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LOCALLY COMPACT F-SPACES, SUB–STONEAN SPACES AND QUOTIENTS OF LOCALLY COMPACT GROUPS

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In [1, p. 387] an F'-space is defined as a completely regular space in which disjoint open subsets of type F_σ have disjoint closures.

In [3, p. 124] a sub–Stonean space is defined as a locally compact Hausdorff space in which disjoint open, σ-compact subsets have disjoint compact closures. It was noted in passing, [3, p. 129], that if a compact group is sub–Stonean, it is finite.

In section 2 below I give a straightforward proof of the stronger result that if a quotient of a locally compact group by a closed subgroup is an F'-space, it is discrete.

When T is a locally compact Hausdorff space let $C_0(T)$ denote the algebra of realvalued continuous functions on T tending to zero at infinity and let $\mathscr{K}(T)$ denote its minimal dense ideal of functions with compact support.

According to [1, p.366] an F-ring is a commutative ring in which each finitely generated ideal is a principal ideal. In section 1 I show that a locally compact space is an F'-space if and only if $\mathscr{K}(T)$ is an F-ring.

1.

THEOREM. *Let T be a locally compact Hausdorff space. The following conditions are equivalent:*

(i) *$\mathscr{K}(T)$ is an F-ring.*
(ii) *For f in $\mathscr{K}(T)$ there exists k in $\mathscr{K}(T)$ such that $f = k|f|$.*
(iii) *Any compact subset of T is a sub–Stonean space.*
(iv) *Any point in T has a sub–Stonean neighbourhood.*
(v) *T is an F'-space.*
(vi) *Disjoint open, σ-compact, relatively compact subsets of T have disjoint closures.*

PROOF. We prove (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi) $\Rightarrow$ (ii) $\Rightarrow$ (i).

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Assume that $\mathscr{K}(T)$ is an F-ring and let f in $\mathscr{K}(T)$ be given. As in [1, p. 370] note that the ideal generated by f and $|f|$ is generated by some function d in $\mathscr{K}(T)$ so there exist functions g, h, s and t in $\mathscr{K}(T)$ with $f = gd$, $|f| = hd$ and $d = sf + t|f|$, and $k = (s + t)^2(gh) \vee 0 + (s - t)^2(gh) \wedge 0$ will do.

Let L be a compact subset of T, and let U and V be disjoint relatively open, σ-compact subsets of L, and so positivity sets for non-negative functions a and b in $C(L)$. Choose by Tietze's theorem an extension f in $\mathscr{K}(T)$ of $a - b$ and choose, assuming (ii), k in $\mathscr{K}(T)$ with $f = k|f|$. Then $\{t \in L \mid k(t) = 1\}$ and $\{t \in L \mid k(t) = -1\}$ are disjoint compact subsets of L containing U and V respectively. Thus (ii) implies (iii).

As (iii) $\Rightarrow$ (iv) is no problem, assume (iv) and let X and Y be open subsets of type F_σ of T. Let t be a point in $\bar{X} \cap \bar{Y}$, let W be a sub–Stonean neighbourhood of t and let K be a compact neighbourhood of t contained in W. By [3, Theorem 1.4] K is a sub–Stonean space. Since $X \cap K$ and $Y \cap K$ are relatively open and σ-compact in K and t belongs to the intersection of their closures, they have non-empty intersection. Hence T is an F'-space.

As (v) $\Rightarrow$ (vi) is no problem, assume (vi) and let f in $\mathscr{K}(T)$ be given. Since $P = \{t \in T \mid f(t) > 0\}$ and $N = \{t \in T \mid f(t) < 0\}$ have disjoint closures, Tietze's theorem gives a function k in $\mathscr{K}(T)$ which is 1 on P and -1 on N, as wanted.

Finally assume (ii). First we show that any ideal I in $\mathscr{K}(T)$ is hereditary. So let h in I and f in $\mathscr{K}(T)$ be given with $|f| \leqq h$. The positivity set S of h is open and σ-compact, and $\bar{S}$ is compact and hence a sub–Stonean space, since (ii) implies (iii). By [3, Theorem 1.10] $\bar{S}$ is homeomorphic the Stone-Čech compactificaton βS of S, so there exists a function g in $\mathscr{K}(T)$ extending the bounded continuous function $s \mapsto f(s)h(s)^{-1}$ on S, cf. [3, Corollary 1.11]. Thus $f = gh$ is in I.

To show that $\mathscr{K}(T)$ is an F-ring it is obviously enough to show that the ideal generated by two functions f and g in $\mathscr{K}(T)$ is the principal ideal generated by $|f| + |g|$, cf. [1, Theorem 2.3]. The ideal generated by $|f| + |g|$ contains f and g by the above. As $f = k|f|$ and hence $|f| = kf$ for some funtion k in $\mathscr{K}(T)$, the ideal generated by f and g contains $|f|$ and likewise $|g|$ and thus $|f| + |g|$.

COROLLARY 1. *Any locally compact subspace of a locally compact F'-space is an F'-space.*

PROOF. Obvious form condition (iii).

COROLLARY 2. *A locally compact Hausdorff space T is sub–Stonean if and only if it is an F'-space with $C_0(T) = \mathscr{K}(T)$ and if and only if $C_0(T)$ is an F-ring.*

PROOF. The condition $C_0(T) = \mathscr{K}(T)$ is equivalent to the condition that any open σ-compact subset of T has compact closure, so T is a sub–Stonean space if and only if T is an F'-space and $C_0(T) = \mathscr{K}(T)$, and this implies that $C_0(T)$ is an F-ring. Assume that $C_0(T)$ is an F-ring and let f in $C_0(T)$ be given; as in the proof of the theorem there exists a function k in $C_0(T)$ with $f = k|f|$; thus f has a compact support. As $\mathscr{K}(T) = C_0(T)$ is an F-ring, T is an F'-space.

2.

LEMMA. *Let T be a locally compact Hausdorff space and t_0 a point in T; assume that there is a basis $\mathscr{B}$ for the neighbourhoods of t_0 consisting of open, compact sets and closed under the formation of intersections of decreasing sequences. Then $\{t_0\}$ is open.*

PROOF. Assume $\{t_0\}$ is not open. Then we can choose a strictly decreasing sequence of sets in $\mathscr{B}$. But a strictly decreasing sequence of compact sets cannot have an open intersection.

PROPOSITION. *Let G be a locally compact group and H a closed subgroup, which is not open. There exists a compact subgroup K of G of type G_δ in G, such that KH is not open. If G is σ-compact, K may be chosen as a normal subgroup.*

PROOF. Assume first that G is σ-compact. Remember that any neighbourhood U of e in G contains a compact normal subgroup of type G_δ [2, Theorem A.9.]. If KH were open for each compact normal subgroup K of type G_δ then the set of images in G/H of the normal compact G_δ subgroups in G would be a basis for the neighbourhoods of H in G/H satisfying the conditions of the lemma above and H would be open.

If G is not σ-compact we choose an open σ-compact subgroup G_0 (generated by any symmetric compact neighbourhood of e in G). Then $H \cap G_0$ is not open so we can choose a compact subgroup K of type G_δ in G_0 and in G with $K(H \cap G_0) = KH \cap G_0$ not open and hence KH not open.

THEOREM. *Let G be a locally compact group and H a closed subgroup. Assume that G/H is an F'-space. Then G/H is discrete.*

PROOF. It is enough to show that some non-empty open subset of G/H is discrete, so we may assume that G is σ-compact. Let K be any compact normal subgroup of type G_δ in G. Then $KH = HK$ is a closed subgroup. As G/K has countable basis for the neighbourhoods of K and the natural map $gK \mapsto gHK$ is open and continuous, $G/(KH)$ has countable basis for the neighbourhoods of KH. The natural map $gH \mapsto gKH$ of G/H onto $G/(KH)$ is continuous, open and proper, so disjoint open, σ-compact, relatively compact subsets of $G/(KH)$ have disjoint closures, i.e. $G/(KH)$ is an F'-space (cf. Theorem in section 1, and [3, Proposition 1.2]). As in [1, Corollary 2.4] and [3, Proposition 1.5] we see that KH is isolated in $G/(KH)$, so KH is open in G.

By the proposition above H is open in G, so G/H is discrete.

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