z_{k+1}, \ldots, z_n)$.*

A. Némethi has proved a similar result in [7] for the case when $I = (f^s)$ where $s \geqq 1$ and $f \in \mathcal{O}$ is an isolated singularity. When $n = 3$, D. Siersma has considered a similar problem for the inner modality (see [12]).

In the last section we derive the list of $\mathcal{R}_I$-simple germs for $I = (z_1, z_2)$.

§2. Proof of Theorem 1.

We recall from [8], [9] that for an ideal $I \subset \mathcal{O}$ and for $f \in \int I$, the tangent space at

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f to the $\mathscr{R}_I$-orbit of f is defined by

$$T_I(f) = \left\{\eta(f) \middle| \eta = \sum_{j=1}^{n} \eta_j \frac{\partial}{\partial z_j} \text{ with } \eta(I) \subset I \text{ and } \eta_j \in m \text{ for } j = 1,\dots,n\right\}$$

and the I-codimension of f is

$$c_I(f) = \dim_C \frac{\int I}{T_I(f)}.$$

Let $f_1,\dots,f_p$ be a minimal set of quasihomogeneous generators of I. Let q be the dimension of the C-vector space $(I + m^2)/m^2$. If $q = p$, we have a) with $k = q = p$.

Suppose that $q < p$. Using a linear change of coordinates, we can assume, without altering the quasihomogeneity of $f_1,\dots,f_p$, that $f_j(z) = z_j +$ higher monomials not containing z_j, for $j = 1,\dots,q$ (we assume that the weights of the coordinates are positive). Thus, we can consider, by subtracting suitable multiples of $f_1,\dots,f_q$, if necessary, that $f_{q+1},\dots,f_p$ are quasihomogeneous polynomials, not depending on $z_1,\dots,z_q$. It follows that, in a suitable system z of coordinates, the ideal I is generated by $f_1 = z_1,\dots,f_q = z_q$, $f_q = z_q$, $f_{q+1},\dots,f_p$, where $f_{q+1},\dots,f_p \in m^2$ are quasihomogeneous polynomials depending only on $z_{q+1},\dots,z_n$.

Since there exist $\mathscr{R}_I$-simple germs in $\int I$, we can find $f \in \int I$ such that $c_I(f) = 0$. (The $\mathscr{R}_I$-simple germs are defined similarly with the simple isolated singularities; see for example [2] or [4].) From [8], [9], we have $\int I = I^2$ and we can write $f = \sum_{i,j=1}^{p} g_{ij} f_i f_j$, with $g_{ij} = g_{ji}$. Let r be the rank of the matrix $(g_{ij}(0))_{i,j=1,q}$. Then r is also the rank of the Hessian matrix evaluated in 0, $\left(\frac{\partial^2 f}{\partial z_i \partial z_j}(0)\right)_{i,j=1,n}$. As in the proof of Morse Lemma (see for example [6]) we can obtain a system $\tilde{z}$ of coordinates, with $\tilde{z}_j = z_j$ for $j > q$, such that I is generated by $\tilde{f}_1 = \tilde{z},\dots,\tilde{f}_q = \tilde{z}_q$, $\tilde{f}_{q+1} = f_{q+1},\dots,\tilde{f}_p = f_p$ and such that

$$f = \tilde{z}_1^2 + \dots + \tilde{z}_r^2 + \sum_{i,j=r+1}^{p} \tilde{g}_{ij} \tilde{f}_i \tilde{f}_j, \tag{1}$$

with $\tilde{g}_{ij} = \tilde{g}_{ji}$ and with $\tilde{g}_{ij} \tilde{f}_i \tilde{f}_j \in m^3$. It is easy to see that for any $i,j \geqq r + 1$, there exists $h_{ij} = h_{ij}(\tilde{z}_{r+1},\dots,\tilde{z}_n)$ with $h_{ij} \tilde{f}_i \tilde{f}_j \in m^3$ and such that for any $k \in \mathrm{N}$, $k \geqq 2$, we have that f is $\mathscr{R}_I$-equivalent to $\tilde{z}_1^2 + \dots + \tilde{z}_r^2 + \sum_{i,j=r+1}^{p} h_{ij} \tilde{f}_i \tilde{f}_j + \sum_{i,j=r+1}^{p} \varphi_{ij} f_i f_j$, for some $\varphi_{ij} \in (\tilde{z}_1,\dots,\tilde{z}_r)^k$. Since $c_I(f) < \infty$, f is I-finitely determined (see [8], [9]). Hence we can assume that in (1) the germs $\tilde{g}_{ij}$ do not depend on $\tilde{z}_1,\dots,\tilde{z}_r$.

We shall write in the sequel z for $\tilde{z}$, f_j for $\tilde{f}_j$ and g_{ij} for $\tilde{g}_{ij}$. Since $c_I(f) = 0$, we must have $T_I(f) = \int I = I^2$; we prove that this equality implies that $r = q = p - 1$.

Let $\theta_I = \left\{\eta = \sum_{j=1}^{n} \eta_j \frac{\partial}{\partial z_j} \middle| \eta(I) \subset I\right\}$ be the $\mathcal{O}$-module of logarithmic vector fields for I. Since $I = (f_1, \ldots, f_p)$ is a reduced quasihomogeneous complete intersection in $(\mathbf{C}^n, 0)$ with isolated singularity, the $\mathcal{O}$-module θ_I is generated by the following vector fields (see for example [3]):

(A) $f_i \frac{\partial}{\partial z_j}$, where $i = 1, \ldots, p$ and $j = 1, \ldots, n$;

(B) the "trivial vector fields"

$$\begin{vmatrix} \frac{\partial}{\partial z_{i_1}} & \cdots & \frac{\partial}{\partial z_{i_{p+1}}} \\ \frac{\partial f_1}{\partial z_{i_1}} & \cdots & \frac{\partial f_1}{\partial z_{i_{p+1}}} \\ \cdots & \cdots & \cdots \\ \frac{\partial f_p}{\partial z_{i_1}} & \cdots & \frac{\partial f_p}{\partial z_{i_{p+1}}} \end{vmatrix}$$

for all $(p + 1)$-tuples $(i_1, \ldots, i_{p+1})$ satisfying $1 \leqq i_1 \leqq i_2 \leqq \ldots \leqq i_{p+1} \leqq n$;

(C) the Euler vector field $E = \sum_{j=1}^{n} w_j z_j \frac{\partial}{\partial z_j}$, where $w_1, \ldots, w_n$ are the weights of the coordinates.

It is clear that $T_I(f) = \theta_I(f)$.

We recall that $f_j = z_j$ for $1 \leqq j \leqq q$ and $f_{q+1}, \ldots, f_p \in m^2$ do not depend on $z_1, \ldots, z_q$. Also we recall that $f = z_1^2 + \ldots + z_r^2 + \sum_{i,j=r+1}^{p} g_{ij} f_i f_j$ with $g_{ij} = g_{ji}$ not depending on $z_1, \ldots, z_r$ and with $g_{ij} f_i f_j \in m^3$.

Suppose first that $r < q$. Then a moment's thought will convince us that for any $\eta \in \theta_I$, if we consider the expansion of $\eta(f)$ in a power series, then the coefficient of z_q^2 is zero. Hence $z_q^2 \notin T_I(f) = I^2$, a contradiction. It follows that $r = q$.

We look now for $f_{q+1}^2, \ldots, f_p^2$. It is easy to see that if $\eta \in \theta_I$ is one of the generators from (A) or (B), then $\eta(f)$ belongs to the ideal $L = m \cdot (f_{q+1}, \ldots, f_p)^2 + (z_1, \ldots, z_q) \cdot (f_{q+1}, \ldots, f_p) + (z_1, \ldots, z_q)^2$. On the other hand, for any germ $g \in m$ we have also $(gE)(f) \in L$. Thus $\theta_I(f) = L + \mathbf{C} \cdot E(f)$. If $p - q \geqq 2$ we have the uniqueness of the weights $w_{q+1}, \ldots, w_n$ (see for example [4]), hence $f_{q+1}^2, \ldots, f_p^2$ can not belong simultaneously to $\theta_I(f)$, in contradiction with the equality $I^2 = \theta_I(f)$. It follows that $q + 1 = p$. The theorem is proved.

§3. The simple germs for $I = (z_1, z_2)$.

D. Siersma has found the $\mathcal{R}_I$-simple germs when $I = (z_1, \ldots, z_{n-1}) \subset \mathcal{O}$ in [10] and for $I = (z_1 z_2, z_3, \ldots, z_n) \subset \mathcal{O}$ in [12]. For the case when $I = (z_1) \subset \mathcal{O}$, the list of $\mathcal{R}_I$-simple germs follows from the work of V.I. Arnold [1] (see for example [13]).

In the sequel we derive the list of $\mathcal{R}_I$-simple germs for $I = (z_1, z_2)$. We shall suppose that $n \geqq 4$ and we shall consider only germs $f \in I^2$ with $j^2 f = 0$. (The simple germs $f \in I^2$ with $j^2 f \neq 0$ are suspensions of those in [13].)

We use the following classical lemma:

LEMMA. *Let $f_t = f + t \cdot \phi \in I^2$ be a family of germs, with $t \in \mathbb{R}$.*

a) *If $\phi \in \mathcal{T}(f_t)$ for every $t \in \mathbb{R}$, then, for any $t \in \mathbb{R}$, f_t is $\mathcal{R}_I$-equivalent with f_0.*

b) *If $\phi \notin \mathcal{T}(f_t)$ for every $t \in \mathbb{R}$, then, for any $t \in \mathbb{R}$, f_t is not $\mathcal{R}_I$-simple.*

If we denote the coordinates $z_3, \ldots, z_n$ by $u_1, \ldots, u_{n-2}$ and the Milnor number of an isolated singularity g by $\mu(g)$, we have the following:

THEOREM 2. *Let $I = (z_1, z_2) \subset \mathcal{O}$ and $f \in I^2$ with $j^2 f = 0$. Then f is $\mathcal{R}_I$-simple if and only if f is $\mathcal{R}_I$-equivalent to a germ in the following table.*

	Normal form of f	$c_I(f)$	Conditions
I_n	$u_1 z_1^2 + u_2 z_2^2 + u_3 z_1 z_2$	3	$n \geqq 5$
I_4	$u_1 z_1^2 + u_2 z_2^2$	3	$n = 4$
II	$u_1 z_1^2 + u_2 z_2^2 + z_1 z_2 \cdot g(u_3, \ldots, u_{n-2})$	$n - 2 + \mu(g)$	$n \geqq 5$; $g \in$ A–D–E
III	$u_1 z_1 z_2 + u_2 z_1^2 + z_2^2(z_2 + u_2^k + u_3^2 + \ldots + u_{n-2}^2)$	$k + n - 2$	$n \geqq 4$; $k \geqq 2$
	$u_1 z_1 z_2 + u_2 z_1^2 + z_2^2(z_2 + u_2 u_3 + u_3^k + u_4^2 + \ldots + u_{n-2}^2)$	$k + n - 2$	$n \geqq 5$; $k \geqq 3$
	$u_1 z_1 z_2 + u_2 z_1^2 + z_2^2(z_2 + u_2^2 + u_3^3 + u_4^2 + \ldots + u_{n-2}^2)$	$n + 2$	$n \geqq 5$
IV	$u_1 z_1 z_2 + u_2 z_1^2 + z_2^2(z_2^k + u_2^2 + \ldots + u_{n-2}^2)$	$k + n - 1$	$n \geqq 4$; $k \geqq 2$
Va	$u_1 z_1 z_2 + u_2 z_1^2 + z_2^2(z_2^2 + u_2^3 + u_3^2 + \ldots + u_{n-2}^2)$	$n + 3$	$n \geqq 4$
Vb	$u_1 z_1 z_2 + u_3 z_1^2 + z_2^2(z_2^2 + u_2^3 + u_3^2 + \ldots + u_{n-2}^2)$	$n + 4$	$n \geqq 5$
VI	$u_1 z_1 z_2 + u_2 z_1^2 + z_2^2(z_2 u_2 + u_2^k + u_3^2 + \ldots + u_{n-2}^2)$	$k + n - 1$	$n \geqq 4$; $k \geqq 3$
VI^3	$u_1 z_1 z_2 + u_3 z_1^2 + z_2^2(z_2 u_2 + u_2^3 + u_3^2 + \ldots + u_{n-2}^2)$	$n + 4$	$n \geqq 5$

PROOF. If $f \in I^2$ has $j^2 f = 0$, then $j^3 f = u_1 Q_1(z_1, z_2) + \ldots + u_{n-2} Q_{n-2}(z_1, z_2) + C(z_1, z_2)$, where $Q_1, \ldots, Q_{n-2}$ are quadrics and C is a cubic in z_1, z_2. We suppose that $c_I(f) < \infty$. Hence f is $\mathcal{R}_I$-equivalent to a jet $j^k f$ for sufficiently large k (see [9]).

Let V be the C-vector space generated by $Q_1,\dots,Q_{n-2}$ in the vector space of quadrics in z_1, z_2.

If $\dim V = 3$ then $n \geqq 5$ and we can find in $\mathscr{D}_I$ a linear isomorphismm of $(\mathbf{C}^n, 0)$ such that $j^3 f = u_1 z_1^2 + u_2 z_2^2 + u_3 z_1 z_2$. It follows by [8], [9] that f is $\mathscr{R}_I$-equivalent with $j^3 f$ (f is a $D(1,1)$-type germ) and f is $\mathscr{R}_I$-simple.

If $\dim V \leqq 1$ then f is not $\mathscr{R}_I$-simple. Namely, any neighbourhood of f contains a germ which is $\mathscr{R}_I$-equivalent to a germ $\tilde{f} = u_1 z_1 z_2 + z_1^3 + z_2^3 + z_1^2(u_2^2 + \dots + u_{n-2}^2) + z_2^2 \cdot \varphi(u_2,\dots,u_{n-2})$ where $\varphi \in m^2$. It is easy to see that for any φ, $\tilde{f}$ is not $\mathscr{R}_I$-simple.

If $\dim V = 2$ then, using the classification of pencils of quadrics in z_1, z_2 we can find in $\mathscr{D}_I$ some linear isomorphisms of $(\mathbf{C}^n, 0)$ such that $j^3 f$ is one of the following cubics:

$$u_1 z_1^2 + u_2 z_2^2;\ u_1 z_1 z_2 + u_2 z_1^2 \text{ or } u_1 z_1 z_2 + u_2 z_1^2 + z_2^3.$$

When $j^3 f = u_1 z_1^2 + u_2 z_2^2$ it follows, directly or using the technique of global transversal from [5], that f is $\mathscr{R}_I$-equivalent to $u_1 z_1^2 + u_2 z_2^2 + z_1 z_2 g(u_3,\dots, u_{n-2})$. Now it is easy to see, for $n \geqq 5$, that f is $\mathscr{R}_I$-simple if and only if g is a simple isolated singularity (g is an A–D–E singularity; see [2], or [4] for the normal forms).

If $j^3 f = u_1 z_1 z_2 + u_2 z_1^2 + z_2^3$, then f is $\mathscr{R}_I$-equivalent to $u_1 z_1 z_2 + u_2 z_1^2 + z_2^2(z_2 + g(u_2,\dots,u_{n-2}))$ with $g \in m^2$. It is easy to see that f is $\mathscr{R}_I$-simple if and only if g is a simple boundary singularity in the sense of Arnold, the boundary being $u_2 = 0$ (see [1]).

The most difficult case is when $j^3 f = u_1 z_1 z_2 + u_2 z_1^2$. In this situation f is $\mathscr{R}_I$-equivalent to $u_1 z_1 z_2 + u_2 z_1^2 + z_2^2 h(z_2, u_2,\dots,u_{n-2})$ with $h \in m^2$. If h is a simple boundary singularity with respect to $z_2 = 0$, we change the coordinates such that h becomes the normal form of a B–C–F singularity. Then f is $\mathscr{R}_I$-equivalent to $u_1 z_1 z_2 + \varphi(u_2,\dots,u_{n-2}) z_1^2 + z_2^2 h$, with $\varphi \in m \backslash m^2$, and we obtain the germs in the table by using the lemma and loocking at $j^1 \varphi$.

If h is not a simple boundary singularity then f can be deformed to a germ which is $\mathscr{R}_I$-equivalent to $u_1 z_1 z_2 + \varphi(u_2,\dots,u_{n-2}) z_1^2 + z_2^2 h(z_2, u_2,\dots,u_{n-2})$ where $\varphi \in m \backslash m^2$ and h is one of the following unimodal boundary singularities (see [2]):

$$F_{1,0}\colon z_2^3 + a z_2 u_2^2 + u_2^3 + u_3^2 + \dots + u_{n-2}^2,\ 4a^3 + 27 \neq 0$$

$$K_{4,2}\colon z_2^2 + a z_2 u_2^2 + u_2^4 + u_3^2 + \dots + u_{n-2}^2,\ a^2 \neq 4$$

or $$L_6\colon z_2^2 u_2 + a z_2 u_3 + u_2^2 u_3 + u_3^3 + u_4^2 + \dots + u_{n-2}^2.$$

Using the lemma with $\mathscr{T}(f)$ replaced by $\mathscr{T}(f) + (u_2,\dots,u_{n-2})^3 (z_1)^2$ we obtain that f is not $\mathscr{R}_I$-simple.

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DEPARTMENT OF MATHEMATICS
INCREST
BD. PĂCII 220
79622 BUCHAREST
ROUMANIA