MATH. SCAND. 68 (1991), 247-268

VECTOR VALUED LOEB MEASURES AND THE LEWIS INTEGRAL

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Dedicated to Professor Kurt Schütte on the occasion of his 80th birthday

(I) Introduction.

It is well known that nonstandard analysis benefits from the fact that the cardinality of the set of open sets in a topological space is small enough to apply saturation arguments. We always have this advantage, when the topological space is inside the standard model and we are working with a polysaturated nonstandard model.

On the other hand, in important applications of nonstandard analysis, topological spaces appear, which don't belong to the standard model; for example: the nonstandard hulls of topological vector spaces or the spaces of integrable functions on Loeb spaces. We want to apply dual space theory to the Banach space of square integrable martingales on an adapted Loeb space. Since the cardinality of the neighbourhoods in the weak topology of this space is too big, saturation arguments seem not to work.

This situation provides the background of this paper.

In the first part we will prove in a quite general setting a theorem about conversions of internal vector measures to σ-additive measures so that the most important property of the Loeb spaces remains valid: The measurable sets are exactly those sets which can be approximated by internal sets.

The general idea behind it comes from Loeb's functional approach to nonstandard measure theory [15]. In this conception, also the scalar valued Loeb measures can be obtained in a very simple way.

In the second part we will give some examples, one of them is the construction of σ-additive measures with values in the space of square integrable martingales on an adapted Loeb space.

Received March 29, 1989; in revised form March 23, 1990

In the last section we will prove lifting theorems for Lewis integrable functions with respect to the constructed measures, which make the notion of the Lewis integral more explicit. These results are applied to obtain a convergence theorem for uniformly Lewis integrable functions and the stochastic integral as a special case of the Lewis integral.

ACKNOWLEDGEMENT. I would like to thank Peter Imkeller for helpful discussions and suggestions. I am also grateful to Michael Reeken and Manfred Wolff for the suggestion to understand the stochastic integral on Loeb spaces as Lewis integral.

(II) Vector valued Loeb measures.

In the whole paper we will work in a κ-saturated nonstandard model, where κ is an uncountable cardinal number. For undefined notions and notations and for the foundation of nonstandard analysis we refer to the book of S. Albeverio, J. E. Fenstad, R. Høegh Krohn, T. L. Lindstrøm [1],

Assume that E is a locally convex topological vector space over the set C of complex numbers or over the set R of real numbers. E does not necessarily belong to the standard space. Let E' denote the topological dual of E.

Let L be an internal vector space over *C or *R and let SL be an *S-subvector space* of L, i.e., whenever $a, b \in SL$ and α, β are finite elements of *C or *R, then $\alpha a + \beta b \in SL$. Of course, SL is then a subvector space of L.
We assume that there exists a linear mapping

$$\mathscr{S}: SL \to E.$$

This mapping $\mathscr{S}$ may be understood as "*standard part map*".

Moreover, let $\mathfrak{U}$ be an internal algebra over Ω. If $\nu := \nu_1 + i\nu_2 : \mathfrak{U} \to {}^*\mathrm{C}$ is an internal measure, then $\nu_\perp$ denotes the *internal total variation* of ν, i.e.,

$$\nu_\perp := \nu_1^+ + \nu_1^- + \nu_2^+ + \nu_2^-,$$

where for $j = 1, 2$ and all $A \in \mathfrak{U}$

$$\nu_j^+(A) := \sup\{\nu_j(C) \mid C \subset A \text{ and } C \in \mathfrak{U}\}$$
$$\nu_j^-(A) := -\inf\{\nu_j(C) \mid C \subset A \text{ and } C \in \mathfrak{U}\}$$

Notice that with ν also $\nu_\perp$ is an internal measure and that ν is finite if and only if $\nu_\perp$ is finite.

The *total variation* $\nu_\perp$ of a standard scalar valued measure ν is defined in a similar way.
Let

$$\mu: \mathfrak{U} \to SL$$

be an internal finitely additive measure such that $\mathscr{S} \circ \mu\colon \mathfrak{U} \to E$ is of *bounded semivariation*, i.e. $(\varphi \circ \mathscr{S} \circ \mu)_\perp(\Omega) < \infty$ for all $\varphi \in E'$.
An internal finite measure

$$\nu\colon \mathfrak{U} \to {}^*\mathsf{R}_0^+$$

is called *absolutely continuous with respect to* μ, if for all $A \in \mathfrak{U}$:

$$(\varphi \circ \mathscr{S} \circ \mu)_\perp(A) = 0 \text{ for all } \varphi \in E' \text{ implies } \nu(A) \approx 0.$$

ν is called a *control measure* for a finitely additive set function

$$\rho\colon \mathfrak{U} \to \mathsf{C}$$

if for all standard $\varepsilon > 0$ there exists a standard $\delta > 0$ such that for all $A \in \mathfrak{U}$:

$$\nu(A) < \delta \text{ implies } \rho_\perp(A) < \varepsilon.$$

Of course, if ν is a control measure for ρ, then $\nu(A) \approx 0$ implies $\rho_\perp(A) = 0$. A set $\mathfrak{C}$ of internal finite measures $\nu\colon \mathfrak{U} \to {}^*\mathsf{R}_0^+$ is called a *control set* for μ, if

(1) every $\nu \in \mathfrak{C}$ is absolutely continuous w.r.t. μ and

(2) for every $\varphi \in E'$ there exists a control measure $\nu \in \mathfrak{C}$ for $\varphi \circ \mathscr{S} \circ \mu$.

The notion "control set" is an extension of the notion "control measure" used in [18]. Let

$$\mathfrak{M} := \bigcup \{\mathfrak{C} \mid \mathfrak{C} \text{ is a control set for } \mu\}.$$

Sometimes, it is possible to obtain control sets by $\mathscr{S}$-liftings of $\varphi \in E'$. An internal linear function

$$\Phi\colon L \to {}^*\mathsf{C}({}^*\mathsf{R})$$

is called an *$\mathscr{S}$-lifing* of $\varphi \in E'$, if

$$\Phi(F) \approx \varphi(\mathscr{S}(F)) \text{ for all } F \in SL.$$

Notice that, if Φ is an $\mathscr{S}$-lifting of φ, then $(\Phi \circ \mu)_\perp$ is a control measure for $\varphi \circ \mathscr{S} \circ \mu$, which is absolutely continuous w.r.t. μ. But, unfortunately, $\mathscr{S}$-liftings don't exist in general. A class of examples, where $\mathscr{S}$-liftings don't exist, is given later in the examples (α) and (γ).

Let $\nu\colon \mathfrak{U} \to {}^*\mathsf{C}$ be an internal finite measure. An internal or external subset $N \subset \Omega$ is called a *ν-nullset*, if for all standard $\varepsilon > 0$ there exists $A \in \mathfrak{U}$ with $N \subset A$ and $\nu_\perp(A) < \varepsilon$.

Lemma 1. *The set of ν-nullsets is closed under countable unions.*

Proof. By a simple saturation argument.

Lemma 2. *Let $\mathfrak{C}$ be a control set for μ of cardinality less than κ. Then $N \subset \Omega$ is a ν-nullset for all $\nu \in \mathfrak{M}$ if and only if N is a λ-nullset for all $\lambda \in \mathfrak{C}$.*

PROOF. The "only if" part is clear. Let N be a λ-nullset for all $\lambda \in \mathfrak{E}$. Then for every $\lambda \in \mathfrak{E}$ there exists a decreasing sequence $A_1^\lambda \supset \ldots \supset A_n^\lambda \supset \ldots$ in $\mathfrak{U}$ such that

$$N \subset \bigcap \{A_n^\lambda \mid n \in \mathsf{N}\} \text{ and } \lim {}^0\lambda(A_n^\lambda) = 0.$$

Let $\mathfrak{D} := \{B_i \mid i < \operatorname{card}(\mathfrak{E} \times \mathsf{N})\}$ be the set of all finite intersection of elements of $\{A_n^\lambda \mid n \in \mathsf{N} \text{ and } \lambda \in \mathfrak{G}\}$. Assume that N is not a ν-nullset for some $\nu \in \mathfrak{M}$. Then there exists a standard $\varepsilon > 0$ such that $\nu(B) > \varepsilon$ for all $B \in \mathfrak{D}$. By κ-saturation, there exists a set $A \in \mathfrak{U}$ such that $A \subset A_n^\lambda$ for all $n \in \mathsf{N}$ and all $\lambda \in \mathfrak{E}$ and such that $\nu(A) > \varepsilon$. But, since $\lambda(A) \approx 0$ for all $\lambda \in \mathfrak{E}$, $(\varphi \circ \mathscr{S} \circ \mu)_\perp(A) = 0$ for all $\varphi \in E'$; thus, since ν is absolutely continuous with respect to μ, $\nu(A) \approx 0$. Contradiction!

Let $\mathfrak{F}$ be a set of internal finite measures ν: $\mathfrak{U} \to {}^*\mathbf{C}$, let $A \in \mathfrak{U}$ and let $B \subset \Omega$ be internal or external. A is called an *$\mathfrak{F}$-approximation of B*, if

$$A \triangle B := (A \setminus B) \cup (B \setminus A) \text{ is a } \nu\text{-nullset for all } \nu \in \mathfrak{F}.$$

Define

$$L_{\mathfrak{F}}(\mathfrak{U}) := \{B \subset \Omega \mid B \text{ has an } \mathfrak{F}\text{-approximation}\}.$$

Define

$$L_\mu(\mathfrak{U}) := L_{\mathfrak{M}}(\mathfrak{U}),$$

and for all $B \in L_\mu(\mathfrak{U})$ and all $\mathfrak{M}$-approximations $A \in \mathfrak{U}$ of B

$$\hat{\mu}(B) := \mathscr{S} \circ \mu(A).$$

LEMMA 3. *$\hat{\mu}$ is well defined, if there exists a control set for μ.*

PROOF. Let A, A' be $\mathfrak{M}$-approximations of B. Since $A \triangle A' \subset (A \triangle B) \cup (A' \triangle B)$, by lemma 1:

$$\nu(A \triangle A') \approx 0 \text{ for all } \nu \in \mathfrak{M}.$$

Hence, $(\varphi \circ \mathscr{S} \circ \mu)_\perp(A \triangle A') = 0$ for all $\varphi \in E'$. We obtain for all $\varphi \in E'$

$$\begin{aligned} &\varphi \circ \mathscr{S} \circ \mu(A) = \\ &\varphi \circ \mathscr{S} \circ \mu((A \setminus (A \setminus A')) \cup (A' \setminus A)) = \\ &\varphi \circ \mathscr{S} \circ \mu(A'). \end{aligned}$$

Hence, $\mathscr{S} \circ \mu(A) = \mathscr{S} \circ \mu(A')$.

PROPOSITION 1. *Assume that there exists a control set $\mathfrak{E}$ for μ of cardinality less than κ. Then*

(1) $L_\mu(\mathfrak{U}) = L_{\mathfrak{G}}(\mathfrak{U})$.

(2) *$\hat{\mu}$ is σ-additive on the σ-algebra $L_\mu(\mathfrak{U})$.*

PROOF. (1) follows from lemma 2. "ad (2)": It is easy to see that $L_\mu(\mathfrak{U})$ is an algebra. In order to show that $L_\mu(\mathfrak{U})$ is a σ-algebra, fix an increasing sequence $B_1 \subset \ldots \subset B_n \subset \ldots$ in $L_\mu(\mathfrak{U})$. There exist $\mathfrak{M}$-approximations A_n of B_n for all

$n \in \mathbb{N}$. We may assume that the A_n are also increasing. Since $\operatorname{card}(\mathfrak{C}) < \kappa$, by κ-saturation, there exists $A \in \mathfrak{U}$ such that

$$\bigcup A_n \subset A \text{ and } \lim {}^0\lambda(A \setminus A_n) = 0 \text{ for all } \lambda \in \mathfrak{C}.$$

Hence, A is an $\mathfrak{C}$-approximation of $\bigcup A_n$. By lemma 1, A is also an $\mathfrak{C}$-approximation of $\bigcup B_n$. Thus, $\bigcup B_n \in L_{\mathfrak{C}}(\mathfrak{U})$. By (1), $\bigcup B_n \in L_\mu(\mathfrak{U})$. This proves that $L_\mu(\mathfrak{U})$ is a σ-algebra.

It is easy to see that $\hat{\mu}$ is finitely additive. To show that $\hat{\mu}$ is weakly σ-additive, fix $\varphi \in E'$ and B_n, A_n and A as above. Let $\varepsilon > 0$ be standard and let $\nu \in \mathfrak{C}$ be a control measure for $\varphi \circ \mathscr{S} \circ \mu$. There exists a standard $\delta > 0$ such that

$$(\varphi \circ \mathscr{S} \circ \mu)_\perp(C) < \varepsilon \text{ for all } C \in \mathfrak{U} \text{ with } \nu(C) < \delta.$$

By the first part of this proof, there exists $n_0 \in \mathbb{N}$ such that $\nu(A \setminus A_n) < \delta$ for all $n > n_0$. So we obtain for all $n > n_0$

$$\begin{gathered}\|\varphi \circ \hat{\mu}((\bigcup B_n) \setminus B_n)\| = \\ \|\varphi \circ \mathscr{S} \circ \mu(A \setminus A_n)\| \leqq \\ (\varphi \circ \mathscr{S} \circ \mu)_\perp(A \setminus A_n) < \varepsilon\end{gathered}$$

So, $\hat{\mu}$ is weakly σ-additive.

To show that $\hat{\mu}$ is also σ-additive in the original topology of E, we apply a result of Grothendieck [7], which says that the weakly σ-additive measure with values in a locally convex topological vector space is also σ-additive in every topology consistent with the duality between E' and E.

$(\Omega, L_\mu(\mathfrak{U}), \hat{\mu})$ is called the *vector valued Loeb space* over $(\Omega, \mathfrak{U}, \mu)$ under the assumption that there exists a control set for μ of cardinality less than κ.

REMARK. If Ω is a *finite set, then, for technical reasons, it is sometimes more convenient to start the theory with the notion of a weak nullset, which we will explain now.

Let $\mu: \mathfrak{U} \to SL$ be as before, but let Ω be *finite. The *atom* of $\omega \in \Omega$ is denoted by $[\omega]$. Recall that

$$[\omega] := \bigcap \{A \in \mathfrak{U} \mid \omega \in A\} \text{ and } [\omega] \in \mathfrak{U}.$$

Then the measure μ is given by an $\mathfrak{U}$-measurable weighted counting measure c_μ. Define

$$c_\mu(\omega) := \mu([\omega]) \cdot |[\omega]|^{-1} \text{ for all } \omega \in \Omega.$$

where $|[\omega]|$ denotes the internal (*finite) cardinality of $[\omega]$. Notice that

$$\mu(A) = \sum_{\omega \in A} c_\mu(\omega) \text{ for all } A \in \mathfrak{U}.$$

(If c_μ is constant, then μ is called a *uniform* or *fair* counting measure.) μ can be extended to an internal measure $\bar{\mu}$ on $^*\mathfrak{P}(\Omega) := \{A \subset \Omega \mid A \text{ is internal}\}$:

$$\bar{\mu}(A) := \sum_{\omega \in A} c_\mu(\omega) \text{ for all internal } A \subset \Omega.$$

Let $\nu\colon \mathfrak{U} \to {}^*\mathbf{C}$ be an internal finite measure. $N \subset \Omega$ is now called a weak ν-nullset, if for all standard $\varepsilon > 0$ there exists an internal $A \subset \Omega$ (not necessarily $A \in \mathfrak{U}$) such that

$$N \subset A \text{ and } \bar{\nu}_\perp(A) < \varepsilon.$$

We obtain a very similar theory with the same proofs. In the examples (δ) and (ε) we will use the notion of weak nullsets. There we will identify $\bar{\mu}$ and μ.

(III) Examples.

(α) *Scalar valued Loeb measures.*

Let

$E := \mathbf{C}$, so $E' = \{\varphi\colon \mathbf{C} \to \mathbf{C} \mid \varphi \text{ is linear}\}$,
$L := {}^*\mathbf{C}$,
$SL := \{\alpha \in {}^*\mathbf{C} \mid \alpha \text{ is finite}\}$,
$\mathscr{S} :=$ the standard part map. So $\mathscr{S}$ is a mapping from SL onto E.

The $*$-image $^*\varphi$ of $\varphi \in E'$ is an $\mathscr{S}$-lifting of φ. If $\mu\colon \mathfrak{U} \to SL$ is an internal measure, then $\{\mu_\perp\}$ is a control set for μ. Hence

$$L_\mu(\mathfrak{U}) = \{B \subset \Omega \mid \exists A \in \mathfrak{U}\,(A \triangle B \text{ is a } \mu\text{-nullset})\},$$

so $(\Omega, L_\mu(\mathfrak{U}), \hat{\mu})$ is the scalar valued finite Loeb space as developed in [14].

It can be easily seen, using the definition of a μ-nullset, that $B \in L_\mu(\mathfrak{U})$ if and only if for all standard $\varepsilon > 0$ there exist $A, C \in \mathfrak{U}$ such that

$$A \subset B \subset C \text{ and } \mu_\perp(C \setminus A) < \varepsilon.$$

This result is due to Loeb [14].

The next results perhaps show that the notion of "$\mathscr{S}$-lifting" is interesting. Let $(\Omega, L_\mu(\mathfrak{U}), \hat{\mu})$ be a scalar valued Loeb space as given above. Define for all $p \in \mathbf{R}$ with $1 \leqq p < \infty$:

$E_p := L^p(\Omega, L_\mu(\mathfrak{U}), \hat{\mu}) := \{f\colon \Omega \to \mathbf{C} \mid \|f\|^p \text{ is } \hat{\mu}\text{-integrable}\}$.
$E_\infty := L^\infty(\Omega, L_\mu(\mathfrak{U}), \hat{\mu}) := \{f\colon \Omega \to \mathbf{C} \mid \|f\| \text{ is } \hat{\mu}\text{-essentially bounded}\}$.
$L := \{F\colon \Omega \to {}^*\mathbf{C} \mid F \text{ is internal and } \mathfrak{U}\text{-measurable with a *finite range}\}$.
$SL^p := \{F \in L \mid \|F\|^p \text{ is } S_\mu\text{-integrable}\}$.

Recall from Anderson [2] that a function $G \in L$ is called *S_μ-integrable*, if for all infinite $H \in {}^*\mathbf{N}$ and all $A \in \mathfrak{U} \cap {}^*\mathfrak{P}(\{\omega \mid \|G(\omega)\| \geqq H\})$

$$\int 1_A \cdot G d\mu \approx 0.$$

Moreover, define

$$SL^\infty := \{F \in L \mid F \text{ is bounded by a standard real number}\}$$

and for all $p \in \mathrm{R}$ with $1 \leqq p \leqq \infty$ and for all $F \in SL^p$:

$$\mathscr{S}(F) := {}^\circ F.$$

Notice that $\mathscr{S}(F)$ is -up to a μ-nullset- well defined.

PROPOSITION 2. (1) *Whenever* $1 \leqq p < \infty$, *every* $\varphi \in E_p'$ *has an* $\mathscr{S}$*-lifting.* (2) $\varphi \in E_\infty'$ *has an* $\mathscr{S}$*-lifting if and only if* $\varphi \in E_1$.
(Here we identify $\varphi \in E_1$ with the canonical image of φ in $E_1'' = E_\infty'$.

The proofs, some extensions to the spaces of p-integrable martingales and applications to the stochastic integral will be given in a paper, which is in preparation.

(β) *Measures with values in vector spaces E inside the standard model.*
Let

$L := {}^*E$,
$SL := \{\alpha \in {}^*E \mid \alpha \text{ is nearstandard in the weak topology}\}$,
$\mathscr{S} :=$ the standard part map with respect to the weak topology,

so again $\mathscr{S}$ is a mapping from SL onto E.
Since $^*\varphi$ is an $\mathscr{S}$-lifting of φ for all $\varphi \in E'$, we can choose the number κ of saturation so large that $\{({}^*\varphi \circ \mu)_\perp \mid \varphi \in E'\}$ is a control set for μ: $\mathfrak{U} \to SL$ of cardinality less than κ.
The following result perhaps shows that this notion of a vector valued Loeb measure is an appropriate and natural conception in the locally convex case. (For non locally convex spaces some results about nonstandard vector valued measures can be found in [16]. There we studied locally solid vector lattices.)

D. R. Lewis [12] defined a vector measure v on an algebra containing the open sets of a Hausdorff space to be *weakly tight*, if $(\varphi \circ v)_\perp$ is tight for all $\varphi \in E'$. In [19] it was shown that every weakly tight vector measure can be parametrized by a hyperfinite vector valued Loeb measure. This result is a vector valued version of Anderson's work [3]. More precisely, we have

PROPOSITION 3. *Let $\mathfrak{D}$ be an algebra, which contains the open sets of a Hausdorff space X, and let m: $\mathfrak{D} \to E$ be a weakly regular measure with relatively weakly compact range. Then there exists an internal *finite subset $\Omega \subset {}^*X$ and an internal measure μ: ${}^*\mathfrak{P}(\Omega) \to {}^*E$ such that for all $B \in \mathfrak{D}$*
$\mathrm{st}^{-1}[B] \cap \Omega \in L_\mu({}^*\mathfrak{P}(\Omega))$ *and* $m(B) = \hat{\mu}(\mathrm{st}^{-1}[B] \cap \Omega)$.

Define
$L_m(\mathfrak{U}) := \{B \subset X \mid \mathrm{st}^{-1}[B] \cap \Omega \in L_\mu(^*\mathfrak{P}(\Omega))\}$ *and*
$\bar{m}: B \mapsto \hat{\mu}(\mathrm{st}^{-1}[B] \cap \Omega)$ *for all* $B \in L_m(\mathfrak{U})$.
Then $\bar{m}$ *is* σ*-additive on the* σ*-algebra* $L_m(\mathfrak{U})$.
If X *is a* T_3*-space, then* $\bar{m}$ *is weakly tight.*

This result, see [19], can be extended to measures with values in the weak nonstandard hull of E. A simple consequence of proposition 3 is the well known result that every such measure m can be extended to a σ-additive measure on a σ-algebra containing $\mathfrak{D}$.

(γ) *Measures with values in the nonstandard hull of a Banach space.*
Let $(D, \| \ \|)$ be a Banach space inside the standard model. Let
$L := {}^*D$,
$SL := \{\alpha \in {}^*D \mid \alpha \text{ is finite}\}$,
$E := \hat{D} :=$ the nonstandard hull of D,
$\mathscr{S}: \alpha \mapsto \bar{\alpha} \in E$ for all $\alpha \in SL$, where $\bar{\alpha} := \{\beta \in L \mid \|\alpha - \beta\| \approx 0\}$.
The next result is due to Y. Sun in [18].

PROPOSITION 4. *For every internal measure* $\mu: \mathfrak{U} \to SL$

$$\mathscr{S} \circ \mu: \mathfrak{U} \to E$$

can be extended to a σ*-additive measure on a* σ*-algebra over* $\mathfrak{U}$ *if and only if there exists a control set for* μ *with one single element.*

Here are again examples, where $\mathscr{S}$-liftings don't exist:
PROPOSITION 5: *Every* $\varphi \in E'$ *has an* $\mathscr{S}$*-lifting if and only if* E *is reflexive.*

PROOF. First notice that

$$i: \bar{\Phi} \mapsto (\varphi: \bar{a} \mapsto {}^0\Phi(a)) \text{ for all finite } \Phi \in L' \text{ and all finite } a \in {}^*D$$

is an embedding from $\hat{D}'$ into $E' = \hat{D}'$ and that every $\mathscr{S}$-lifting of some $\varphi \in E'$ is *continuous. Moreover, notice that Φ is an $\mathscr{S}$-lifting of $i(\bar{\Phi})$ for all finite $\Phi \in L'$. Thus, every $\varphi \in E'$ has an $\mathscr{S}$-lifting if and only if the mapping i is surjective. By theorem 8.5 in Henson and Moore [8], this mapping i is surjective if and only if E is reflexive.

(δ) *Measures with values in the space of square integrable martingales.*
Recall that we will now work with the conception of weak nullsets described at the end of section (II).
First we recall the main notions which lead to the conception of *adapted hyperfinite Loeb probability spaces*

$$(\Lambda, L_\nu(\Lambda), \hat{\nu}, (\mathfrak{b}_t)_{t \in [0,1]}).$$

(See for example Lindstrøm II [13].)
The set function

$$\nu\colon {}^*\mathfrak{P}(\Lambda) \to {}^*[0,1]$$

is an internal measure with $\nu(\Lambda) \approx 1$, where Λ is an internal hyperfinite set.

Let

$$T := \{0, \Delta t, 2\Delta t, \ldots, 1\}$$

be a hyperfinite time line such that $\mathbb{Q} \cap [0,1] \subset T$ and $\Delta t^{-1} \in {}^*\mathbb{N}$. Define for all s, $t \in T$

$$[s,t[_T := \{r \in T \mid s \leqq r < t\}$$

Let

$$(\mathfrak{U}_t) := (\mathfrak{U}_t)_{t\in T}$$

be an internal filtration on Λ, i.e. $\mathfrak{U}_s \subset \mathfrak{U}_t$ for all $s, t \in T$ with $s \leqq t$. Suppose that $\mathfrak{U}_1 = {}^*\mathfrak{P}(\Lambda)$. Define for all $t \in [0,1]$

$$\mathfrak{b}_t := \sigma(\bigcup \{\mathfrak{U}_{\underline{t}} \mid \underline{t} \in T \text{ and } \underline{t} \approx t\} \cup \mathfrak{N}),$$

where $\mathfrak{N}$ is the set of all (weak) ν-nullsets. Recall that $(\mathfrak{b}_t) := (\mathfrak{b}_t)_{t\in[0,1]}$ is right continuous. $(\mathfrak{b}_t)$ is called the *standard part* of the internal filtration $(\mathfrak{U}_t)$.

Let

$$E := \{m\colon \Lambda \times [0,1] \to \mathbb{R} \mid m \text{ is a square integrable } (\mathfrak{b}_t)\text{-martingale}\}$$

Notice that E, equipped with the norm

$$\|m\| := (\mathrm{E}(m(\cdot,1)^2)^{\frac{1}{2}},$$

is a Banach space over $\mathbb{R}$. We identify every $m \in E$ with its equivalence class. Since every $m \in E$ has a cadlag version, we may assume that for all $m \in E$ $m(\omega,\cdot)$ is right continuous for almost all $\omega \in \Lambda$.

In order to avoid difficulties at 0, we let the internal martingales start not at time 0 but a little bit later at some time point $\Phi \in T$ infinitely close to 0. Let

$$T_\Phi := \{s \in T \mid \Phi \leqq s\}$$
$$L := L_\Phi := \{M\colon \Lambda \times T_\Phi \to {}^*\mathbb{R} \mid M \text{ is an internal } (\mathfrak{U}_t)_{t\geqq\Phi}\text{-martingale}\},$$

In the following, suppose that $M \in L$ and $M(\cdot,t)^2$ is S_ν-integrable for all $t \geqq \Phi$.

By a result of Lindstrøm ([13] I Theorem 9) there exists a set $U \subset \Lambda$ of $\hat{\nu}$-measure 1 such that for all $\omega \in U$ and all $t \in [0,1]$

$$M(\omega,s) \text{ is nearstandard for all } s \in T_\Phi \text{ and}$$

$$^\circ M^+(\omega, t) := \lim_{\substack{^\circ s \downarrow t \\ s \in T}} {}^\circ M(\omega, s) \text{ exists for all } t < 1;$$

define

$^\circ M^+(\omega, 1) := {}^\circ M(\omega, 1)$, if $M(\omega, 1)$ is nearstandard.
Moreover, Lindstrøm proved (Proposition 9 II [13]) that $^\circ M^+ \in E$.
M is called *S-right continuous at Φ*, if $^\circ M^+(\cdot, 0) = {}^\circ M(\cdot, \Phi)$ $\hat{\nu}$-almost sure.
Let

$$SL := SL_\Phi := \{M \in L_\Phi \mid M(\cdot, t)^2 \text{ is } S_\nu\text{-integrable for all } t \in T_\Phi\}.$$

Define the standard part mapping $\mathscr{S}$: $SL \to E$ by

$$\mathscr{S}: M \mapsto ((\omega, \cdot) \mapsto {}^\circ M^+(\omega \cdot)) \text{ for almost all } \omega \in \Lambda.$$

We give a simple proof of the next result, essentially due to Lindstrøm ([13] III Theorem 10), which says that $\mathscr{S}$ is a mapping onto E.

PROPOSITION 6. *For every $m \in E$ there exists $\Phi \in T$ with $\Phi \approx 0$ and $M \in SL_\Phi$ such that*

$$M \text{ is right continuous at } \Phi$$

and such that for almost all $\omega \in \Lambda$

$$m(\omega, \cdot) = {}^\circ M^+(\omega, \cdot)$$

PROOF. By Loeb theory, there exists a lifting $Y: \Lambda \to {}^*\mathbb{R}$ of $m(\cdot, 1)$ such that Y^2 is S_ν-integrable. Define

$$N(\cdot, t) := E(Y \mid \mathfrak{U}_t) \text{ for all } t \in T.$$

(Notice that, since Λ can be handled as a finite set, $E(Y \mid \mathfrak{U}_t)(\omega)$ is defined for all $\omega \in \Lambda$.) By the hyperfinite version of Jensen's inequality, $N(\cdot, t)^2$ is S_ν-integrable for all $t \in T$. Hence $^\circ N^+(\omega, \cdot)$ exists for almost all $\omega \in \Lambda$.
By a saturation argument, see Lindstrøm ([13] III Lemma 8), there exists $\Phi \in T$, $\Phi \approx 0$, such that $^\circ N^+(\omega, 0) = {}^\circ N(\omega, \Phi)$ for almost all $\omega \in \Lambda$.
Hence

$$M := N \text{ restricted to } \Lambda \times T_\Phi.$$

belongs to SL_Φ and is right continuous at Φ.
It remains to show that $m(\omega, \cdot)$ for almost all $\omega \in \Lambda$.
Let $t \in \mathbb{Q} \cap [0, 1]$ and let $(s_n)_{n \in \mathbb{N}}$ be a sequence in T_Φ such that

$$t < {}^\circ s_n \text{ for all } n \in \mathbb{N} \text{ and } \lim {}^\circ s_n = t.$$

Then there exists a set U_t of $\hat{\nu}$-measure 1 such that for all $\omega \in U_t$

$^\circ M^+(\omega, t) =$
$\lim {}^\circ M(\omega, s_n) =$
$\lim {}^\circ E(Y|\mathfrak{U}_{s_n})(\omega) =$ (by Loeb theory)
$\lim E(m(\cdot, 1) \mid \sigma(\mathfrak{U}_{s_n} \cup \mathfrak{N}))(\omega) =$
(by Theorem 4.3 in Doob [5] and since $\mathfrak{b}_t = \bigcap \sigma(\mathfrak{U}_{s_n} \cup \mathfrak{N})$)
$E(m(\cdot, 1) \mid \mathfrak{b}_t)(\omega) =$
$m(\omega, t)$.
So, we obtain for all $\omega \in \bigcap \{U_t \mid t \in \mathbb{Q} \cap [0, 1]\}$ and such that $m(\omega, \cdot)$ and $^\circ M^+(\omega, \cdot)$ are right continuous and $m(\omega, \cdot) = {}^\circ M^+(\omega, \cdot)$.

The algebra $\mathfrak{U}$ is defined to be the set of all nonantipicating subsets of Ω. Recall that an internal subset

$$A \subset \Lambda \times T_\Phi =: \Omega$$

is called *nonanticipating*, see [1], if

$$\forall (\omega, t) \in \Omega((\omega, t) \in A \text{ implies } [\omega]_t \times \{t\} \subset A),$$

where $[\omega]_t$ denotes the atom of ω in $\mathfrak{U}_t$.

Before we will give the definition of the internal measure, we need some more notations and well known results.

Suppose that $M \in SL_\Phi$.

(A) The total variation $[M]$ of M is defined by

$$[M](\omega, t) := \sum_{s \in [\Phi, t[_T} \Delta M(\omega, s)^2 \text{ for all } (\omega, t) \in \Lambda \times T_\Phi.$$

where

$$\Delta M(\omega, s) := M(\omega, s + \Delta t) - M(\omega, s), \text{ if } s < t,$$
$$\Delta M(\omega, 1) := 0.$$

By Proposition 17 in Lindstrøm I [13] and since the expectation of the internal stochastic integral is 0, we have for all $t \in T_\Phi$

$$E([M](\cdot, t)) = E(M(\cdot, t)^2 - M(\cdot, \Phi)^2.$$

(B) The internal Dolean's measure λ_M: $^*\mathfrak{P}(\Lambda \times T_\Phi) \to {}^*R_0^+$ of M is defined by

$$\lambda_M(A) := \sum_{(\omega, s) \in A} (\Delta M(\omega, s))^2 \cdot \nu(\omega)$$

By (A) and because of the S_ν-integrability of $M(\cdot, t)^2$ for all $t \in T_\Phi$

$$\lambda_M(A) \leqq E[M](\cdot, 1) < \infty \text{ for all internal } A \subset \Omega.$$

(C) Let F: $\Omega \to {}^*\mathsf{R}$ be internal and $\mathfrak{U}$-measurable. The *internal stochastic* *integral*

$$\int F \Delta M \colon \Omega \to {}^*\mathrm{R}$$

is defined by

$$\int F \Delta M \colon (\omega, t) \mapsto \sum_{s<t} F(\omega, s) \Delta M(\omega, s).$$

By Proposition 3 in Lindstrøm II [13] we have:
If F^2 is S_{λ_M}-integrable, then $\int F \Delta M(\cdot, t)^2$ is S_ν-integrable for all $t \in T_\Phi$.

(D) We obtain for all $A \in \mathfrak{U}$ and all $t \in T_\Phi$ by internal complete induction: $E((\int 1_A \Delta M)^2(\cdot, t)) = E(\sum_{s<t} (1_A \Delta M^2(\cdot, s))$, thus, $E((\int 1_A \Delta M)^2(\cdot, 1)) = \lambda_M(A)$.

(E) (Keisler [10]) If $B \in \mathfrak{b}_s$, then there exist $\underline{s} \in T_\Phi$, $\underline{s} \approx s$, and $A \in \mathfrak{U}_{\underline{s}}$ such that

$$A \,\Delta B \text{ is a } \nu\text{-nullset.}$$

For the proof of Proposition 9 we need

LEMMA 4. *If $N \subset \Lambda$ is a ν-nullset, then $N \times T_\Phi$ is a λ_M-nullset.*

PROOF. Fix a standard $\varepsilon > 0$. We must show that there exists an internal set $B \subset \Lambda \times T_\Phi$ such that

$$N \times T_\Phi \subset B \text{ and } \lambda_M(B) < \varepsilon.$$

Since $M(\cdot, t)^2$ is S_ν-integrable for all $t \in T_\Phi$, by Theorem 2 in Lindstrøm II [13] $[M](\cdot, 1)$ is S_ν-integrable. Hence, there exists a standard $\delta > 0$ such that

$$\sum_{\omega \in A} [M](\omega, 1) \cdot \nu(\omega) < \varepsilon \text{ for all internal } A \subset \Lambda \text{ with } \nu(A) < \delta.$$

Choose an internal $A \subset \Lambda$ such that $N \subset A$ and $\nu(A) < \delta$
and define $B := A \times T_\Phi$. Then

$$\lambda_M(A \times T_\Phi) = \sum_{\omega \in A} [M](\cdot, 1) \cdot \nu(\omega) < \varepsilon.$$

Now we define the measure $\mu := \mu_M$ where $M \in SL_\Phi$:

$$\mu := \mu_M \colon A \mapsto \int 1_A \Delta M \text{ for all } A \in \mathfrak{U}.$$

By (C), $\mu \colon \mathfrak{U} \to SL_\Phi$.

PROPOSITION 7. *Every $\varphi \in E'$ has an $\mathcal{S}$-lifting ${}^+\varphi$.*

PROOF. Fix $\varphi \in E'$. By the Riesz representation theorem, there exists a function $g_\varphi \in L^2$ such that

$$\tilde{\varphi}(f) = Ef \cdot g_\varphi \text{ for all } f \in L^2.$$

By Loeb theory, there exists a ν-lifting G_φ: $\Lambda \to {}^*\mathsf{R}$ of g_φ such that G_φ^2 is S_ν-intgrable. Define for all $M \in L$

$$^+\varphi(M) := \int_\Lambda G_\varphi \cdot M(\cdot, 1)\, \mathrm{d}\nu.$$

Of course, $^+\varphi$ is linear. Fix $M \in SL$. By Hölder's inequality, $G_\varphi \cdot M(\cdot, 1)$ is an S_ν-integrable ν-lifting of $g_\varphi \cdot \mathscr{S}(M)(\cdot, 1)$. Hence, by Loeb theory,

$$\varphi(\mathscr{S}(M)) = \tilde{\varphi}(\mathscr{S}(M)(\cdot, 1)) = \int_\Lambda g_\varphi \cdot \mathscr{S}(M)(\cdot, 1)\mathrm{d}\hat{\nu} \approx \int_\Lambda G_\varphi \cdot M(\cdot, 1)\mathrm{d}\nu = {}^+\varphi(M).$$

With the same notation as in the proof of proposition 7 we define for all $\varphi \in B'$

$$\|{}^+\varphi\| := \Big(\int_\Lambda G_\varphi^2\, d\nu\Big)^{\frac{1}{2}}.$$

$\|{}^+\varphi\|$ *is finite, since* G_φ^2 is S_ν-integrable.

Lemma 5. *For all $\varphi \in B'$ and all $A \in \mathfrak{U}$*

$$({}^+\varphi \circ \mu)_\perp^2(A) \leqq 4 \cdot \|{}^+\varphi\|^2 \cdot \lambda_M(A).$$

Proof. Let $D \subset A$ and $D \in \mathfrak{U}$. Then

$$\begin{aligned} &({}^+\varphi \circ \mu(D))^2 \leqq \text{(by Hölder inequality)}\\ &\|{}^+\varphi\|^2 \cdot \Big(\int 1_D \Delta M\Big)^2(\cdot, 1)\, d\nu = \text{(by (D))}\\ &\|{}^+\varphi\|^2 \cdot \lambda_M(D). \end{aligned}$$

Hence, $({}^+\varphi \circ \mu)_\perp^2(A) \leqq 4 \cdot \|{}^+\varphi\|^2 \cdot \lambda_M(A)$.

Proposition 8. *$\{\lambda_M\}$ is a control set for μ.*

Proof. From lemma 5 follows that λ_M is a control measure for every $\varphi \circ \mathscr{S} \circ \mu$, $\varphi \in E'$. Let $\varphi \circ \mathscr{S} \circ \mu(A) = 0$ for all $\varphi \in E'$. Then $\mathscr{S} \circ \mu(A) = 0$. Hence, for almost all $\omega \in \Lambda : \int 1_A \Delta M(\omega, 1) \approx 0$. Since $(\int 1_A \Delta M(\cdot, 1))^2$ is S_ν-integrable by (C), we obtain by (D)

$$\lambda_M(A) = E\Big(\int 1_A \Delta M\Big)^2 (\cdot, 1)) \approx 0.$$

Remark. Using methods due to Dunford and Schwartz [6], one can show that there exists a countable subset $\{\varphi_i \mid i \in \mathsf{N}\} \subset E'$ such that $\{({}^+\varphi_i \circ \mu)_\perp \mid i \in \mathsf{N}\}$ is a control set for μ.
These measures can be put together to a single control measure. The proof is similar to the proof of lemma 3.3. in [18]. We omit the proof, because we don't use this result.

From proposition 8 follows:

Corollary 1. $L_{\lambda_M}(\mathfrak{U}) = L_{\mu_M}(\mathfrak{U})$.

So, we can apply the second part of proposition 1 to obtain a measure $\widehat{\mu_M}$ on the σ-algebra $L_{\mu_M}(\mathfrak{U})$ on $\Lambda \times T_\Phi$ with values in the space E of square integrable martingales.

Now we want to compare this nonstandard theory with the standard theory of measures on the σ-algebra of predictable sets with values in E.

Recall that the set $\mathfrak{P}_0$ of *predictable rectangles* on $\Lambda \times [0,1]$ contains exactly the sets of the form

$B \times]s,t]$ with $B \in \mathfrak{b}_s$, $t \in [0,1]$, $s < t$,

$B \times \{0\}$ with $B \in \mathfrak{b}_0$.

Let $m \in E$. Recall that the vector valued set function μ_m: $\mathfrak{P}_0 \to E$ is defined by

$\mu_m(B \times]s,t]) := ((\omega, r) \mapsto 1_B(\omega) \cdot (m(\omega, t \wedge r) - m(\omega, s \wedge r)))$

$\mu_m(B \times \{0\}) := 0.$

Choose Φ and M as in Proposition 6 and define for all $C \subset \Lambda \times [0,1]$

$$\mathrm{St}^{-1}[C] := \{(\omega, t) \in \Lambda \times T_\Phi \mid (\omega, {}^\circ t) \in C\}.$$

The proof of the next result is similar to the proof of Theorem 4.4.5 in [1]. We will prove this result, because it is a vector valued version of 4.4.5.

PROPOSITION 9. *For all* $C \in \mathfrak{P}_0$

$$\mathrm{St}^{-1}[C] \in L_{\mu_M}(\mathfrak{U})$$

and

$$\mu_m(C) = \widehat{\mu_M}(St^{-1}[C]).$$

PROOF. Let $C := B \times]s,t]$ be a predictable rectangle. Choose $\underline{s}$ and A as in (E) and $\underline{t} \in T$ with $\underline{t} \approx t$. Then

$$\mathrm{St}^{-1}[B \times]s,t]] = \bigcup_{k \in \mathrm{P}} \bigcap_{l \in \mathrm{N}} B \times [\underline{s} + \frac{1}{k}, \underline{t} + \frac{1}{l}[$$

So, $A \times [\underline{s} + \frac{1}{k}, \underline{t} + \frac{1}{l}[$ is a λ_M-approximation of $B \times [\underline{s} + \frac{1}{k}, \underline{t} + \frac{1}{l}[$ by Lemma 4. Hence, $\mathrm{St}^{-1}[B \times]s,t]] \in L_{\lambda_M}(\mathfrak{U}) = L_{\mu_M}(\mathfrak{U})$. So, we obtain for almost all $\omega \in \Lambda$ and all $r \in [0,1]$:

$$\widehat{\mu_M}(\mathrm{St}^{-1}[B \times]s,t]]) =$$

$$\lim_{k \to \infty} \lim_{l \to \infty} \mathscr{S} \circ \mu_M\left(A \times [\underline{s} + \frac{1}{k}, \underline{t} + \frac{1}{l}[_T\right) =$$

$$\lim_{k \to \infty} \lim_{l \to \infty} ((\omega, r) \mapsto \lim_{{}^\circ\rho \downarrow r} {}^\circ\sum_{\sigma < \rho} 1_{A \times [\underline{s} + \frac{1}{k}, \underline{t} + \frac{1}{l}[_T}(\omega, \sigma) \cdot \Delta M(\omega, \sigma)) =$$

$$\lim_{k\to\infty}\lim_{l\to\infty}((\omega,r)\mapsto\lim_{{}^\circ\rho\downarrow r}{}^\circ 1_A(\omega)\left(M\left(\omega,\left(\underline{t}+\frac{1}{l}\right)\wedge\rho\right)-M\left(\omega,\left(\underline{s}+\frac{1}{k}\right)\wedge\rho\right)\right))=$$

$$\lim_{k\to\infty}\lim_{l\to\infty}N_{k,l},$$

$$\text{with } N_{k,l}:=((\omega,r)\mapsto\lim_{{}^\circ\rho\downarrow r}1_B(\omega)\left({}^\circ M\left(\omega,\left(\underline{t}+\frac{1}{l}\right)\wedge\rho\right)-{}^\circ M\left(\omega,\left(\underline{s}+\frac{1}{k}\right)\wedge\rho\right)\right)).$$

Define

$$N:=((\omega,r)\mapsto 1_B(\omega)\cdot(m(\omega,t\wedge r)-m(\omega,s\wedge r)).$$

We have to show that

$$\lim_{k\to\infty}\lim_{l\to\infty}E(N_{k,l}(\cdot,1)-N(\cdot,1))^2=0.$$

This can be done, using the following facts:

$$\left.\begin{aligned}&\lim_{k\to\infty}\lim_{l\to\infty}{}^\circ M\left(\omega,\underline{s}+\frac{1}{k}\right)-m(\omega,s)=0,\\&\lim_{k\to\infty}\lim_{l\to\infty}{}^\circ M\left(\omega,\underline{t}+\frac{1}{l}\right)-m(\omega,t)=0\end{aligned}\right\}\quad\text{follows from proposition 6}$$

and Doob's inequality in order to apply the dominated convergence theorem. In a similar way it can be shown that for $B\in\mathfrak{b}_0$

$$\mathrm{St}^{-1}[B\times\{0\}]\in L_{\mu_M}(\mathfrak{U})\text{ and }\widehat{\mu_M}(\mathrm{St}^{-1}[B\times\{0\}])=0.$$

We can use Proposition 9 to obtain a simple proof of the well known fact that the set function μ_m can be extended to a σ-additive vector measure $\overline{\mu_m}$ on $\sigma(\mathfrak{P}_0)$: Define

$$\mathfrak{P}:=\{C\subset\Lambda\times[0,1]\mid St^{-1}[C]\in L_{\mu_M}(\mathfrak{U})\}$$

and

$$\overline{\mu_m}(C):=\widehat{\mu_M}(\mathrm{St}^{-1}[C]).$$

Then $\sigma(\mathfrak{P}_0)\subset\mathfrak{P}$, and since $L_{\mu_M}(\mathfrak{U})$ is a σ-algebra and $\widehat{\mu_M}$ is σ-additive, the same holds for $\mathfrak{P}$ and $\overline{\mu_m}$.

(ε) *Measures with values in the Banach space of continuous martingales.* Here

$$E:=\{m\mid\text{is a square integrable a.e. continuous }(\mathfrak{b}_t)\text{-martingale}\}.$$

The norm on E and $L:=L_T:=L_0$ is defined as in (δ). Set

$SL_T := \{M \in L \mid M$ is a.e. S-continuous and $M(\cdot,t)^2$ is S_ν-integrable for all $t \in T\}$,

$$\mathscr{S}: SL_T \to E$$

$$M \mapsto ((\omega, {}^\circ t) \mapsto {}^\circ M(\omega, t)) \text{ for almost all } \omega \text{ and all } t \in T.$$

We can proceed as in (δ), since by Panetta's result [20], for every $m \in E$ there exists a hyperfinite time line T and $M \in SL_T$ such that M is a uniform lifting of m in Keisler's sense [10], i.e. for almost all $\omega \in \Lambda$ and all $t \in T$ $M(\omega, t) \approx m(\omega, {}^\circ t)$.

(IV) The Lewis integral.

First we will give a lifting theorem for scalar valued $L_\mu(\mathfrak{U})$-measurable functions. We are in the general situation of section (II), and we assume that μ has a control set $\mathfrak{C}$ of cardinality less than κ.

An internal $\mathfrak{U}$-measurable function $F: \Omega \to {}^*\mathrm{C}$ with *finite range is called a *μ-lifting of f*: $\Omega \to \mathrm{C}$, if

$$\{\omega \mid F(\omega) \not\approx f(\omega)\} \text{ is a } \nu\text{-nullset for all } \nu \in \mathfrak{M}.$$

Notice that, if F is a μ-lifting of f, then the internal integral $\int_\Lambda F d\mu$ of F w.r.t. the internal vector measure μ is well defined, since F is a simple function.

PROPOSITION 10. *$f: \Omega \to \mathrm{C}$ is $L_\mu(\mathfrak{U})$-measurable if and only if f has a μ-lifting F: $\Omega \to {}^*\mathrm{C}$.*

PROOF. Similar to the proof of the Loeb-Anderson lifting theorem. See Keisler [10].

R. D. Lewis [12] defined an $L_\mu(\mathfrak{U})$-measurable function $f: \Omega \to \mathrm{C}$ to be *integrable w.r.t. the vector measure $\hat{\mu}$*, if

(1) f is integrable w.r.t. $\varphi \circ \hat{\mu}$ for all $\varphi \in E'$ and

(2) for all $B \in L_\mu(\mathfrak{U})$ there exists a vector $a_B \in E$ such that for all $\varphi \in E'$

$$\varphi(a_B) = \int_B f \, d\varphi \circ \hat{\mu}.$$

Then a_B is called the *Lewis integral of f over B*, denoted by $L\text{-}\int_B f \, \mathrm{d}\hat{\mu}$.

PROPOSITION 11. *Suppose that every $\varphi \in E'$ has an $\mathscr{S}$-lifting ${}^+\varphi$ and that there exists a μ-lifting $F: \Omega \to {}^*C$ of f: $\Omega \to \mathrm{C}$ such that*

(1) F is $S_{{}^+\varphi \circ \mu}$-integrable for all $\varphi \in E'$ and

(2) $\int_A F \, d\mu \in SL$ for all $A \in \mathfrak{U}$.

Then f is $\hat{\mu}$-integrable, and for all $B \in L_\mu(\mathfrak{U})$ and all $\mathfrak{M}$-approximations A of B

$$L - \int_B f\, d\hat{\mu} = \mathscr{S}(\int_A F\, d\mu).$$

PROOF. Fix $\varphi \in E'$. Notice that for all $B \in L_\mu(\mathfrak{U})$ and all $\mathfrak{M}$-approximations A of B

$$\widehat{{}^+\varphi \circ \mu}(A) = \varphi \dot{\circ} \hat{\mu}(B)$$

Since $\{\omega \mid F(\omega) \not\approx f(\omega)\}$ is a ${}^+\varphi \circ \mu$-nullset and because of (1), by Loeb theory, f is $\varphi \circ \hat{\mu}$ integrable. We obtain for all $B \in L_\mu(\mathfrak{U})$ and all $\mathfrak{M}$-approximations A of B

$$\varphi \circ \mathscr{S}(\int_A F\, d\mu) \approx {}^+\varphi(\int_A F\, d\mu) = \int_A F\, d^+\varphi \circ \mu \approx \int_B f\, d\varphi \circ \hat{\mu}.$$

Hence, $\mathscr{S}(\int_A F\, d\mu) = L - \int_B f\, d\hat{\mu}$.

EXAMPLES. ad (α): We have the well established integration theory on Loeb spaces developed by Loeb [14] and Anderson [2].

ad $(\beta) < \kappa$. Then the condition in proposition 11 for f to be $\hat{\mu}$-integrable is not only sufficient but also necessary:

PROPOSITION 12. *$f\colon \Omega \to \mathbf{C}$ is $\hat{\mu}$-integrable if and only if f has a μ-lifting $F\colon \Omega \to {}^*\mathbf{C}$ such that*

(1) F is $S_{*\varphi \circ \mu}$-integrable for all $\varphi \in E'$ and

(2) $\int_A F\, d\mu$ is nearstandard in the weak topology for all $A \in \mathfrak{U}$,

in which case for all $B \in L_\mu(\mathfrak{U})$ and all $\mathfrak{M}$-approximations A of B

$$L\text{-}\int_B f\, d\hat{\mu} = {}^\circ\int_A F\, d\mu \text{ (in the weak topology).}$$

PROOF. By proposition 11, we must only show the "only if" part. Suppose that f is $\hat{\mu}$-integrable. Since f is then $L_\mu(\mathfrak{U})$-measurable, by proposition 10, f has a μ-lifting $G\colon \Omega \to {}^*\mathbf{C}$. By (1) and Loeb theory, for every $\varphi \in E'$ there exists $H_\varphi \in {}^*\mathbf{N} \setminus \mathbf{N}$ such that $G \wedge H_\varphi$ is $S_{*\varphi \circ \mu}$-integrable, where for all $\omega \in \Omega$

$$G \wedge H_\varphi(\omega) := \begin{cases} G(\omega), \text{ if } \|G(\omega)\| \leqq H_\varphi \\ 0, \text{ otherwise.} \end{cases}$$

By κ-saturation, there exists $H \in {}^*\mathbf{N} \setminus \mathbf{N}$ with $H \leqq H_\varphi$ for all $\varphi \in E'$. Hence, $F := G \wedge H$ is $S_{*\varphi \circ \mu}$-integrable for all $\varphi \in E'$. We obtain for all $A \in \mathfrak{U}$

$$\varphi(L - \int_A f\, d\hat{\mu}) = \int_A f\, d\varphi \circ \hat{\mu} \approx \int_A F d\, {}^*\varphi \circ \mu = {}^*\varphi(\int_A F\, d\mu) \text{ for all } \varphi \in E'.$$

Thus, $L\text{-}\int_A f\, d\hat{\mu}$ is the standard part of $\int_A F\, d\mu$ in the weak topology.

COROLLARY. (1) $\{L\text{-}\int_A f\, d\hat{\mu} \mid B \in L_\mu(\mathfrak{U})\}$ *is weakly compact.*

(2) (*Lewis* [12]) ξ: $B \mapsto L\text{-}\int_B f\, d\hat{\mu}$ *for all* $B \in L_\mu(\mathfrak{U})$ *is* σ*-additive, if* f: $\Omega \to \mathbf{C}$ *is a* $\hat{\mu}$*-integrable function.*

PROOF. (1) follows immediately from proposition 12 and Luxemburg's result [17] that the weak standard part of an internal set of weak nearstandard points is weakly compact.

"ad (2)". According to proposition 12, let F: $\Omega \to {}^*\mathbf{C}$ be a μ-lifting of f such that (1) and (2) of proposition 12 are true and define

$$v: A \mapsto \int_A F\, d\mu \text{ for all } A \in \mathfrak{U}$$

Then v: $\mathfrak{U} \to SL$ is finitely additive and for all $\varphi \in E'$ every ${}^*\varphi \circ \mu$-nullset is also a ${}^*\varphi \circ v$-nullset, thus, $L_\mu(\mathfrak{U}) \subset L_v(\mathfrak{U})$. Because $\xi = \hat{v}$ restricted to $L_\mu(\mathfrak{U})$ and $\hat{v}$ is σ-additive, ξ is also σ-additive.

Lewis showed that this integral, although it is defined by the weak topology, is not a weak integral. In [12] he proved a dominated convergence theorem under the assumption that E is weakly sequentially complete.

We extend Lewis' result to uniformly Lewis integrable functions combining the lifting result of proposition 12 and methods known in nonstandard analysis to handle uniformly integrable functions. See for example, Hoover Perkins [9] or Cutland [4].

A sequence (f_n) of $\hat{\mu}$-integrable functions is called *uniformly* $\hat{\mu}$*-integrable*, if for all $\varphi \in E'$, (f_n) is uniformly $\varphi \circ \hat{\mu}$-integrable, i. e.

$$\lim_{k \to \infty} \int_{B_k^i} \|f_i\|\, d(\varphi \circ \hat{\mu})_\perp = 0 \text{ uniform for all } i \in \mathbf{N},$$

where $B_k^i := \{\omega \mid \|f_i\|(\omega) \geqq k\}$.

We say that a sequence (f_n) of $L_\mu(\mathfrak{U})$-measurable functions *convergens to* f: $\Omega \to \mathbf{C}$ *in* $\hat{\mu}$*-measure*, if (f_n) converges to f in $(\varphi \circ \hat{\mu})_\perp$-measure for all $\varphi \in E'$.

LEMMA 6: *A function* f: $\Omega \to \mathbf{C}$ *is* $\hat{\mu}$*-integrable, if there exists a sequence* (f_n) *of* $\hat{\mu}$*-integrable functions* f_n *such that* (f_n) *converges to* f *in* $\hat{\mu}$*-measure and such that* $\int_B f_n\, d\hat{\mu}$ *converges in the weak topology for all* $B \in L_\mu(\mathfrak{U})$.

PROOF. Obvious

PROPOSITION 13. *Suppose that* E *is weakly sequentially complete. If* (f_n) *is uniformly* $\hat{\mu}$*-integrable and converges to* f *in* $\hat{\mu}$*-measure, then* f *is* $\hat{\mu}$*-integrable and*

$$L\text{-}\int_B f\, d\hat{\mu} = \lim L\text{-}\int_B f_n\, d\hat{\mu} \textit{ in the weak topology.}$$

PROOF. By proposition 10, there exists a μ-lifting G of f. By proposition 12, for all $n \in \mathsf{N}$ there exists a μ-lifting F_n of f_n such that (1) of proposition 12 is true. Thus, by hypothesis, there exists a standard function g: $E' \times \mathsf{N} \to \mathsf{N}$ such that for all $\varphi \in E'$, $n \in \mathsf{N}$ and $k \geqq g(\varphi, n)$, $k \in \mathsf{N}$

$$(^*\varphi \circ \mu)_\perp(\{\omega \mid \|F_k - G\| \geqq n^{-1}\}) < n^{-1} \text{ and}$$

$$\forall i \in \mathsf{N}\Big(\int_{A_k^i} \|F_i\|\, d(^*\varphi \circ \mu)_\perp < \mathrm{n}^{-1}\Big), \text{ where } A_k^i := \{\omega \mid \|F_i(\omega)\| \geqq k\}.$$

By κ-saturation, one can extend the sequence $(F_n)_{n \in \mathsf{N}}$ to an internal sequence $(F_n)_{n \in {}^*\mathsf{N}}$ such that for all $\varphi \in E'$, $n \in \mathsf{N}$ and $k \geqq g(\varphi, n)$, $k \in {}^*\mathsf{N}$,

$$(^*\varphi \circ \mu)_\perp(\{\omega \mid \|F_k - G\| \geqq n^{-1}\}) < n^{-1} \text{ and}$$

$$\forall i \in {}^*\mathsf{N}\Big(\int_{A_k^i} \|F_i\|\, d(^*\varphi \circ \mu)_\perp\Big) < n^{-1}.$$

It follows that for all infinite $k \in {}^*\mathsf{N}$, F_k is a μ-lifting of f and that F_i is $S_{{}^*\varphi \circ \mu}$-integrable for all $i \in {}^*\mathsf{N}$ and all $\varphi \in E'$. Hence, for all infinite $i, j \in {}^*\mathsf{N}$ and all $\varphi \in E'$

$$(+) \int_\Omega \|F_i - F_j\|\, d(^*\varphi \circ \mu)_\perp \approx 0.$$

Now we will prove that $(L\text{-}\int_B f_n\, d\hat{\mu})$ is a weak Cauchy sequence for all $B \in L_\mu(\mathfrak{U})$.

By hypothesis and lemma 6, then f is $\hat{\mu}$-integrable. Assume that $(L\text{-}\int_B f_n\, d\hat{\mu})$ fails to be a weak Cauchy sequence. Then there exist $\varphi \in E'$, $A \in \mathfrak{U}$ and a standard $\varepsilon > 0$ such that

$$\forall n \in \mathsf{N}\, \exists i, j \geqq n\Big(\int_A \|F_i - F_j\|\, d(^*\varphi \circ \mu)_\perp \geqq \varepsilon.$$

By κ-saturation, there are $i, j \in {}^*\mathsf{N} \setminus \mathsf{N}$ such that

$$\int_A \|F_i - F_j\|\, d(^*\varphi \circ \mu)_\perp \geqq \varepsilon, \text{ which contradicts } (+).$$

ad (γ): Since in general $\mathfrak{G}$-liftings don't exist, I have no results on Lewis' integration theory using the weak topology. So, I refer to the work of Y. Sun [21], who developed integration theory with respect to measures with values in the nonstandard hulls of Banach spaces, using the norm topology.

ad (δ): Fix $m \in E$ and let M be as in proposition 6. We will first show that the

Lewis integral of f w. r. to $\overline{\mu_m}$ is the "standard part" of an internal integral w. r. t. μ_M. Assume that $f\colon \Lambda \times [0,1] \to \mathbb{R}$ is $\mathfrak{P}$-measurable and $\tilde{f}^2$ is $\widehat{\lambda_M}$-integrable, where

$$\tilde{f}\colon \Lambda \times T_\Phi \to \mathbb{R}\colon (\omega, t) \mapsto f(\omega, {}^0t).$$

By Loeb theory, there exists an $\mathfrak{U}$-measurable λ_M-lifting $F\colon \Lambda \times T_\Phi \to {}^*\mathbb{R}$ of $\tilde{f}$ such that F^2 is S_{λ_M}-integrable. We obtain

PROPOSITION 14. *For all $P \in \mathfrak{P}$ and all λ_M-approximations A of* $\mathrm{St}^{-1}[P]$

$$L\text{-}\int_P f\, d\overline{\mu_m} = \mathscr{S}\left(\int 1_A F\, d\mu_M\right) = \mathscr{S}\left(\int 1_A F\, \Delta M\right).$$

PROOF. In order to apply proposition 11, we will show two facts:
claim 1: F is $S_{{}^+\varphi \circ \mu_M}$-integrable for all $\varphi \in E'$.

PROOF. Let $H \in {}^*\mathbb{N} \setminus \mathbb{N}$ and let $A \subset \{\omega \in \Lambda \mid \|F(\omega)\| \geqq H\}$ with $A \in \mathfrak{U}$. The set A can be written as a disjoint union $A = [\omega_1, t_1] \cup \ldots \cup [\omega_k, t_k]$ of atoms $[\omega_i, t_i] \subset A$ in $\mathfrak{U}$. Then we have for all $\varphi \in E'$:

$$\|\int_A F\, d\, {}^+\varphi \circ \mu_M\|^2 =$$

$$\|\sum_{i=1}^k F(\omega_i, t_i) \cdot {}^+\varphi \circ \mu_M([\omega_i, t_i])\|^2 =$$

$$\|\sum_{i=1}^k F(\omega_i, t_i) \cdot \int_\Lambda G_\varphi \cdot \mu_M([\omega_i, t_i])(\cdot, 1)\, \mathrm{d}\nu\|^2 =$$

$$\|\sum_{i=1}^k F(\omega_i, t_i) \cdot \int_\Lambda G_\varphi \cdot \left(\sum_{s<1} 1_{[\omega_i, t_i]}(\cdot, s)\, \Delta M(\cdot, s)\right) d\nu\|^2 =$$

$$\|\sum_{i=1}^k \sum_{(\omega,s)\in[\omega_i,t_i]} G_\varphi(\omega) \cdot F(\omega, s) \cdot \Delta M(\omega, s) \cdot \nu(\omega)\|^2 \leqq$$

$$\|{}^+\varphi\|^2 \cdot \left(\sum_{(\omega,s)\in A} F(\omega, s)^2 \cdot \Delta M(\omega, s)^2 \cdot \nu(\omega)\right) =$$

$$\|{}^+\varphi\|^2 \cdot \left(\int_A F^2\, d\lambda_M\right) \approx 0, \text{ since } F^2 \text{ is } S_{\lambda_M}\text{-integrable.}$$

This proves claim 1.

claim 2: $\int_A F\, d\mu_M = \int 1_A F\, \Delta M$ for all $A \in \mathfrak{U}$.

PROOF. Let $A = [\omega_1, t_1] \cup \ldots \cup [\omega_k, t_k]$ as in the proof of claim 1 and let $(\omega, t) \in \Omega$.

Then

$$\int_A F\,d\mu_M(\omega,t) =$$

$$\sum_{i=1}^{k} F(\omega_i,t_i)\cdot\mu_M([\omega_i,t_i])(\omega,t) =$$

$$\sum_{i=1}^{k} F(\omega_i,t_i)\cdot\left(\sum_{s<t} 1_{[\omega_i,t_i]}(\omega,s)\cdot\Delta M(\omega,s)\right) =$$

$$\sum_{i=1}^{k}\sum_{s<t} 1_{[\omega_i,t_i}{}^{]}(\omega,s)\cdot F(\omega,s)\cdot\Delta M(\omega,s) =$$

$$\sum_{s<t} 1_A(\omega,s)\cdot F(\omega,s)\cdot\Delta M(\omega,s) =$$

$$\int 1_A\cdot F\cdot\Delta \mathrm{M}\,(\omega,t).$$

This proves claim 2.

Hence, by (C), $\int_A F\,d\mu_M \in SL$. By proposition 11, $\tilde{f}$ is $\widehat{\mu_M}$-integrable and

$$L\text{-}\int_B \tilde{f}\,d\widehat{\mu_M} = \mathscr{S}(\int_A F\mu_M) = \mathscr{S}(\int_A 1_A\cdot F\cdot\Delta M)$$

for all $B\in L_{\mu_M}(\mathfrak{U})$ and all $\mathfrak{M}$-approximations $A\in\mathfrak{U}$ of B. Hence, we obtain for all $P\in\mathfrak{P}$ and all $\mathfrak{M}$-approximations A of $\mathrm{St}^{-1}[P]$

$$L\text{-}\int_P f\,d\overline{\mu_m} = \text{(by the transformation rule for scalar valued integrals)}$$

$$L\text{-}\int_{\mathrm{St}^{-1}[P]} \tilde{f}\,d\widehat{\mu_M} =$$

$$\mathscr{S}(\int 1_A\cdot F\,d\mu_M) = \text{(by claim 2)}$$

$$\mathscr{S}(\int 1_A\cdot F\,\Delta M).$$

By Theorem 17 in Lindstrøm II [13] we obtain the

COROLLARY. *The Lewis integral of f w. r. t. $\overline{\mu_m}$ is exactly the stochastic integral of f w. r. t. m.*

ad (ε): We obtain similar results as in (δ) for continuous martingales.

REFERENCES

1. S. Albeverio, J. E. Fenstad, R. Høegh Krohn, T. L. Lindstrøm, *Nonstandard method in stochastic analysis and mathematical physics*, Academic Press, 1986.

2. R. A. Anderson, *A nonstandard representation of Brownian motion and Ito-integration*, Israel J. Math. 15 (1976).
3. R. A. Anderson, *Star finite representation of measure spaces*, Trans. Amer. Math. Soc. 271 (1982).
4. N. J. Cutland, *Nonstandard measure theory and its applications*, Bull. London Math. Soc. 15 (1983).
5. J. L. Doob, *Stochastic Processes*, J. Wiley & Sons, Inc. New York, 1952.
6. N. Dunford, J. T. Schwartz, *Linear Operators*, Vol. 1, Interscience, New York, 1958.
7. A. Grothendieck, *Sur les applications linearies faiblement compactes d'espaces du type $C(K)$*, Canad. J. Math. 5 (1953).
8. C. W. Henson, L. C. Moore, *The nonstandard theory of topological vector spaces*, Trans. Amer. Math. Soc. 172 (1972).
9. D. H. Hoover, E. A. Perkins, *Nonstandard construction of the stochastic integral and applications to stochastic differential equations*, Trans Amer. Math. Soc. 275 (1983).
10. H. J. Keisler, *An infinitesimal approach to stochastic analysis*, Mem. Amer. Math. Soc. (1984).
11. A. U. Kussmaul, *Stochastic Integration and Generalized Martingales*, Pitman Publishing, Research Notes in Math., London, 1977.
12. D. R. Lewis, *Integration with respect to vector measures*, Pacific J. Math. 33 (1970).
13. T. L. Lindstrøm, *Hyperfinite stochastic integration* I, II, III *and addendum Math. Scand.* 46 (1980).
14. P. A. Loeb, *Conversion from nonstandard to standard measure spaces and applications to probability theory*, Trans. Amer. Math. Soc. 211 (1975).
15. P. A. Loeb, *A functional approach to nonstandard measure theory*; Contemp. Math. 26 (1984).
16. P. A. Loeb, H. Osswald, *Nonstandard integration theory in solid Riesz spaces, in preparation.*
17. W. A. Luxemburg, *A General Theory of Monads, Applications of Model Theory to Algebra, Analysis and Probability*, Holt Rinehart and Winston, New York, 1969.
18. H. Osswald, Y. Sun, *On the extension of vector valued Loeb measures*, to appear in Proc. A.M.S.
19. H. Osswald, Y. Sun, *Star finite representation of weakly regular vector measures*, preprint 1988.
20. R. L. Planetta, *Hyperreal Probability Spaces: Some Applications of the Loeb Construction*, Ph. D. thesis, Univ. of Wisconsin, Madison, 1978.
21. Y. Sun, *On the theory of vector valued Loeb measures and integration*, in preparation.

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