MATH. SCAND. 68 (1991), 210–220

MAXIMUM PRINCIPLES FOR SUBHARMONIC FUNCTIONS

STEPHEN J. GARDINER

1. Main results.

Let s be a subharmonic function on the unit ball B of R^n $(n \geqq 2)$. A simple form of the maximum principle states that, if

$$\limsup_{X \to Y} s(X) \leqq 0 \tag{1}$$

for all $Y \in \partial B$, then $s \leqq 0$ on B. The same conclusion is true when (1) holds for almost every (surface area measure) $Y \in \partial B$, provided s is bounded above on B. Dahlberg [4] has shown that this boundedness hypothesis can be relaxed if the exceptional set for which (1) does not hold satisfies a Hausdorff measure condition. In this paper we present a different type of maximum principle based on weighted volume integrals of s^+ near the boundary. This, in turn, has a number of corollaries including the results mentioned above and some new ones.

Let $X = (X', x_n) = (x_1, \ldots, x_n)$ be a typical point of $\mathsf{R}^n = \mathsf{R}^{n-1} \times \mathsf{R}$, let $|X|$ denote the Euclidean norm of X, and $B(X, r)$ be the open ball of centre X and radius r. Thus $B = B(O, 1)$, where O denotes the origin of R^n. The main result is as follows.

THEOREM 1. *Let s be a subharmonic function on B and let $\gamma \geqq -1$. If, for each $\delta, \varepsilon > 0$, there is a covering of ∂B by balls $B(X_i, r_i)$ with centres $X_i \in \partial B$ and radii $r_i < \delta$ such that*

$$\sum_i r_i^{-\gamma-1} \int_{B(X_i, 2r_i) \cap B} (1 - |X|)^\gamma s^+(X)\, \mathrm{d}X < \varepsilon,$$

then $s \leqq 0$ on B.

Theorem 1 is in the same spirit as a removable singularity result of Kaufman

Received June 27, 1990

and Wu [9; Theorem 2], but the proof is completely different. The main tool used in the proof of Theorem 1 is estimation of harmonic measure. We remark that the case $\gamma = -1$ of the above result corresponds to the elementary fact that, if $(1-|X|)^{-1}s^+(X)$ is Lebesgue integrable on B, then (the mean of s^+ over $\partial B(O,r)$ has lower limit 0 as $r \to 1-$ and so) $s \leqq 0$ on B.

We now give a number of applications of Theorem 1, beginning with a version of the Schwarz reflection principle. Let Ω be an open set in R^n, symmetrical about the hyperplane $\{x_n = 0\}$ such that $\Omega^0 = \Omega \cap \{x_n = 0\}$ is non-empty. The classical reflection principle asserts that, if h is a harmonic function on $\Omega^+ = \Omega \cap \{x_n > 0\}$ which continuously vanishes on Ω^0, then h can be continued as a harmonic function into Ω by writing

$$h(X) = \begin{cases} 0 & (X \in \Omega, x_n = 0) \\ -h(X', -x_n) & (X \in \Omega, x_n < 0). \end{cases} \tag{2}$$

Corollary 1. *Let Ω be as above and h be a harmonic function on Ω^+. If, for each $\delta, \varepsilon > 0$, there is a covering of Ω^0 by balls $B(X_i, r_i)$ with centres $X_i \in \Omega^0$ and radii $r_i < \delta$ such that*

$$\sum_i r_i^{-\gamma-1} \int_{B(X_i,2r_i)\cap\Omega^+} x_n^\gamma |h(X)|\, dX < \varepsilon,$$

then h can be continued as a harmonic function into Ω by (2).

The proofs of Theorem 1 and Corollary 1 may be found in §2.

Let $\omega:(0,\infty) \to (0,\infty)$ be an non-decreasing function. The *(Hausdorff) ω-measure* of a bounded set E is defined by

$$m^\omega(E) = \lim_{\varepsilon \to 0+} \left\{ \inf \sum_i \omega(r_i) \right\},$$

where the infimum is taken over all countable coverings of E by open balls of radii $r_i < \varepsilon$. The limit always exists, but may take the value $+\infty$. A set E is said to have *σ-finite* ω-measure if E is a countable union of sets of finite ω-measure. In the special case $\omega(t) = t^\alpha$, where $\alpha \geqq 0$, the measure m^ω is denoted by m_α and called *α-dimensional Hausdorff measure.* Accounts of Hausdorff measures may be found in [8; Chapter 5] and [3; Section II].

Corollary 2. *Let s be a subharmonic function on B and let E be a subset of ∂B such that* (1) *holds for all $Y \in \partial B \backslash E$. If $m^\omega(E) = 0$ and*

$$\sup\{s^+(X): |X| = r\} = O[(1-r)^{1-n}\omega(1-r)] \quad (r \to 1-), \tag{3}$$

then $s \leqq 0$ on B.

Corollary 3. *Let s be a subharmonic function on B and let E be a subset of ∂B such that* (1) *holds for all $Y \in \partial B \backslash E$. If E has σ-finite ω-measure and*

$$(4) \qquad \sup\{s^+(X): |X| = r\} = o[(1-r)^{1-n}\omega(1-r)] \quad (r \to 1-),$$

then $s \leqq 0$ *on* B.

Corollary 2 is similar to [2; Theorem 10], but the proof indicated there is incorrect (cf. the Editor's note added in proof). In the case where $\omega(t) = t^\alpha$ and E is closed, the result had previously been established by Dahlberg [4; Theorem], who dealt more generally with Lipschitz domains. The proof of Theorem 1 is related to Dahlberg's work.

Another way of stating the case $\omega(t) = t^\alpha$ of Corollary 2 is as follows. *If* $(1-|X|)^\gamma s^+(X) \in L^\infty(B)$ *and* (1) *holds except for some set* E *such that* $m_{n-1-\gamma}(E) = 0$, *then* $s \leqq 0$ *on* B. It is thus natural to ask what happens if we replace L^∞ by L^p, where $p > 1$.

COROLLARY 4. *Let s be a subharmonic function on B and let E be a subset of* ∂B *such that* (1) *holds for all* $Y \in \partial B \backslash E$. *Let* $-1 < \gamma < n-1$, *let* $p \geqq n/(n-1-\gamma)$ *and* $1/p + 1/p' = 1$. *If* $(1-|X|)^\gamma s^+(X) \in L^p(B)$ *and E has σ-finite* $(n-(1+\gamma)p')$-*dimensional Hausdorff measure, then* $s \leqq 0$ *on* B.

In the case where E is a closed set, $\gamma = 0$, and $s = |h|$ (where h is harmonic on B), Corollary 4 is due to Gaĭdenko [6], who dealt more generally with solutions of second order elliptic differential equations. Otherwise the result is new.

The proofs of Corollaries 2–4 are given in §3, and the sharpness of these results is discussed in §4.

2. Proofs of Theorem 1 and Corollary 1.

2.1. Let D denote the halfspace $\{(X', x_n): x_n > 0\}$. For any Z in the hyperplane ∂D let $\beta(Z,r) = B(Z,r) \cap D$ and $\tau(Z,r) = B(Z,r) \cap \partial D$. We will establish the following.

THEOREM 1′. *Let s be a subharmonic function on* $\beta(Z,r)$, *where* $Z \in \partial D$, *and let* $\gamma \geqq -1$. *If, for each* $\delta, \varepsilon > 0$, *there is a covering of* $\tau(Z,r)$ *by balls* $B(X_i, r_i)$ *with centres* $X_i \in \partial D$ *and radii* $r_i < \delta$ *such that*

$$(5) \qquad \sum_i r_i^{-\gamma-1} \int_{\beta(X_i,2r_i)\cap\beta(Z,r)} x_n^\gamma s^+(X)\, dX < \varepsilon,$$

then (1) *holds for all* $Y \in \tau(Z,r)$.

Once Theorem 1′ is established, Theorem 1 may be easily deduced using the Kelvin transformation. Also, Corollary 1 may be obtained by taking $s = |h|$ and using Theorem 1′ to show that h vanishes on any $\tau(Z,r)$ for which $\beta(Z,r) \subset \Omega^+$.

2.2. This section contains some preliminary material required in the proof of

Theorem 1′. Let $r > 0$, let $0 < \phi \leqq \pi/4$ and

$$R_\phi = \{(X', x_n): 0 < x_n < |X'| \tan \phi\},$$

$$V_\phi(r) = \{(X', x_n): 0 < x_n < (r - |X'|) \tan \phi\}.$$

Since $\phi \leqq \pi/4$, it follows that $V_\phi(r) \subset \beta(0, r)$. If $Z \in \partial D$, let

$$A_\phi(Z, r) = \{(X', x_n) \in \overline{\beta(Z, r)}: x_n \geqq (|X' - Z'| - 3r/4) \tan \phi\}.$$

In fact, we want a version of $A_\phi(Z, r)$ for which the "edge" in the surface $\partial A_\phi(Z, r) \cap D$ is "rounded off". More precisely, let $A_\phi^*(Z, r)$ denote the union of all the sets of the form $\overline{B(Y, 7r/128) \cap D}$ which are contained in $A_\phi(Z, r)$. (See Figure 1.)

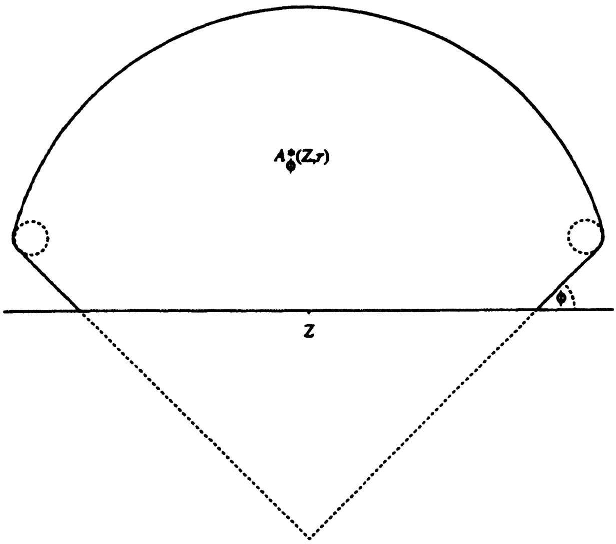

Figure 1.

We will use $C(a, b, c, \ldots)$ to denote a positive constant, depending at most on $a, b, c, \ldots$, not necessarily the same on any two occurrences.

LEMMA 1. *There is a positive harmonic function on R_ϕ of the form*

$$|X|^{k(\phi)} F_\phi(\tan^{-1}(x_n/|X'|)),$$

where $k(\phi) > 0$, $F_\phi(0) = 0 = F_\phi(\phi)$ and F'_ϕ is continuous on $(0, \pi/2)$. Further, $k(\phi) \to \infty$ as $\phi \to 0+$.

LEMMA 2. *Let $g(X, Y)$ denote the Green function for the set $V_\phi(1)$. Then the normal derivative $\partial g/\partial n_Y$ at $Y \in \partial V_\phi(1)$ satisfies*

(6) $$\frac{\partial g}{\partial n_Y}((O', [\tan \phi]/2), (Y', y_n)) \leqq C(n, \phi)\, y_n^{k(\phi)-1}$$

whenever $0 < y_n < (\tan \phi)/2$.

LEMMA 3. *Let* $G(X, Y)$ *denote the Green function for the set* $D \backslash A_\phi^*(O, 1)$. *Then*

(7) $$\frac{\partial G}{\partial n_Y}((X', x_n), (Y', y_n)) \leqq C(n, \phi)\, y_n^{k(\phi)-1}\, x_n |X|^{-n}$$

whenever $Y \in \partial A_\phi^*(O, 1)$, *where* $0 < y_n < (\tan \phi)/8$, *and* $X \in D \backslash \beta(O, 2)$.

Lemmas 1, 2, and 3 can be justified along the lines of [7; §§2.1, 2.2]. When $n \geqq 3$, (6) and (7) are true for higher powers of y_n than $y_n^{k(\phi)-1}$, but the stated estimates are sufficient for our purposes. Because of the smooth nature of $\partial A_\phi^*(O, 1) \cap D$, the estimate (7) is valid for all $Y \in \partial A_\phi^*(O, 1) \cap D$.

2.3. In proving Theorem 1′ we may take $Z = O$ and $r = 1$. It is sufficient to prove that (1) holds for all $Y \in \tau(O, \rho)$, where $\rho \in (0, 1)$ is arbitrary. In the light of Lemma 1 we can choose ϕ small enough to ensure that $k(\phi) \geqq \gamma + 1$.

From (5) the function $f(X) = x_n^\gamma s^+(X)$ is Lebesgue integrable on $\beta(O, \rho')$ for any $\rho' \in (0, 1)$. Thus there exists $\rho_0 \in (\rho, 1)$ for which f is integrable with respect to surface area measure on $S_0 = \partial V_\phi(\rho_0) \cap D$. It follows from Lemma 2 and a result of Dahlberg [5; Theorem 3] that s^+ is integrable on S_0 with respect to harmonic measure for the Lipschitz domain $V_\phi(\rho_0)$. Let h_0 denote the (Perron-Wiener-Brelot) solution to the Dirichlet problem on $V_\phi(\rho_0)$ with boundary data s^+ on S_0 and 0 on $\tau(O, \rho_0)$. By a strong regularity property of Lipschitz domains [1; Theorem 2] the function h_0 continuously vanishes on $\tau(O, \rho_0)$. Thus Theorem 1′ will be established if we can show that $s^+ \leqq h_0$ on $V_\phi(\rho_0)$.

Now let $0 < \delta < (1 - \rho_0)/4$, let $\varepsilon > 0$, and let $\{B(X_i, r_i)\}$ be a covering of $\tau(O, 1)$ as described in Theorem 1′. By compactness we can choose a finite subset $B(X_1, r_1), \ldots, B(X_m, r_m)$ of the balls $B(X_i, r_i)$ to cover $\overline{\tau(O, \rho_0)}$ and each of the balls $B(X_1, 2r_1), \ldots, B(X_m, 2r_m)$ is contained in $B(O, 1)$. For each i we can choose $\rho_i \in [4r_i/3, 2r_i]$ such that the surface integral, I_i, of f over $S_i = \partial A_\phi^*(X_i, \rho_i) \cap D$ satisfies

$$I_i \leqq C(n, \phi)\, r_i^{-1} \int_{A_\phi^*(X_i, 2r_i) \backslash A_\phi^*(X_i, 4r_i/3)} f(X)\, dX$$

$$\leqq C(n, \phi)\, r_i^{-1} \int_{\beta(X_i, 2r_i)} f(X)\, dX.$$

Hence, from (5),

$$\sum_i r_i^{-\gamma} I_i < C(n, \phi)\, \varepsilon. \tag{8}$$

Since I_i is finite, we see from Lemma 3 (and the final observation of §2.2) that s^+ is integrable with respect to harmonic measure for $W_i = D \backslash A_\phi^*(X_i, \rho_i)$. Let h_i denote the solution to the Dirichlet problem on W_i with boundary data s^+ on S_i and 0 on $\partial D \backslash \tau(X_i, \rho_i)$. Let $x_n \geqq 4r_i$. Using σ to denote surface area measure on S_i, and G_i to denote the Green function for W_i, (7) gives

$$\begin{aligned} h_i(X) &= C(n) \int_{S_i} s^+(Y)(\partial G_i/\partial n_Y)(X, Y)\, d\sigma(Y) \\ &\leqq C(n, \phi) x_n |X - X_i|^{-n} \int_{S_i} s^+(Y)(y_n/\rho_i)^{k(\phi)-1}\, d\sigma(Y', y_n) \\ &\leqq C(n, \phi) x_n^{1-n} \int_{S_i} s^+(Y)(y_n/\rho_i)^{\gamma}\, d\sigma(Y', y_n) \\ &\leqq C(n, \phi) x_n^{1-n} r_i^{-\gamma} I_i. \end{aligned}$$

Hence, if $x_n \geqq 4\delta$, it follows from (8) that

$$\sum_i h_i(X) \leqq C(n, \phi)\, \delta^{1-n} \varepsilon.$$

Let h denote the solution to the Dirichlet problem in

$$W = V_\phi(\rho_0) \backslash \bigcup_{i=1}^{m} A_\phi^*(X_i, \rho_i)$$

with boundary data s^+. (Clearly $\overline{W} \subset D$.) It follows that, if $(X', x_n) \in V_\phi(\rho_0)$ and $x_n \geqq 4\delta$, then

$$s^+(X) \leqq h(X) \leqq h_0(X) + \sum_{i=1}^{m} h_i(X) \leqq h_0(X) + C(n, \phi)\, \delta^{1-n} \varepsilon.$$

Since ε can be arbitrarily small, $s^+(X) \leqq h_0(X)$ when $X \in V_\phi(\rho_0)$ and $x_n \geqq 4\delta$. Since δ can be arbitrarily small, the proof of Theorem 1′ is complete.

3. Proof of Corollaries 2–4.

3.1. Let $\delta, \varepsilon > 0$, let E be as in Corollary 2, and suppose that (3) holds. There exists a countable collection of balls $\{B(X_i, r_i)\}$ such that $X_i \in \partial B$ and $r_i < \delta$ for each i, and

$$E \subset \bigcup_i B(X_i, r_i), \quad \sum_i \omega(r_i) < \varepsilon.$$

The maximum principle and (3) imply that there is a positive constant C such that

$$s^+(X) \leqq C r_i^{1-n} \omega(r_i) \quad (X \in B(O, 1 - r_i)).$$

Splitting the integral below into integrals over

$$B(X_i, 2r_i) \cap \{|X| < 1 - r_i\} \text{ and } B(X_i, 2r_i) \cap \{1 - r_i \leqq |X| < 1\},$$

and then using (3) again, we obtain

$$\text{(9)} \qquad \sum_i r_i^{-n} \int_{B(X_i, 2r_i) \cap B} (1 - |X|)^{n-1} s^+(X)\, dX \leqq C(n) \sum_i \omega(r_i) \leqq C(n)\varepsilon.$$

Now let

$$W_\varepsilon = \{X \in B: s(X) < \varepsilon\} \text{ and } F_\varepsilon = \{Y \in \partial B: \limsup_{X \to Y} s(X) < \varepsilon\}.$$

The set $W_\varepsilon \cup F_\varepsilon$ is relatively open in the topology of $\bar{B}$ and contains the compact set $\partial B \backslash \cup_i B(X_i, r_i)$. Thus we can find a finite collection of balls $\{B(Y_i, \rho_i)\}$ with

$$Y_i \in \partial B, \ \rho_i < \delta, \ B(Y_i, 2\rho_i) \cap B \subset W_\varepsilon$$

for each i, and

$$\partial B \backslash \bigcup_i B(X_i, r_i) \subset \bigcup_i B(Y_i, \rho_i), \quad \sum_i \rho_i^{n-1} < C(n).$$

It follows that

$$\text{(10)} \qquad \begin{aligned} &\sum_i \rho_i^{-n} \int_{B(Y_i, 2\rho_i) \cap B} (1 - |X|)^{n-1} s^+(X)\, dX \\ &\leq \varepsilon \sum_i \rho_i^{-n} \int_{B(Y_i, 2\rho_i) \cap B} (1 - |X|)^{n-1}\, dX \\ &\leq C(n)\varepsilon \sum_i \rho_i^{n-1} < C(n)\varepsilon. \end{aligned}$$

The balls $\{B(X_i, r_i), (B(Y_i, \rho_i)\}$ cover ∂B. Combining (9) and (10), we can thus apply Theorem 1 (with $\gamma = n - 1$) to deduce that $s \leqq 0$ in B. This completes the proof of Corollary 2.

3.2. Let $\delta, \varepsilon > 0$, let E be as in Corollary 3 and suppose that (4) holds. There exists a countable collection of sets E_k such that $E = \cup_k E_k$ and each $M_k = m^\omega(E_k)$ is finite. Without loss of generality we can assume that $M_k > 0$ for each k.

For each k, let $\delta_k \leqq \delta$ be chosen such that

$$\sup\{s^+(X): |X| = r\} \leqq \varepsilon 2^{-k} M_k^{-1} (1 - r)^{1-n} \omega(1 - r) \qquad (1 - \delta_k < r < 1).$$

We can now choose a countable collection of balls $\{B(X_{k,i}, r_{k,i})\}_i$ such that $X_{k,i} \in \partial B$ and $r_{k,i} < \delta_k$ for each i, and

$$E_k \subset \bigcup_i B(X_{k,i}, r_{k,i}), \quad \sum_i \omega(r_{k,i}) < 2M_k.$$

Arguing as in (9) it follows that

$$\sum_k \sum_i r_{k,i}^{-n} \int_{B(X_{k,i}, 2r_{k,i}) \cap B} (1 - |X|)^{n-1} s^+(X) dX \leqq C(n)\varepsilon \sum_k \sum_i 2^{-k} M_k^{-1} \omega(r_{k,i}) \leqq C(n)\varepsilon.$$

The remainder of the proof of Corollary 3 is similar to the second paragraph of §3.1.

3.3. Let s, γ, p, p' be as in Corollary 4, let $\alpha = n - (1 + \gamma)p'$, and let $\delta, \varepsilon > 0$. A proof will first be given for the case where E has finite α-dimensional Hausdorff measure. In the next section we will indicate how the argument can be modified to deal with the σ-finite case.

Let $M > m_\alpha(E)$, and let $\delta_0 < \delta$ be such that

$$\int_{B \setminus B(O, 1-2\delta_0)} [(1 - |X|)^\gamma s^+(X)]^p dX < M^{1-p} \varepsilon^p. \tag{11}$$

Then there is a countable collection of balls $\{B(X_i, r_i)\}$ such that $X_i \in \partial B$ and $r_i < \delta_0$ for each i, and

$$E \subset \bigcup_i B(X_i, r_i), \quad \sum_i r_i^\alpha \leqq M.$$

We next introduce a covering of E by dyadic cubes Q which have the property that, for each Q, there exists $k \in \mathsf{Z}$ such that $\{2^k X : X \in Q\}$ is a cube of side 1 whose vertices have integral co-ordinates. Each ball $B(X_i, r_i)$ can be covered by 2^n such cubes of side ρ_i, where $\rho_i < 4r_i$. This yields a covering of E by cubes $Q(Y_j, \rho_j)$, of centre Y_j and side ρ_j, such that $\Sigma \rho_j^\alpha \leqq C(n, p, \gamma)M$. Further, for each i, there exists j such that $B(X_i, 2r_i) \subseteq Q(Y_j, 5\rho_j)$. Also, there is a constant a_n, depending only on n, such that each point of R^n is in at most a_n of the cubes $Q(Y_j, 5\rho_j)$ of a given size.

Now let

$$F(X) = \sum_j \rho_j^{-\gamma-1} \chi_{Q(Y_j, 5\rho_j)}(X) \qquad (X \in \mathsf{R}^n),$$

where χ_A denotes the characteristic function of a set A, valued 1 on A and 0 elsewhere. If $F(X) > t$, then X must belong to a cube $Q(Y_j, 5\rho_j)$, where $\rho_j^{-\gamma-1} > t/b_n$ and $b_n = a_n/(1 - 2^{-1-\gamma})$. Hence

$$|\{X : F(X) > t\}| \leqq 5^n \Sigma^{(t)} \rho_j^n,$$

where $|A|$ denotes the n-dimensional Lebesgue measure of a measurable set A, and $\Sigma^{(t)}$ denotes the sum over all j for which $t\rho_j^{1+\gamma} < b_n$. Thus

$$\int_0^\infty p' t^{p'-1} |\{X: F(X) > t\}| \, dt \leqq 5^n \int_0^\infty p' t^{p'-1} (\Sigma^{(t)} \rho_j^n) \, dt$$

$$= 5^n \sum_j \rho_j^n \int_0^{b_n \rho_j^{-1-\gamma}} p' t^{p'-1} \, dt$$

$$= 5^n b_n^{p'} \sum_j \rho_j^{n-(1+\gamma)p'},$$

and so

$$\int_{\mathbf{R}^n} \{F(X)\}^{p'} \, dX \leqq C(n, p, \gamma) \sum_j \rho_j^\alpha.$$

Using Hölder's inequality and (11) we now have

$$\sum_i r_i^{-\gamma-1} \int_{B(X_i, 2r_i) \cap B} (1 - |X|)^\gamma s^+(X) \, dX$$

$$= \int_B \left\{ \sum_i r_i^{-\gamma-1} \chi_{B(X_i, 2r_i)}(X) \right\} \{(1 - |X|)^\gamma s^+(X)\} \, dX$$

$$\leqq C(\gamma) \int_{\mathbf{R}^n} F(X) \{(1 - |X|)^\gamma s^+(X) \chi_{B \backslash B(O, 1-2\delta_0)}(X) \, dX$$

$$\leqq C(n, p, \gamma) \left\{ \sum_j \rho_j^\alpha \right\}^{1/p'} M^{-1/p'} \varepsilon \leqq C(n, p, \gamma) \varepsilon.$$

The remainder of the argument is similar to the second paragraph of §3.1.

3.4. We now indicate how the above proof can be modified to deal with the case where E has σ-finite α-dimensional Hausdorff measure. In this case we can write $E = \cup_k E_k$, where each $m_\alpha(E_k)$ is finite. For each k let $M_k > m_\alpha(E_k)$ and let $\delta_k < \delta$ be chosen such that

$$\int_{B \backslash B(O, 1-2\delta_k)} [(1 - |X|)^\gamma s^+(X)]^p \, dX < 2^{-kp} M_k^{1-p} \varepsilon^p.$$

Arguing as in §3.3, there is a countable collection of balls $\{B(X_{k,i}, r_{k,i})\}_i$ covering E_k such that $r_{k,i} < \delta_k$ for each i and

$$\sum_i r_{k,i}^{-\gamma-1} \int_{B(X_{k,i}, 2r_{k,i}) \cap B} (1 - |X|)^\gamma s^+(X) \, dX \leqq C(n, p, \gamma) 2^{-k} \varepsilon,$$

whence

$$\sum_k \sum_i r_{k,i}^{-\gamma-1} \int_{B(X_{k,i},2r_{k,i})\cap B} (1-|X|)^\gamma s^+(X)\,dX \leqq C(n,p,\gamma)\,\varepsilon.$$

4. Sharpness of results.

4.1. Dahlberg [4; Proposition] has shown that Corollary 2 is sharp in the case $\omega(t) = t^\alpha$, where $0 < \alpha < n-1$. He showed that, if E is a closed subset of ∂B for which $m_\alpha(E) > 0$, then there is a positive harmonic function s on B such that (1) holds for all $Y \in \partial B\backslash E$ and (3) holds.

4.2. The sharpness of Corollary 3 is similarly demonstrated by the following.

THEOREM 2. *Let $0 < \alpha < n-1$ and let E be a closed subset of ∂B which is not σ-finite with respect to m_α. Then there is a positive harmonic function s on B such that* (1) *holds for all $Y \in \partial B\backslash E$ and* (4) *holds.*

To prove Theorem 2 we note (see [10; pp. 83, 84]) that, since E is not σ-finite with respect to m_α, there is a positive non-decreasing continuous function ω on $(0,\infty)$ such that $t^{-\alpha}\omega(t) \to 0$ as $t \to 0+$ and E is not σ-finite with respect to m^ω. By Frostman's Lemma [8; Lemma 5.4] there is a finite positive measure μ supported by E such that $\mu(B(X,r)) \leqq \omega(r)$ for all $X \in \mathsf{R}^n$ and $r > 0$. If we let s be the Poisson integral of μ in B, then (1) holds for all $Y \in \partial B\backslash E$. The proof of [4; Proposition] is now easily modified to show that (4) also holds.

4.3. Corollary 4 is sharp in the following sense.

THEOREM 3. *Let $-1 < \gamma < n-1$, let $p \geqq n/(n-1-\gamma)$ and $1/p + 1/p' = 1$. If E is a closed subset of ∂B such that $m_\alpha(E) > 0$ for some $\alpha > n-(1+\gamma)p'$, then there is a positive harmonic function s on B such that* (1) *holds for all $Y \in \partial B\backslash E$ and* $(1-|X|)^\gamma s(X) \in L^p(B)$.

By Frostman's Lemma we can choose a finite positive measure μ supported by E such that $\mu(B(X,r)) \leqq r^\alpha$ for all X and $r > 0$. If we let s be the Poisson integral of μ in B, then (1) holds for all $Y \in \partial B\backslash E$. It remains to check that $(1-|X|)^\gamma s(X)$ belongs to $L^p(B)$.

Let f be a positive step function on B which satisfies $\int_B f^{p'} = 1$ and let $t = (n-(1+\gamma)p')/p' + \alpha/p$ so that $n-(1+\gamma)p' < t < \alpha$. Following the approach of Carleson [3; p. 76] we define a function F of a complex variable z on the strip $\Omega = \{z: 0 \leqq \operatorname{Re} z \leqq 1\}$ by

$$F(z) = \int_B (1-|X|)^\gamma \int_E [f(X)]^{(1-z)p'}(1-|X|^2)\,|X-Y|^{-t-\gamma-1+(\alpha-n)z}\,d\mu(Y)\,dX.$$

Minor modification to the estimates in [6; §3] now shows that $|F(z)| \leqq M$ on $\partial\Omega$, where M is independent of the choice of f. Since F is bounded and analytic in the interior of Ω, it follows from the maximum principle that

$$F(1/p) = C(n) \int_B (1 - |X|)^\gamma s(X) f(X) \, dX \leqq M$$

for any f as above. Hence, by the converse to Hölder's inequality,

$$\int_B [(1 - |X|)^\gamma s(X)]^p \, dX \leqq C(n,p) M^p,$$

and the proof is complete.

ACKNOWLEDGEMENTS. The proof of Corollary 4 is a slight modification of an argument of Kaufman and Wu, implicit in [9, Example 3]. I am grateful to Professor Kaufman for helpful correspondence in this regard. I would also like to thank Professor Pertti Mattila for pointing out an error in the original version of this paper.

REFERENCES

1. D. H. Armitage, *A strong type of regularity for the* PWB *solution of the Dirichlet problem*, Proc. Amer. Math. Soc. 61 (1976), 285–289.
2. R. D. Berman and W. S. Cohn, *Phragmén-Lindelöf theorems for subharmonic functions on the unit disk*, Math. Scand. 62 (1988), 269–293.
3. L. Carleson, *Selected Problems on Exceptional Sets*, Van Nostrand, Princeton, N. J., 1967.
4. B. E. J. Dahlberg, *On exceptional sets at the boundary for subharmonic functions*, Ark. Mat. 15 (1977), 305–312.
5. B. E. J. Dahlberg, *Estimates of harmonic measure*, Arch. Rational Mech. Anal. 65 (1977), 275–288.
6. S. V. Gaĭdenko, *On exceptional sets on the boundary and the uniqueness of solutions of the Dirichlet problem for a second order elliptic equation*, Math. USSR Sbornik 39 (1981), 107–123.
7. S. J. Gardiner, *Removable singularities for subharmonic functions*, Pacific J. Math, 147 (1991), 71–80.
8. W. K. Hayman and P. B. Kennedy, *Subharmonic Functions, Vol.* 1, Academic Press, London, 1976.
9. R. Kaufman, and J.-M. Wu, *Removable singularities for analytic or subharmonic functions*, Ark. Mat. 18 (1980), 107–116. Correction, ibid. 21 (1983), 1.
10. C. A. Rogers, *Hausdorff Measures*, Cambridge University Press, London, 1970.

DEPARTMENT OF MATHEMATICS,
UNIVERSITY COLLEGE,
DUBLIN 4,
IRELAND.