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EMBEDDINGS OF n-DIMENSIONAL LOCALLY COMPACT METRIC SPACES TO $2n$-MANIFOLDS

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1. Introduction.

The classical Pontrjagin-Tolstowa-Hurewicz theorem [16, Theorem 1.11.4] that the set of all embeddings of an n-dimensional compact metric space X to R^q is dense with respect to the compact-open topology in the space of all continuous maps of X to R^q whenever $n \geqq 0$ and $q \geqq 2n + 1$ has recently been complemented by the following theorem.

1.1. THEOREM. *Let $n > 0$ and $q > 0$ be integers with $q \leqq 2n$, and let X be an n-dimensional compact metric space. Then the set of all embeddings of X to R^q is dense with respect to the compact-open topology in the space of all continuous maps of X to R^q if and only if $n \geqq 2$, $q = 2n$, and* $\dim(X \times X) < 2n$.

Recall that $\dim(X \times Y) \leqq \dim X + \dim Y$ for all non-empty separable metric spaces X and Y, with the equality if X is σ-compact and $\dim Y \leqq 1$ [16, Problem 1.9.E(b) and Theorem 1.5.3]. Moreover, $\dim(X \times X) \geqq 2n - 1$ if X is σ-compact and $n = \dim X \geqq 2$ [29, Theorem 41.5], and for each n there is a compact space X realizing the equality. The first example of such kind, for $n = 2$, was constructed in [2], and multiplying this space with the cube I^{n-2} gives an example whenever $n \geqq 3$. Obviously, for each $n \geqq 2$ there is then also a non-compact n-dimensional locally compact separable metric space X with $\dim(X \times X) = 2n - 1$.

Many authors contributed to the proof of Theorem 1.1. The existence of n-dimensional compact metric spaces X for all $n \geqq 2$ satisfying the density condition of 1.1 with $q = 2n$ answers positively to a question of F. D. Ancel and negatively to a conjecture of Y. Sternfeld [36]. Ancel's question was first erroneously answered negatively by D. McCullough and L. R. Rubin [26], but J. Krasinkiewicz and K. Lorentz [23] showed that a lemma of [26] is incorrect if $n \geqq 2$. Theorem 1.1 for $n = 1$ follows, however, from [26]; see [23] and [27].

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Then McCullough and Rubin [27], generalizing ideas of [23], constructed for each $n \geqq 2$ a counterexample to their previous claim (further such examples were constructed by Z. Karno and Krasinkiewicz [20]) and asked whether this is related to the phenomenon that $\dim(X \times X) < 2 \dim X$. That such is indeed the case was answered by Krasinkiewicz [22] who proved the "only if" part of 1.1 for all $n > 0$ with $q = 2n$. Subsequently the "if" part of 1.1 for $n > 2$ was proved by S. Spiez [32] and again, independently and in a different way, by A. N. Dranishnikov and E. V. Shchepin [12], [10]. The case $n = 2$ was then established by Spiez [33–34] and, independently and using different methods, by Dranishnikov, D. Repovš, and Shchepin [10] (announced in [12]). Dranishnikov and Shchepin [12], [10] (the first version) also gave a proof for the "only if" part of 1.1 for all $q \leqq 2n$, but they applied a recent result of D. O. Kiguradze [21], which is based on a classical theorem of G. Chogoshvili [6], whose proof contains a gap as R. Pol noted in 1988 (see the review 90k:54047 about [21] by R. Engelking in Mathematical Reviews). However, in early 1990 Kiguradze proved a theorem which, when applied to the proof of 1.1 in the first version of [10], gives the "only if" part of 1.1 for $q = 2n$. The proof of Kiguradze's theorem is included in the corrected version of [10]. Finally, the "only if" part of 1.1 for all $q \leqq 2n$ was recently proved by Dranishnikov and J. West [13]. In Appendix A we present a short alternative proof of it due to H. Toruńczyk. Both proofs are based on a result of Dranishnikov [9]. The results in [32], [34], [12], and [10] about unstability of intersections of continuous maps of compact metric spaces to R^q which gave the "if" part of 1.1 have since been strengthened by Dranishnikov [7–8], Spież and Toruńczyk [35], and Dranishnikov, Repovš, and Shchepin [11].

The purpose of this paper is to generalize Theorem 1.1 by replacing R^q by an arbitrary topological q-manifold without boundary and by allowing X to be an arbitrary n-dimensional locally compact separable metric space; we then consider majorant approximation, also relative, of proper maps by closed embeddings. Our first main result, Theorem 2.5, is a direct generalization of 1.1 and an application of it. Earlier R. E. Heisey and Toruńczyk [17] and, independently, the author [24] have generalized the classical result similarly. In Theorem 3.11 we show that in the case of Theorem 2.5 where approximation is possible, it is also possible to approximate relatively proper maps extending a fixed closed embedding which is locally homotopically 1-co-connected (1-LCC), i.e., whose image is a 1-LCC subset of the range manifold in the sense of [14]. As shown in Theorem 3.15, at least in the case $n \geqq 3$ it is necessary to assume this 1-LCC-property. If $n \geqq 3$, the approximations can be chosen 1-LCC, even without resorting to the more general results in [31] and [15]. The case $q \geqq 2n + 1$ is due to [17]. The main new ingredient of the proof, Lemma 3.10, follows from each one of the above-mentioned papers [8], [10], [12], and [35].

We adopt the convention that a manifold means a non-empty separable metric topological manifold without boundary.

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2. Approximating proper maps by embeddings.

2.1. FUNCTION SPACES. We recall some well-known notions and facts; see [38, §1], [39, §1], and [24, 1.2]. Let X and Y be metric spaces. We define two topologies for the set $C(X, Y)$ of all continuous maps of X to Y. In the source majorant topology $\mathcal{T}_s$ an open neighbourhood basis of $f \in C(X, Y)$ is given by the sets $U(f, \varepsilon) = \{g \in C(X, Y) \mid \forall x \in X\colon d(f(x), g(x)) < \varepsilon(x)\}$ where $\varepsilon \in C(X, (0, \infty))$. In the target majorant topology $\mathcal{T}_t \subset \mathcal{T}_s$ a neighbourhood basis of $f \in C(X, Y)$ is given by the sets $V(f, \delta) = U(f, \delta f)$ where $\delta \in C(Y, (0, \infty))$. These topologies do not depend on the metric of Y. Let $P(X, Y)$ denote the set of all proper maps of X to Y; by a proper map we mean a continuous map for which the inverse image of every compact set is compact. Then $\mathcal{T}_s$ and $\mathcal{T}_t$ induce the same topology on $P(X, Y)$, by which we topologize $P(X, Y)$. If X is compact, this is the compact-open topology and given by the supremum metric $d(f, g)$. If X and Y are locally compact, $P(X, Y)$ is an open and closed subset of $C(X, Y)$ in $\mathcal{T}_s$ and in $\mathcal{T}_t$.

Let $E(X, Y)$ and $E_c(X, Y)$ be the sets of all embeddings or of all closed embeddings, respectively, of X to Y. Then $E_c(X, Y) = E(X, Y) \cap P(X, Y)$, and $E_c(X, Y)$ is a G_δ-set in $P(X, Y)$. If F denotes any of C, P, E, and E_c, if $X_0 \subset X$ is closed, and if $f_0 \in F(X_0, Y)$, we let $F(X, Y; f_0)$ denote the set of all extensions $f \in F(X, Y)$ of f_0.

In the following two lemmas and their proofs, the topology is $\mathcal{T}_s$ or $\mathcal{T}_t$, whichever is more familiar to the reader.

2.2. LEMMA. *Let X and Y be locally compact metric spaces, $X_0 \subset X$ closed, and $f_0 \in P(X_0, Y)$. Then $P(X, Y; f_0)$ is a Baire space.*

PROOF. Since Y is topologically complete, $C(X, Y; f_0)$ is a Baire space by [18, Theorem 2.4.2] (for $\mathcal{T}_s$) or by [39, Lemma 1.1] (= [5, Lemma 4.1]) (for $\mathcal{T}_t$). Since $P(X, Y; f_0)$ is open in $C(X, Y; f_0)$, the lemma follows.

2.3. LEMMA. *Let X be a metric space and Y an* ANR, *let X_0 and A be disjoint closed subsets of X, and let $f_0 \in C(X_0, Y)$. Then the map $C(X, Y; f_0) \to C(A, Y)$, $f \mapsto f \mid A$, is continuous and open.*

PROOF. The continuity being clear, consider the openness first in the case $X_0 = \emptyset$. For $\mathcal{T}_t$ this is proved, e.g., in [38, (C) in §1], and for $\mathcal{T}_s$ it follows, e.g., from [24, Lemma 3.5]. In the general case, letting $B = X_0 \cup A$, the map $C(B, Y; f_0) \to C(A, Y)$, $f \mapsto f \mid A$, is the inverse of the continuous map $C(A, Y) \to C(B, Y; f_0)$, $f \mapsto f_0 \cup f$, and, thus, a homeomorphism. Hence, it suffices to show that the map $C(X, Y; f_0) \to C(B, Y; f_0)$, $f \mapsto f \mid B$, is open. Let $U \subset C(X, Y; f_0)$ be open. We have $U = V \cap C(X, Y; f_0)$ with $V \subset C(X, Y)$ open. By the above, $V \mid B$ is open in $C(B, Y)$. Hence, the set $F = (V \mid B) \cap C(B, Y; f_0)$ is open in $C(B, Y; f_0)$. On the other hand, $F = U \mid B$.

2.4. COROLLARY. *If X and Y are locally compact and f_0 is proper in 2.3, then the map $P(X, Y; f_0) \to P(A, Y)$, $f \mapsto f \mid A$, is continuous and open.*

The following theorem is the main result of this section. It characterizes through embeddings the locally compact separable metric spaces X for which $\dim(X \times X) < 2 \dim X$.

2.5. THEOREM. *Let $n > 0$ and $q > 0$ be integers with $q \leqq 2n$, and let X be an n-dimensional locally compact separable metric space. Then the following conditions are equivalent:*

(1) $n \geqq 2$, $q = 2n$, *and* $\dim(X \times X) < 2n$.

(2) *For each q-manifold M the set $E_c(X, M)$ is dense in $P(X, M)$.*

(3) *There is a q-manifold M having a non-compact component if X is non-compact such that $E_c(X, M)$ is dense in $P(X, M)$ in the compact-open topology.*

PROOF. (1) $\Rightarrow$ (2): Suppose that $\dim(X \times X) < q = 2n \geqq 4$ and that M is a q-manifold. We may assume that every bounded closed set in M is compact. Choose a countable family $W_i \subset V_i \subset U_i$ $(i \in I)$ of open subsets of M such that (U_i, V_i, W_i) is homeomorphic to $(\mathrm{R}^q, 2B^q, B^q)$ and $d(U_i) \leqq 1$ for each i, where $B^q = \{x \in \mathrm{R}^q : |x| < 1\}$, such that the family $(U_i)_{i \in I}$ is locally finite in M, and such that $M = \bigcup_{i \in I} W_i$. Then the sets $U_i^* = \{x \in M \mid d(x, U_i) \leqq 1\}$ are compact, and the family $(U_i^*)_{i \in I}$ is locally finite in M. It follows that there is a continuous function $\delta \colon M \to (0, 1]$ such that $\delta(x) \leqq d(\bar{V}_i, M \backslash U_i)$ and $\delta(x) \leqq \frac{1}{2} d(\bar{W}_i, M \backslash V_i)$ for all $i \in I$ and $x \in U_i^*$.

Now let $f \in P(X, M)$. Then $N = P(X, M) \cap V(f, \delta)$ is an open neighbourhood of f in $P(X, M)$. Hence, N is a Baire space by 2.2. Define compact sets $A_i = f^{-1} \bar{V}_i \subset X$ and $B_i = f^{-1} \bar{W}_i \subset A_i$ for $i \in I$. We show that $g A_i \subset U_i$ and $g B_i \cap g[X \backslash A_i] = \emptyset$ whenever $g \in N$ and $i \in I$. First, if $x \in A_i$, then $d(g(x), f(x)) < d(\bar{V}_i, M \backslash U_i) \leqq d(f(x), M \backslash U_i)$, and so $g(x) \in U_i$. Now consider $x \in B_i$ and $y \in X \backslash A_i$. If $f(y) \notin U_i^*$, then $d(g(y), U_i) > d(f(y), U_i) - 1 > 0$, whereas if $f(y) \in U_i^*$, then $d(g(x), f(x)) + d(g(y), f(y)) < d(\bar{W}_i, M \backslash V_i) \leqq d(f(x), f(y))$. Thus, $g(x) \neq g(y)$ in both cases.

Let $i \in I$. Then restriction defines a map φ_i of N to the open subspace $C(A_i, U_i)$ of $C(A_i, M)$, and φ_i is continuous and open by 2.4. Since $\dim(A_i \times A_i) < q$, Theorem 1.1 or its classical analogue implies that $E(A_i, U_i)$ is a dense G_δ-set in

$C(A_i, U_i)$. Hence, $E_i = \varphi_i^{-1}E(A_i, U_i)$ is a dense G_δ-set in N. It follows that $E^* = \bigcap_{i\in I} E_i$ is a dense G_δ-set in N. If $g \in E^*$ and $x, y \in X$, $x \neq y$, then, choosing $i \in I$ such that $x \in B_i$, we conclude, considering the cases $y \in A_i$ and $y \notin A_i$ separately, that $g(x) \neq g(y)$. Thus, g is injective. It follows that $E^* = E_c(X, M) \cap N$. Hence, $f \in \mathrm{cl}\, E_c(X, M)$.

(2) ⇒ (3): Choose $M = \mathsf{R}^q$.

(3) ⇒ (1): Choose an open subset U of M homeomorphic to R^q such that U is contained in a non-compact component M_1 of M if X is non-compact. If X is compact, then (3) implies that $E(X, U)$ is dense in $C(X, U)$, and consequently (1) follows from 1.1. Suppose now that X is non-compact. By the sum theorem for dimension, there is a compact set $A \subset X$ such that $\dim A = n$ and $\dim(X \times X) = \dim(A \times A)$. Hence, by 1.1 it suffices to prove that $E(A, U)$ is dense in $C(A, U)$. By (3) this reduces to showing that every $f \in C(A, U)$ has an extension $g \in P(X, M)$. Choose first a continuous extension $f'\colon X \to U$ of f. Now, as well-known, there exists a proper map $\alpha\colon [0, \infty) \to M_1$, and we can choose α such that $\alpha[0, 2] \subset U$. Choose a homeomorphism $\varphi\colon U \to \mathsf{R}^q$ with $\varphi(\alpha(0)) = 0$. We may assume that every bounded closed set in X is compact. Then by setting

$$g(x) = \varphi^{-1}(\max\{1 - d(x, A), 0\}\varphi(f'(x)) + \varphi(\alpha(d(x, A))))$$

if $d(x, A) \leqq 2$ and $g(x) = \alpha(d(x, A))$ if $d(x, A) \geqq 1$ we get the desired map $g \in P(X, M; f)$.

2.6. Corollary. *Let X be an n-dimensional locally compact separable metric space such that $n \geqq 2$ and $\dim(X \times X) < 2n$. Then X is homeomorphic to a closed subset of R^{2n}.*

Proof. There is a proper map $f\colon X \to [0, \infty) \hookrightarrow \mathsf{R}^{2n}$, and so 2.5 applies. Through one-point compactifications 1.1 applies also.

2.7. Remark. It is known that for all $n \geqq 0$ and $q \geqq 2n + 1$, every n-dimensional locally compact separable metric space X has the property (2) of 2.5. This follows from [17, Corollary 5] and [37, Lemma] (see [24, Theorem D]), and it is also proved in [24, Theorem 5.1]. The proof of 2.5 shows how this result also follows from its classical special case where X is compact and $M = \mathsf{R}^q$. See Appendix B for a corrected proof of [37, Lemma].

2.8. Problem. If X is an n-dimensional separable metric space with $\dim(X \times X) < 2n > 0$, is $E(X, \mathsf{R}^{2n})$ dense in $(C(X, \mathsf{R}^{2n}), \mathcal{T}_t)$?

2.9. Remark. We don't know the answer to 2.8 even if X were locally compact. A similar generalization of 2.7 is true; see [24, Theorems E and 5.6]. Related to 2.8 we finally give an example and a consequence of 2.5.

2.10. Example. We construct for each $n \geqq 2$ an n-dimensional non-compact separable metric space X such that $E(X, M)$ is dense in $(C(X, M), \mathscr{T}_t)$ for every $2n$-manifold M. In fact, choose an n-dimensional compact metric space X_0 with $\dim(X_0 \times X_0) < 2n$ and a non-compact separable metric space X_1 with $\dim X_1 < n$; then define X to be the free union of X_0 and X_1. Suppose that M is a $2n$-manifold, $f \in C(X, M)$, and $\delta \in C(M, (0, \infty))$. Then [16, Lemma 1.13.3] implies (see [24, Lemma 3.8]) that there is a locally compact metric space Y containing X_1 topologically as a dense subspace with $\dim Y = \dim X_1$ and such that $f \mid X_1$ is extendable to a proper map $f_1: Y \to M$. Now the free union $Z \supset X$ of X_0 and Y is an n-dimensional locally compact separable metric space with $\dim(Z \times Z) < 2n$, and $g = (f \mid X_0) \cup f_1: Z \to M$ is a proper map extending f. Hence, by 2.5 there is an embedding $h: Z \to M$ in $V(g, \delta)$. Then $h \mid X$ is an embedding in $V(f, \delta)$. Note that the construction gives both locally compact examples and examples which are not locally compact.

2.11. COROLLARY. *Let $n \geqq 2$, let X be an n-dimensional σ-compact metric space with $\dim(X \times X) < 2n$, let M be a $2n$-manifold, and let $C(X, M)$ be equipped with either one of the topologies $\mathscr{T}_s$ and $\mathscr{T}_t$. Then the set $I(X, M)$ of all continuous injections of X to M is a dense G_δ-set in $C(X, M)$.*

PROOF. Write $X = \bigcup_{i=1}^\infty X_i$ with $X_1 \subset X_2 \subset \ldots$ compact. Then $E(X_i, M)$ is a dense G_δ-set in $C(X_i, M)$ by 2.5 or 2.7. Hence, $E_i = \{f \in C(X, M) \mid f \mid X_i$ is injective$\}$ is a dense G_δ-set in $C(X, M)$ by 2.3. Since $C(X, M)$ is a Baire space and $I(X, M) = \bigcap_{i=1}^\infty E_i$, the assertion follows.

3. Relative approximation by 1-LCC embeddings.

3.1. TAMENESS. We recall three notions concerning tameness of a set A in a manifold M with some of their basic properties and show their equivalence under suitable conditions. As in [14], we say that A is locally homotopically 1-co-connected (1-LCC) in M if for each $x \in M$ and for each neighbourhood U of x in M, there is a neighbourhood $V \subset U$ of x in M such that every continuous map $\alpha: S^1 \to V \setminus A$ is null-homotopic in $U \setminus A$. If $k \geqq 0$ and if $C(I^k, M \setminus A)$ is dense in $C(I^k, M)$, we say as in [17] that A is a Z^k-set in M. If X is a metric space, an embedding $f: X \to M$ is said to be 1-LCC or Z^k if fX is 1-LCC or a Z^k-set, respectively, in M. For the definition of the number demension, $\operatorname{dem} A$, (= Shtan'ko's embedding dimension) of a σ-compact set A in M we refer to [14].

Let A be a σ-compact subset of a q-manifold M, and let $\dim A \leqq n \geqq 0$. By [14, Proposition 2.3] we conclude that if A is 1-LCC and $n \leqq q - 2$, then, in fact, $\dim A \leqq q - 3$, and every subset of A is 1-LCC in M, and that, conversely, if $n \leqq q - 3$ and A is the union of a countable family of σ-compact 1-LCC sets, then A is 1-LCC. Let $k \geqq 0$. Then every subset of a Z^k-set is a Z^k-set. Conversely, it is easy to see that if A is the union of a countable family of σ-compact Z^k-sets, then

A is a Z^k-set. If $n < q$, then clearly A is a Z^0-set. It is easy to see that if $n \leqq q - 2$, then A is a Z^1-set.

Let A be a σ-compact subset of a q-manifold M, let $0 \leqq n \leqq q - 3$, and let $\dim A \leqq n$. Then the following three conditions are equivalent: (1) A is 1-LCC; (2) A is a Z^{q-n-1}-set; and (3) A is a Z^2-set. In fact, the implication (1) $\Rightarrow$ (2) reduces to its special case where A is compact and has a neighbourhood homeomorphic to R^q, which then easily follows from [14, Proposition 1.3]; the implication (2) $\Rightarrow$ (3) obtains because a Z^k-set is a Z^l-set whenever $0 \leqq l \leqq k$ as I^l is a retract of I^k; and the implication (3) $\Rightarrow$ (1) follows from its easy special case where A is compact. Further, at least if, in addition, $q \neq 4$, we have by [14, Theorem 2.4] that A is 1-LCC if and only if $\operatorname{dem} A = \dim A$.

For a metric space X and a manifold M, we let $E^1(X, M)$ and $E^1_c(X, M)$ be the sets of all 1-LCC embeddings or of all closed 1-LCC embeddings, respectively, of X to M. If F denotes E^1 or E^1_c, if $X_0 \subset X$ is closed, and if $f_0 \in F(X_0, M)$, we set $F(X, M; f_0) = F(X, M) \cap E(X, M; f_0)$.

3.2. Remark. The Heisey-Toruńczyk theorem referred to in 2.7 is, in fact, the following stronger result: Let X be a locally compact separable metric space, $\dim X \leqq n \geqq 0$, M a q-manifold, $q \geqq 2n + 1$, $X_0 \subset X$ closed, and $f_0: X_0 \to M$ a closed Z^n-embedding; then the set of all closed Z^n-embeddings $f: X \to M$ extending f_0 is dense in $P(X, M; f_0)$. By 3.1, the Z^n-property here is equivalent to the 1-LCC-property if $n \geqq 2$ and automatically true if $n \leqq 1$. Moreover, if $n \leqq 1$, $q \geqq n + 3$, and f_0 is 1-LCC, then by 3.6 it follows that $E^1_c(X, M; f_0)$ is dense in $P(X, M; f_0)$. H. G. Bothe [3] proved earlier a form of this theorem for the manifold I^q. With the aid of [3], the case $n \leqq 1$ of the theorem was also proved in [24, Theorem 3.6].

In Theorem 3.11 we prove an analogue of the Heisey-Toruńczyk theorem for $q = 2n$.

We need the following result on 1-LCC approximation of embeddings. Alternatively, we could use its special case 3.6.

3.3. Theorem (Shtan'ko and Edwards). *Let X be a locally compact separable metric space, let Q be a q-manifold, let $\dim X \leqq q - 3$, let $Y \subset X$ be closed, let $f: X \to Q$ be an embedding, let $f \mid Y$ be* 1-LCC, *and let $\varepsilon \in C(X \setminus Y, (0, \infty))$. Then there is a* 1-LCC *embedding $g: X \to Q$ such that $g \mid Y = f \mid Y$ and $d(g(x), f(x)) < \varepsilon(x)$ for each $x \in X \setminus Y$.*

Proof. The theorem is a special case of [15, Approximation theorem], which concerns σ-compact metric spaces X. We indicate a simpler proof. By replacing Q by a suitable open subset we may assume by 3.1 that X is a closed subset of Q and f the inclusion map.

Consider first the absolute case $Y = \emptyset$. Then the proof on p. 105 of [15] for the special case of the theorem of [15] where X is compact and $\partial Q = \emptyset = Y$ also works now if only there the embeddings $g_{(i,j)}$ are chosen so close to the inclusion that they are closed. This proof is based on [15, Fundamental lemma], which by [15, Proposition 3] (cf. also [31, Lemma 1]) is equivalent to [31, Theorem 3].

Consider now the relative case. By the absolute case there is a 1-LCC embedding $h: X\setminus Y \to Q\setminus Y$ such that $d(h(x), x) < \min\{\varepsilon(x), \frac{1}{2}d(x, Y)\}$ for each $x \in X\setminus Y$. Then $g = \mathrm{id}_Y \cup h: X \to Q$ is an embedding. Now gY and $g[X\setminus Y]$ are locally compact 1-LCC sets in Q, and, hence, such is gX, too.

3.4. REMARK. The proof of the "if" part of 1.1 in [12] and [10] applies the special case of 3.3 due to [31] where X is compact, $Q = \mathrm{R}^q$, and $Y = \emptyset$. However, the alternative proof given in [32–34] is independent of 3.3. We therefore consider it advisable to next deduce a sufficient special case of 3.3 from 1.1 and its classical analogue. See [28, Remarks on p. 459 and Corollary 3.2] for related ideas.

3.5. LEMMA. *Let either $n \geqq 0$ and $q \geqq \max\{2n+1, n+3\}$ or $n \geqq 3$ and $q = 2n$, let X be an n-dimensional locally compact separable metric space, let M be a q-manifold, and let $\dim(X \times X) < 2n$ if $q = 2n$. Then $E^1_c(X, M)$ is dense in $P(X, M)$.*

PROOF. We first assume that X is compact and show that $E^1(X, \mathrm{R}^q)$ contains a dense G_δ-set in $C(X, \mathrm{R}^q)$. Choose a dense set $\{\varphi_i \mid i \geqq 1\}$ in $C(I^2, \mathrm{R}^q)$ such that each φ_i is PL. Let $E_i = \{f \in E(X, \mathrm{R}^q) \mid fX \cap \varphi_i I^2 = \emptyset\}$ for $i \geqq 1$. Then each E_i is a G_δ-set in $C(X, \mathrm{R}^q)$ as an open subset of $E(X, \mathrm{R}^q)$, and, obviously, each $f \in \bigcap_{i=1}^{\infty} E_i$ is a Z^2-embedding and, thus, a 1-LCC embedding. Hence, it suffices to prove that each E_i is dense in $C(X, \mathrm{R}^q)$. Let $f \in C(X, \mathrm{R}^q)$ and $\varepsilon > 0$. It is well-known (see, e.g., [25, Lemma 2, p. 99]) that there are a compact polyhedron L with $\dim L \leqq n$, a continuous map $g: X \to L$, and a PL map $h: L \to \mathrm{R}^q$ such that $d(hg, f) < \frac{1}{2}\varepsilon$. Since $K = hL$ and $K' = \varphi_i I^2$ are compact polyhedra in R^q with $\dim K + \dim K' \leqq n + 2 < q$, by [30, Theorem 5.3] there is a homeomorphism $\psi: \mathrm{R}^q \to \mathrm{R}^q$ such that $\psi K \cap K' = \emptyset$ and $d(\psi, \mathrm{id}) < \frac{1}{2}\varepsilon$. Let $f' = \psi hg \in C(X, \mathrm{R}^q)$; then $f'X \cap K' = \emptyset$ and $d(f', f) < \varepsilon$. Hence, 1.1 or its classical analogue yields $f^* \in E_i$ with $d(f^*, f) < \varepsilon$.

In the general case we follow the proof of the implication (1) $\Rightarrow$ (2) in 2.5, also if $q \geqq 2n + 1$. It suffices to show that $E^1_c(X, M) \cap N$ is dense in N. By the above, $E^1(A_i, U_i)$ contains a dense G_δ-set D_i in $C(A_i, U_i)$ for each $i \in I$. Then $D^1 = \bigcap_{i \in I} \varphi_i^{-1} D_i \subset E_c(X, M)$ is dense in N. By 3.1, $D^1 \subset E^1_c(X, M)$.

3.6. COROLLARY. *Theorem 3.3 holds under the additional assumption $\dim(X \times X) < q$.*

To proving a relative form of 2.5 we need the results 3.7 and 3.9 or 3.10.

3.7. LEMMA. *Let* $K \subset \mathsf{R}^4$ *be a* 1-LCC *compact set with* $\dim K \leqq 2$, *and let* X *be a compact metric space with* $\dim X \leqq 2$. *Then* $C(X, \mathsf{R}^4 \backslash K)$ *is dense in* $C(X, \mathsf{R}^4)$.

PROOF. Let $f \in C(X, \mathsf{R}^4)$ and $\varepsilon > 0$. Choose a compact polyhedron L with $\dim L \leqq \dim X$ and continuous maps $g: X \to L$ and $h: L \to \mathsf{R}^4$ such that $d(hg, f) < \frac{1}{2}\varepsilon$. By [14, Proposition 1.3], there is $h' \in C(L, \mathsf{R}^4 \backslash K)$ such that $d(h', h) < \frac{1}{2}\varepsilon$. Let $f' = h'g$; then $f' \in C(X, \mathsf{R}^4 \backslash K)$ and $d(f', f) < \varepsilon$.

The other lemma is a corollary to the following result related to Theorem 1.1.

3.8. THEOREM. *Let* K *be a compact set in* R^q, $q > 4$, *and* X *a compact metric space which satisfy the following conditions:*

(1) $\dim K + \dim X \leqq q$.

(2) $\dim(K \times X) < q$.

(3) $\operatorname{dem} K = \dim K \leqq q - 3$.

Then $C(X, \mathsf{R}^q \backslash K)$ *is dense in* $C(X, \mathsf{R}^q)$.

This theorem is the same as [12, Theorem 5] and [10, Theorem 4.1]. There in the proof it is assumed that (1) is an equality, but if (1) is a strict inequality, the theorem follows at once from the condition $\operatorname{dem} K + \dim X < q$: Proceed as in the proof of 3.7, but choose h to be PL; then resort to [14, Proposition 1.2]. Dranishnikov has since proved 3.8 with (1) omitted altogether [8] as also the necessity of (2) in this more general result [9].

3.9. COROLLARY. *Let* $n \geqq 3$, $q \geqq 2n$, $K \subset \mathsf{R}^q$ *compact with* $\operatorname{dem} K = \dim K \leqq n$, X *a compact metric space with* $\dim X \leqq n$, *and* $\dim(K \times X) < q$. *Then* $C(X, \mathsf{R}^q \backslash K)$ *is dense in* $C(X, \mathsf{R}^q)$.

We give an alternative proof, based on 3.5 and [35], for the following special case of 3.9 sufficient for us.

3.10. LEMMA. *Corollary* 3.9 *holds under the additional assumption* $\dim(X \times X) < q$.

PROOF. Let $f_1 \in C(X, \mathsf{R}^q)$ and $\varepsilon > 0$. By 3.5 choose $f \in E^1(X, \mathsf{R}^q)$ with $d(f, f_1) < \frac{1}{2}\varepsilon$. Let $X' = fX$; then $\operatorname{dem} X' = \dim X' \leqq n$ by 3.1. Since $\dim(X' \times K) < q$, the map $(x, y) \mapsto x - y$ of $X' \times K$ to R^q can be uniformly approximated by continuous maps to $\mathsf{R}^q \backslash \{0\}$ [16, Problem 1.9.B]. Hence, [35, Theorem 4] yields a map $f' \in E(X', \mathsf{R}^q \backslash K)$ with $d(f', \mathrm{id}) < \frac{1}{2}\varepsilon$. Then $g = f'f$ is in $E(X, \mathsf{R}^q \backslash K)$, and $d(g, f_1) < \varepsilon$.

The following theorem is the main result of this section. It shows that in the situation (1) of Theorem 2.5 the condition (2) can be strengthened to a relative form.

3.11. THEOREM. *Let $n \geqq 2$, let X be an n-dimensional locally compact separable metric space with* $\dim(X \times X) < 2n$, *let M be a $2n$-manifold, let $X_0 \subset X$ be closed and let $f_0: X_0 \to M$ be a* 1-LCC *closed embedding. Then $E_c^1(X, M; f_0)$ for $n \geqq 3$ and $E_c(X, M; f_0)$ for $n = 2$ is dense in $P(X, M; f_0)$.*

PROOF. By 3.6, the set $E_c^1(X, M; f_0)$ is dense in $E_c(X, M; f_0)$ if $n \geqq 3$. Hence, it suffices to show that $E_c(X, M; f_0)$ is dense in $P(X, M; f_0)$ for all $n \geqq 2$. Let P_0 be the set of all maps $g \in P(X, M; f_0)$ for which $g[X \backslash X_0] \cap Y = \emptyset$, where $Y = f_0 X_0 = g X_0$. We first show that P_0 is dense in $P(X, M; f_0)$. We may assume that every bounded closed set in M is compact. Let the family $(U_i, V_i, W_i, U_i^*)_{i \in I}$ and the function δ be as in the proof of 2.5. Then the compact sets $Y_i = Y \cap \bar{W}_i$, $i \in I$, cover Y, and $\delta(x) \leqq d(Y_i, M \backslash V_i)$ for all $i \in I$ and $x \in U_i^*$.

Now let $f \in P(X, M; f_0)$. Then $N = P(X, M; f_0) \cap V(f, \delta)$ is an open neighbourhood of f in $P(X, M; f_0)$. Hence, N is a Baire space by 2.2. Let $i \in I$ and $N_i = \{g \in N \mid g[X \backslash X_0] \cap Y_i = \emptyset\}$. Then $X_i = f^{-1} \bar{V}_i$ is compact, and $g X_i \subset U_i$ for each $g \in N$. We claim that $N_i = \{g \in N \mid g[X_i \backslash X_0] \cap Y_i = \emptyset\}$. In fact, if $g \in N$, then for each $x \in f^{-1} U_i^* \backslash X_i$ we have $d(g(x), f(x)) < d(f(x), Y_i)$, and for each $x \in X \backslash f^{-1} U_i^*$ we have $d(g(x), Y_i) > d(f(x), Y_i) - 1 > 0$; hence, $g[X \backslash X_i] \cap Y_i = \emptyset$, and the claim follows. Choose a sequence $A_j \subset X_i \backslash X_0, j \geqq 1$, of compact sets covering $X_i \backslash X_0$. Then the set $N_{ij} = \{g \in N \mid g A_j \cap Y_i = \emptyset\}$ is open in N for each $j \geqq 1$, and $N_i = \bigcap_{j \geqq 1} N_{ij}$. Consider $j \geqq 1$. Then restriction defines a map φ_j of N to the open subspace $C(A_j, U_i)$ of $C(A_j, M)$, and $N_{ij} = \varphi_j^{-1} C(A_j, U_i \backslash Y_i)$. Now Y_i is 1-LCC in U_i; see 3.1. If $n \geqq 3$, then $\dim Y_i \leqq n \leqq 2n - 3$ and $2n \geqq 6$, whence $\operatorname{dem} Y_i = \dim Y_i \leqq n$ in U_i; see 3.1. Hence, $C(A_j, U_i \backslash Y_i)$ is dense in $C(A_j, U_i)$ by 3.7 if $n = 2$ and by 3.10 if $n \geqq 3$. Since φ_j is open by 2.4, this implies that N_{ij} is dense in N. It follows that N_i is a dense G_δ-set in N. From this we conclude that $P_0 \cap N = \bigcap_{i \in I} N_i$ is a dense G_δ-set in N. Hence, $f \in \operatorname{cl} P_0$.

It now suffices to show that $E_c(X, M; f_0) \subset P_0$ is dense in P_0. Let $f \in P_0$, and let $\varepsilon \in C(X, (0, \infty))$ with $U(f, \varepsilon) \subset P(X, M)$. We have $f^{-1}[M \backslash Y] = X \backslash X_0$. Hence, f defines a proper map $f_1: X \backslash X_0 \to M \backslash Y$. Since $X_1 = X \backslash X_0$ is a locally compact separable metric space such that $\dim X_1 \leqq n$ and $\dim(X_1 \times X_1) < 2n$, it follows from 2.5 or 2.7 that there is $g \in E_c(X_1, M \backslash Y)$ such that $d(g(x), f(x)) < \min\{\varepsilon(x), d(x, X_0)\}$ for each $x \in X_1$. Then the extension $f^*: X \to M$ of g by f_0 is a closed embedding in $U(f, \varepsilon)$.

3.12. REMARK. In Theorem 3.11 with $n \geqq 3$ and in its complement in 3.2 with $q \geqq n + 3$ and f_0 1-LCC we have that $E_c^1(X, M; f_0)$ is a dense G_δ-set in $P(X, M; f_0)$. The G_δ-property follows from [15, Corollary (1), p. 96]. Since the part of the manuscript of Edwards containing the proof of this fact has apparently never appeared (cf. [14, p. 196] and [15, p. 96]), we provide a proof as the following theorem; see 3.1.

3.13. THEOREM. *Let X be a σ-compact metric space, let Y be a complete separable metric space, let $k \geqq 0$, let $C_k(X, Y) = \{f \in C(X, Y) \mid fX$ is a Z^k-set in $Y\}$, and let $C(X, Y)$ be equipped with the compact-open topology. Then $C_k(X, Y)$ is a G_δ-set in $C(X, Y)$.*

PROOF. Write $X = \bigcup_{i=1}^{\infty} X_i$ with $X_i \subset X$ compact. Let $C_i = \{f \in C(X, Y) \mid fX_i$ is a Z^k-set in $Y\}$ for $i \geqq 1$. Then $C_k(X, Y) = \bigcap_{i=1}^{\infty} C_i$ as Y is complete. Since Y is separable, there is a sequence $(\varphi_j)_{j \geqq 1}$ in $C(I^k, Y)$ such that $\{\varphi_j \mid j \geqq j_1\}$ is dense in $C(I^k, Y)$ for each $j_1 \geqq 1$. For $i,j \geqq 1$, let G_{ij} be the set of all maps $f \in C(X, Y)$ for which there is a map $\varphi \in C(I^k, Y)$ such that $d(\varphi, \varphi_j) < \frac{1}{j}$ and $\varphi I^k \cap fX_i = \emptyset$. Then, obviously, $C_i = \bigcap_{j=1}^{\infty} G_{ij}$. Thus, it suffices to show that each G_{ij} is open in $C(X, Y)$. Let $f \in G_{ij}$. Choose $\varphi \in C(I^k, Y)$ such that $d(\varphi, \varphi_j) < \frac{1}{j}$ and $r = d(\varphi I^k, fX_i) > 0$. Then $U = \{g \in C(X, Y) \mid d(g \mid X_i, f \mid X_i) < r\}$ is an open neighbourhood of f in $C(X, Y)$, and $\varphi I^k \cap gX_i = \emptyset$ for each $g \in U$ implying that $U \subset G_{ij}$.

3.14. COROLLARY. *The set of all Z^k-embeddings of X to Y is a G_δ-set in $(C(X, Y), \mathscr{T}_t)$.*

PROOF. With respect to the topology of uniform convergence, $E(X,Z)$ is a G_δ-set in $C(X,Z)$ for every metric space Z; see [4, Lemma 2.1].

Our last result together with Theorem 3.11 and its complement in 3.2 characterize in a certain situation the closed embeddings which are 1-LCC. It generalizes [3, Theorem 3].

3.15. THEOREM. *Let $q \geqq n \geqq 0$, and let f_0 be a closed embedding of an at most n-dimensional locally compact separable metric space X_0 to a q-manifold M. Suppose that for every at most n-dimensional locally compact separable metric space X containing X_0 as a closed subspace, $E_c(X, M; f_0)$ is dense in $P(X, M; f_0)$ in the compact-open topology. Then $q \geqq 2n + 1$, and f_0 is* 1-LCC *whenever $n \geqq 2$.*

Analogously, if $n \geqq 2$, $\dim(X_0 \times X_0) < 2n$, *and the density condition is only assumed for spaces X with the additional property* $\dim(X \times X) < 2n$, *then $q \geqq 2n$, and f_0 is* 1-LCC *whenever $n \geqq 3$.*

PROOF. We first prove the claims concerning q. In the second part of the theorem, choose a compact metric space X_1 with $\dim X_1 = n$ and $\dim(X_1 \times X_1) < 2n$ such that X_1 is a subset of X_0 if $\dim X_0 = n$. Then $\dim(X \times X) < 2n$ for the free union X of X_0 and X_1. It now easily follows that $E(X_1, M)$ is dense in $C(X_1, M)$. Thus, $q \geqq 2n$ by 2.5. In the first part of the theorem, $E(I^n, M)$ is dense in $C(I^n, M)$, and so $q \geqq 2n + 1$ by 2.5 if $n \geqq 1$. In the case $n = 0$ the space $X_1 = \{i^{-1} \mid i \geqq 1\} \cup \{0\} \subset \mathrm{R}^1$ embeds to M, and so $q \geqq 1$.

To proving the 1-LCC-property of f_0 we apply the pertinent density condition with X being the free union of X_0 and the closed disc $\bar{B}^2$; note that

$\dim(X \times X) < 2n$ if $\dim(X_0 \times X_0) < 2n$ and $n \geqq 3$. Assume, on the contrary, that $A = f_0 X_0$ is not 1-LCC in M. Then there is a point $x \in A$ with a neighbourhood U in M such that for each neighbourhood $V \subset U$ of x in M there is a continuous map $\alpha: S^1 \to V \backslash A$ which is not null-homotopic in $U \backslash A$. Choose an open neighbourhood $V \subset U$ of x in M homeomorphic to R^q, and let α be the respective map. Then α has a continuous extension $\beta: \bar{B}^2 \to V$. Now $f = f_0 \cup \beta: X \to M$ is a proper map. Hence, for each $\varepsilon > 0$ there is $g \in E_c(X, M; f_0)$ with $d(g, f) < \varepsilon$. Then $g\bar{B}^2 \cap A = \emptyset$. Choosing ε small enough we have that $g\bar{B}^2 \subset V \backslash A$ and that $g \mid S^1$ is homotopic to α in $V \backslash A$. This implies that α is null-homotopic in $V \backslash A$ and, thus, in $U \backslash A$, too, contrary to the definition of α.

3.16. REMARK. Consider the situation of the first part of 3.15 with $n \leqq 1$ and $q \geqq 2n + 1$, in which case f_0 always has the density property by 3.2. Now f_0 is 1-LCC if $q = 1$ but not if $q = 2$ and $X_0 \neq \emptyset$. More interestingly, by [1] there is for each $q \geq 3$ an embedding f_0 of the Cantor set X_0 to R^q such that $\mathsf{R}^q \backslash f_0 X_0$ is not simply connected; then f_0 is not a Z^2-embedding and, hence, not 1-LCC.

3.17. PROBLEM. In the second part of Theorem 3.15 with $n = 2$, does it follow that f_0 is 1-LCC or, if $q = 4$, that at least $\dim X_0 \leqq 1$?

Appendix A. A necessary condition for unstable intersections.

A. N. Dranishnikov and J. West [13] have recently proved the following theorem. The purpose of this appendix is to present a short alternative proof of it. This proof is the outcome of the author's conversation with H. Toruńczyk during his visit to Helsinki in September 1990.

We say that two compact metric spaces X and Y *intersect unstably* in a metric space Z if for all $f \in C(X, Z)$, $g \in C(Y, Z)$, and $\varepsilon > 0$ there are $f' \in C(X, Z)$ and $g' \in C(Y, Z)$ such that $d(f, f') < \varepsilon$, $d(g, g') < \varepsilon$, and $f'X \cap g'Y = \emptyset$.

A.1. THEOREM. *Let X and Y be compact metric spaces which intersect unstably in R^q, where $q \geqq 0$. Then* $\dim(X \times Y) < q$.

A.2. COROLLARY. *If X is a compact metric space, $q > 0$, and $E(X, \mathsf{R}^q)$ is dense in $C(X, \mathsf{R}^q)$, then* $\dim(X \times X) < q$ *and, thus,* $q \geqq 2 \dim X$.

The proof as also that in [13] are based on the following theorem due to Dranishnikov [9, Theorem 1].

A.3. THEOREM. *Let $X \subset \mathsf{R}^q$ be compact with $q \geqq 0$, let Y be a compact metric space, and suppose that $C(Y, \mathsf{R}^q \backslash X)$ is dense in $C(Y, \mathsf{R}^q)$. Then* $\dim(X \times Y) < q$.

Dranishnikov assumed in his proof that $q \geqq 3$. We give a proof for the case $q \leqq 2$. We may assume that $X \times Y \neq \emptyset$; then $q \geqq 1$. Now $\operatorname{int} X = \emptyset$, and no map $g \in C(Y, \mathsf{R}^q)$ has a stable value. Hence, $\dim X < q$ and $\dim Y < q$. The theorem with $q = 1$ follows. Suppose that $q = 2$ and $\dim Y = 1$. Then Y contains a connected set B with at least two points. If $G \subset \mathrm{R}^2$ is a domain and $x, y \in G \backslash X$, choose $g \in C(Y, G)$ with $x, y \in gB$. Then there is $g' \in C(Y, G \backslash X)$ with $x, y \in g'B$. Hence, $G \backslash X$ is connected. It is well-known that this implies that $\dim X = 0$ proving the case $q = 2$.

A.4. LEMMA. *Let (X, Y, q) satisfy the assumptions of Theorem* A.1. *Then*

$$K = \{f \in C(X, \mathrm{R}^q) \mid C(Y, \mathrm{R}^q \backslash fX) \text{ is dense in } C(Y, \mathrm{R}^q)\}$$

is a dense G_δ-set in $C(X, \mathrm{R}^q)$.

PROOF. Choose a sequence $(g_j)_{j \geqq 1}$ in $C(Y, \mathrm{R}^q)$ such that $\{g_j \mid j \geqq i\}$ is dense in $C(Y, \mathrm{R}^q)$ for each $i \geqq 1$. For each $j \geqq 1$, let K_j be the set of all maps $f \in C(X, \mathrm{R}^q)$ for which there is a map $h \in C(Y, \mathrm{R}^q)$ such that $d(h, g_j) < 1/j$ and $fX \cap hY = \emptyset$. Let $L = \bigcap_{j=1}^{\infty} K_j$. We show that $K = L$. Clearly $K \subset L$. Conversely, let $f \in L$, $g \in C(Y, \mathrm{R}^q)$, and $\varepsilon > 0$. Choose $j_0 \geqq 1$ with $1/j_0 < \varepsilon/2$ and $j \geqq j_0$ with $d(g_j, g) < \varepsilon/2$. Since $f \in K_j$, there is $h \in C(Y, \mathrm{R}^q)$ such that $d(h, g_j) < 1/j < \varepsilon/2$ and $hY \subset \mathrm{R}^q \backslash fX$. Now $d(h, g) < \varepsilon$. Thus, $f \in K$. Hence, $K = L$.

We show that each K_j is open in $C(X, \mathrm{R}^q)$. Consider $f \in K_j$. Choose $h \in C(Y, \mathrm{R}^q)$ such that $d(h, g_j) < 1/j$ and $fX \cap hY = \emptyset$. Then, if $f' \in C(X, \mathrm{R}^q)$ and $d(f', f) < d(fX, hY)$, we have that $f'X \cap hY = \emptyset$. Thus, $f' \in K_j$. This implies that K_j is open.

We show that each K_j is dense in $C(X, \mathrm{R}^q)$. Let $f \in C(X, \mathrm{R}^q)$ and $\varepsilon > 0$. Then there are $f' \in C(X, \mathrm{R}^q)$ and $h \in C(Y, \mathrm{R}^q)$ such that $d(f', f) < \varepsilon$, $d(h, g_j) < 1/j$, and $f'X \cap hY = \emptyset$. We conclude that $f' \in K_j$. Hence, K_j is dense.

It follows that K is a dense G_δ-set in $C(X, \mathrm{R}^q)$.

A.5. LEMMA. *Let X be a compact metric space and $q > \dim X$. Then*

$$F(X, \mathrm{R}^q) = \{f \in C(X, \mathrm{R}^q) \mid f^{-1}(x) \text{ is finite for each } x \in \mathrm{R}^q\}$$

contains a dense G_δ-set in $C(X, \mathrm{R}^q)$.

Lemma A.5 is due to Hurewicz [19], who showed that the set $R(X, \mathrm{R}^q) \subset F(X, \mathrm{R}^q)$ of all regularly branched maps of X to R^q in the sense of [10] is a dense G_δ-set in $C(X, \mathrm{R}^q)$. In [10, Theorem 3.1] a simpler proof is given for the weaker result that $R(X, \mathrm{R}^q)$ contains a dense G_δ-set in $C(X, \mathrm{R}^q)$.

A.6. PROOF OF THEOREM A.1. We may assume that $\dim X \leqq \dim Y$. If $q \leqq \dim X$, there are $f \in C(X, \mathrm{R}^q)$ and $g \in C(Y, \mathrm{R}^q)$ with a common stable value, which leads to a contradiction. Thus, $q > \dim X$. Hence, by A.4 and A.5 there is $f \in C(X, \mathrm{R}^q)$ such that $C(Y, \mathrm{R}^q \backslash fX)$ is dense in $C(Y, \mathrm{R}^q)$ and such that $f^{-1}(x)$ is finite for each $x \in fX$. Then $\dim(fX \times Y) < q$ by A.3. Define $h: X \times Y \to fX \times Y$ by $h(x, y) = (f(x), y)$. Then $h^{-1}(z)$ is finite for each

$z \in fX \times Y$. Hence, $\dim(X \times Y) \leqq \dim(fX \times Y)$ by [16, Theorem 1.12.4]. Thus, $\dim(X \times Y) < q$.

Appendix B. Localness of the disjoint n-cube property.

A metric space Y has the *disjoint n-cube property* for an integer $n \geqq 0$ if every continuous map of $I^n \times \{1,2\}$ to Y is uniformly approximable by continuous maps sending $I^n \times \{1\}$ and $I^n \times \{2\}$ onto disjoint sets. The following theorem is due to H. Toruńczyk [37, Lemma]; cf. Remark 2.7.

B.1. THEOREM. *Let Y be an* ANR, *and let $\mathcal{U}$ be an open cover of Y consisting of sets having the disjoint n-cube property. Then Y has the disjoint n-cube property.*

The original proof of this result in [37] is not quite satisfying as was observed by the author in [24, p. 63], but in a letter to him, in 1980, Toruńczyk gave a correction. The purpose of this appendix is to present a slightly modified form of this corrected proof.

We use the following well-known property of ANR's (see, e.g., [24, Lemma 3.5]).

B.2. LEMMA. *Let X be a compact metric space, let Y be an* ANR, *let $f: X \to Y$ be continuous, and let $\varepsilon > 0$. Then there is $\delta > 0$ such that for each closed set $A \subset X$, every continuous map $g: A \to Y$ with $d(g, f \mid A) < \delta$ has a continuous extension $g': X \to Y$ with $d(g', f) < \varepsilon$.*

B.3. PROOF OF THEOREM B.1. For $p, q \in \{0, 1, \ldots, n\}$ consider the following condition:

$(*)_{p,q}$ If $X = K \cup L$ is a compact polyhedron with K and L disjoint closed subpolyhedra such that either $\dim K < p$ and $\dim L \leqq n$ or $\dim K = p$ and $\dim L \leqq q$, then every continuous map of X to Y is uniformly approximable by continuous maps sending K and L onto disjoint sets.

It is easy to see that $(*)_{0,0}$ is satisfied. We show that $(*)_{p,q}$ implies $(*)_{p,q+1}$ whenever $q < n$. As $(*)_{p,n}$ implies $(*)_{p+1,0}$ whenever $p < n$, we then inductively get $(*)_{n,n}$, which completes the proof.

To this end, assume $(*)_{p,q}$, let $X = K \cup L$ be a compact polyhedron with K and L disjoint closed subpolyhedra such that $\dim K = p$ and $\dim L = q + 1$, let $f: X \to Y$ be a continuous map, and let $\varepsilon > 0$. Choose $\delta \in (0, \varepsilon)$ such that if $g: X \to Y$ is a continuous map with $d(f, g) < \delta$ and $C \subset gX$ is a set with $d(C) < 9\delta$, then $C \subset U$ for some $U \in \mathcal{U}$. Choose triangulations S of K and T of L such that $d(f\sigma) < \delta$ for each $\sigma \in S \cup T$. Let A be the $(p-1)$-skeleton of S and B the q-skeleton of T. Then by $(*)_{p,q}$ and B.2, there is a continuous map $f_0: X \to Y$ with $d(f, f_0) < \frac{1}{2}\delta$ such that $f_0 A \cap f_0 L = \emptyset = f_0 K \cap f_0 B$. Choose $\eta > 0$ with

$$\eta \leqq \tfrac{1}{2}\min\{\delta, d(f_0 A, f_0 L), d(f_0 K, f_0 B)\}.$$

Let $\sigma_1, \ldots, \sigma_k$ be the $(q+1)$-simplices of T. We construct inductively continuous maps $f_1, \ldots, f_k: X \to Y$ such that if $1 \leqq i \leqq k$, then

$(1)_i$ $f_i K \cap f_i[\sigma_1 \cup \ldots \cup \sigma_i] = \emptyset$,
$(2)_i$ $d(f_i, f_{i-1}) < \eta/k$.

Note that for each i, since $d(f_0, f_i) < \eta$, we have that $f_i A \cap f_i L = \emptyset = f_i K \cap f_i B$ and $d(f, f_i) < \delta$. Thus, $f_k K \cap f_k L = \emptyset$ and $d(f, f_k) < \varepsilon$, implying $(*)_{p,q+1}$.

Suppose that $1 \leqq i \leqq k$ and that $f_0, f_1, \ldots, f_{i-1}$ have already been constructed. Let Σ_0 denote the set of all p-simplices $\sigma \in S$ with $f_{i-1}\sigma \cap f_{i-1}\sigma_i \neq \emptyset$, let $\Sigma = \Sigma_0 \cup \{\sigma_i\}$ and let $F = \bigcup_{\sigma \in \Sigma} f_{i-1}\sigma$. Since $d(f_{i-1}\sigma) < 3\delta$ for each $\sigma \in S \cup T$ and, hence, $d(F) < 9\delta$, there is $U \in \mathscr{U}$ with $F \subset U$. Choose $\varrho \in (0, \eta/k)$ such that $\varrho \leqq d(f_{i-1}\sigma, f_{i-1}\sigma_j)$ for all $\sigma \in \Sigma_0$ and $j < i$ and such that $\varrho \leqq d(f_{i-1}\sigma, f_{i-1}\sigma_i)$ for each p-simplex $\sigma \in S \backslash \Sigma_0$. We construct a continuous map $f_i: X \to Y$ such that $f_i(x) = f_{i-1}(x)$ for each $x \in X \backslash (\bigcup_{\sigma \in \Sigma} \operatorname{int} \sigma)$, such that $d(f_i, f_{i-1}) < \varrho$, and such that $f_i\sigma \cap f_i\sigma_i = \emptyset$ for each $\sigma \in \Sigma_0$. Then f_i satisfies $(1)_i$ and $(2)_i$.

For that purpose, denote for $\lambda \in (0,1)$ and $\sigma \in \Sigma$ by σ^λ the image of σ under the $(1-\lambda)$-homothety of σ with respect to the barycentre of σ. Since U has the disjoint n-cube property, for each fixed $\lambda \in (0,1)$ there are continuous maps $\varphi_\sigma: \sigma^\lambda \to U$ for $\sigma \in \Sigma$ with disjoint images such that $d(\varphi_\sigma, f_{i-1} \mid \sigma^\lambda) < \lambda$ if $\sigma \in \Sigma$. By B.2, for each $\kappa > 0$ there is $\lambda_0 \in (0,1)$ such that if $\lambda \in (0, \lambda_0)$, then $\varphi_\sigma \cup (f_{i-1} \mid \partial\sigma)$ has a continuous extension $\psi_\sigma: \sigma \to Y$ with $d(\psi_\sigma, f_{i-1} \mid \sigma) < \kappa$ whenever $\sigma \in \Sigma$. Now, if κ and λ are small enough, letting $f_i \mid \sigma = \psi_\sigma$ for $\sigma \in \Sigma$ we get the desired map f_i.

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