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RIESZ DISTRIBUTIONS

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Introduction.

In this note we study some properties of the Riesz distributions associated to a quadratic form of signature (m, n). Although there is an extensive literature on this subject, the extent to which the Riesz distributions determine the space of all invariant distributions does not appear to have received the attention it deserves. This note is addressed to this issue. For clarity of exposition we have written this article in two parts: Section 1 deals with the hyperbolic case (of $m = 1$), while Section 2 takes up the general case (including the situation of $m = 0$).

This note is closely related to the longer paper [KV] on Lorentz invariant distributions supported on the forward light cone associated with a form of signature $(1, n)$.

1.0. Riesz distributions in the hyperbolic case. In this section we work in $\mathsf{R}^{1,n}$, for $n \geqq 2$, with coordinates $(x_0, x_1, \ldots, x_n)$. We set $d = n + 1$, and we have $\mathsf{R}^{1,n} \simeq \mathsf{R}^d$. We denote by ω the (normalized indefinite) quadratic form of signature $(1, n)$:

$$\omega(x) = x_0^2 - x_1^2 - \ldots - x_n^2.$$

Since ω: $\mathsf{R}^{1,n} \setminus \{0\} \to \mathsf{R}$ is a surjective submersion, we have the operation ω_* of pushforward mapping $C_c^\infty(\mathsf{R}^{1,n} \setminus \{0\})$ onto $C_c^\infty(\mathsf{R})$, and the dual operation ω^* of pullback sending distributions on R to distributions on $\mathsf{R}^{1,n} \setminus \{0\}$ (see [KV, §5.3-4] for more details). We write G for $\mathrm{SO}(1, n)^\circ$, the connected component containing the identity of the subgroup of $\mathrm{GL}(\mathsf{R}^{1,n})$ of elements fixing ω. The cone $X_0 = \{x \in \mathsf{R}^{1,n} \mid \omega(x) = 0\}$ is the union of the following G-orbits: $\{0\}$, the open forward, and the open backward light cone resp., given by

$$X_0^\pm = \{x \in \mathsf{R}^{1,n} \mid \omega(x) = 0, x_0 \gtrless 0\}.$$

The classical Riesz distributions, cf. [Ri] (and [D] for a contemporary review), are intimately tied up with the problem of finding G-invariant fundamental

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solutions for the associated constant-coefficient hyperbolic differential operator:

$$\square = \frac{\partial^2}{\partial x_0^2} - \frac{\partial^2}{\partial x_1^2} - \dots - \frac{\partial^2}{\partial x_n^2},$$

that is, of finding G-invariant distributions T on $\mathsf{R}^{1,n}$ satisfying $\square T = \delta$, where δ denotes the delta function on $\mathsf{R}^{1,n}$ at 0. Along with ω and $\square$ we have their commutator; up to constants it is equal to the radial or Euler vector field

$$\mathscr{E} = x_0 \frac{\partial}{\partial x_0} + \dots + x_n \frac{\partial}{\partial x_n}.$$

These three operators generate a 3-dimensional simple subalgebra $\mathfrak{a}$ of the algebra of polynomial differential operators. In fact, let $M(\omega)$ be the operator of multiplication by ω. Then, if we write

$$H = \mathscr{E} + \frac{d}{2}, \ X = \frac{1}{2} M(\omega), \ Y = -\frac{1}{2}\square,$$

we have the commutation rules $[H, X] = 2X, [H, Y] = -2Y, [X, Y] = H$; so that

$$\mathfrak{a} = \mathsf{C} \cdot H + \mathsf{C} \cdot X + \mathsf{C} \cdot Y$$

is a three-dimensional simple subalgebra of the algebra of polynomial differential operators on $\mathsf{R}^{1,n}$.

We denote by $\bar{J}(\mathsf{C})$ the linear space of all distributions on $\mathsf{R}^{1,n}$ that are invariant under G and supported on $\mathrm{Cl}(X_0^+) = X_0^+ \cup \{0\}$, the closed forward light cone. It is clear that $\mathfrak{a}$ operates on $\bar{J}(\mathsf{C})$ so that $\bar{J}(\mathsf{C})$ is an $\mathfrak{a}$-module.

The Riesz distributions are supported on the closed *solid* forward light cone Γ^+, i.e.

$$\Gamma^+ = \{(x_0, x_1, \dots, x_n) \,|\, x_0 \geqq (x_1^2 + \dots + x_n^2)^{1/2}\}.$$

They are tempered distributions R_s on $\mathsf{R}^{1,n}$ depending complex-analytically on the complex parameter s (in fact R_s is the restriction to Γ^+ of ω^s with a suitable normalization); they form a one-parameter group for the operation of convolution of distributions. The element R_2 provides a fundamental solution for the wave operator $\square$; more generally we have $\square R_{s+2} = R_s$.

We determine which of the R_s have their supports in $\partial\Gamma^+ = \mathrm{Cl}(X_0^+)$. Taken in conjunction with the results of [KV] our present results show that the Riesz family determines in a very explicit fashion the *entire* $\mathfrak{a}$-module structure of $\bar{J}(\mathsf{C})$. We also show that a corresponding result is true for *all* G-invariant distributions supported on Γ^+, namely that all of these are (generalized) superpositions of the R_s.

More precisely, for d odd we can take as generators for $\bar{J}(\mathsf{C})$ the distributions R_{d-2} and R_0. Now R_{d-2} is up to a scalar factor equal to α_0^+, the (normalized) invariant measure on X_0^+, which is a distribution homogeneous of degree -2, while $R_0 = \delta$, which is homogeneous of degree $-d$. On the other hand, for d even, we can take as a generator for $\bar{J}(\mathsf{C})$ the generalized Riesz distribution

$$\rho := \frac{d}{ds}\bigg|_{s=0} R_s$$

on $\mathsf{R}^{1,n}$, which is the infinitesimal generator of the group of the R_s. This distribution ρ is uniquely determined by the following properties

(1) $\rho = \pi^{-\frac{d-2}{2}} \omega^*(\delta_0^{(\frac{d-2}{2})})$, outside 0;

(2) $(\mathscr{E} + d)\rho = \delta$;

(3) $\langle \rho, e^{-x_0} \rangle = 0$.

The distribution ρ is thus associated with the quadratic form ω, and also normalized, in quite a natural fashion.

1.1. *Mellin potentials on* R. For $\alpha \in \mathsf{C}$ with $\operatorname{Re}\alpha > 0$ the tempered distribution $t_+^{\alpha-1}$ on R with support contained in $[0, \infty)$ is given by

$$\langle t_+^{\alpha-1}, f \rangle = \int_0^\infty t^{\alpha-1} f(t)dt \quad (f \in S(\mathsf{R})).$$

Then

$$\alpha \mapsto M_\alpha := \frac{1}{\Gamma(\alpha)} t_+^{\alpha-1} \in S'(\mathsf{R}) \tag{1.1}$$

defines a complex-analytic mapping. In fact, integration by parts gives

$$M_\alpha = (M_{\alpha+k})^{(k)} \quad (k \in \mathsf{Z});$$

and this formula immediately leads to the analytic continuation from the domain $\operatorname{Re}\alpha > 0$ to all of C. In particular, we notice

$$M_{-k} = \delta_0^{(k)}, \ \operatorname{supp}(M_\alpha) \subset \{0\} \Leftrightarrow \alpha \in \mathsf{Z}_{\leqq 0}. \tag{1.2}$$

1.2. *Distributions supported on* Γ^+. For any open set $U \subset \mathsf{R}^{1,n}$, we shall write $S(U)$ for the Schwartz space of U, that is, the Fréchet space of all C^∞ functions f defined on U for which the usual seminorms are bounded. We denote by $D'(\Gamma^+)$, and $S'(\Gamma^+)$, the space of distributions, and tempered distributions, respectively, on $\mathsf{R}^{1,n}$ with support contained in Γ^+; and by $D'(\Gamma^+)^G$, and $S'(\Gamma^+)^G$, resp. the subspaces of G-invariant elements. As usual, these spaces are provided with the weak topology.

In order to maintain Riesz's classical normalization using the function $x \mapsto e^{-x_0}$, we must make sure it is the restriction to Γ^+ of a Schwartz function on $\mathbf{R}^{1,n}$. This is easy to do and we state the result for the reader's convenience.

LEMMA. *For $a, b > 0$, we set $U_{a,b} = \{x \in \mathbf{R}^{1,n} \mid \omega(x) > -a, x_0 > -b\}$. Then we have, for all $\tau > 0$, that $x \mapsto e^{-\tau x_0}$ belongs to $S(U_{a,b})$. There exists $f \in S(\mathbf{R}^{1,n})$ that restricts to $x \mapsto e^{-\tau x_0}$ on $U_{a,b}$.*

COROLLARY. *Let $p \in \mathbf{C}^{1,n}$ be such that $\operatorname{Re}(p) \in (\Gamma^+)^{\text{int}}$. Then $e^{-\langle \cdot, p \rangle}$ is the restriction to a neighborhood of Γ^+ of an element of $S(\mathbf{R}^{1,n})$, which neighborhood stays fixed when p varies in compact sets.*

PROOF. We may assume p real, and in $(\Gamma^+)^{\text{int}}$. As $\mathbf{R}_+ \cdot G \cdot (1, 0, \ldots, 0) = (\Gamma^+)^{\text{int}}$, we come down to $p = (1, 0, \ldots, 0)$. Then $e^{-\langle x, p \rangle} = e^{-x_0}$.

Next we observe that, for $T \in D'(\Gamma^+)$, (resp. $S'(\Gamma^+)$) and f in $D(\mathbf{R}^{1,n})$, (resp. $S(\mathbf{R}^{1,n})$) we have $T(f) = 0$ whenever f vanishes on Γ^+. It follows, for such distributions T and p as above, that $T(e^{-\langle \cdot, p \rangle})$ is well-defined. Indeed, there are $\phi_m \in D(\mathbf{R}^{1,n})$ such that $\phi_m f \to f$, for all $f \in S(\mathbf{R}^{1,n})$. So it is enough to prove the result for $T \in D'(\Gamma^+)$, and to show that $T(\phi f) = 0$, for all $\phi \in D(\mathbf{R}^{1,n})$. So we may assume that $\operatorname{supp}(T)$ is compact. Applying [H, Th. 2.3.3] (and taking for k the order of T) we now get $T(f) = 0$, because $f = 0$ on Γ^+ implies $\partial^\alpha f = 0$ on Γ^+, for all α.

1.3. *Riesz distributions on Γ^+.* We define for $s \in \mathbf{C}$ with $\operatorname{Re} s > d - 2$, the distribution $R_s \in S'(\Gamma^+)^G$ by

$$\langle R_s, f \rangle = \frac{1}{H_d(s)} \int_{(\Gamma^+)^{\text{int}}} \omega^{\frac{s-d}{2}}(x) f(x)\, d^{n+1}x \quad (f \in S(\Gamma^+)), \tag{3.1}$$

where

$$H_d(s) = \pi^{\frac{d-2}{2}} 2^{s-1} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s-d+2}{2}\right).$$

A simple calculation using polar coordinates on $\mathbf{R} \times \mathbf{R}^n$ shows that the integral converges absolutely and represents a holomorphic function in this domain. The constant $H_d(s)$ is determined by requiring that

$$1 = \langle R_s, e^{-x_0} \rangle, \tag{3.2}$$

so that

$$H_d(s) = \int_{(\Gamma^+)^{\text{int}}} e^{-x_0} (x_0^2 - (x_1^2 + \ldots + x_n^2))^{\frac{s-d}{2}}\, dx.$$

We now take up the *analytic continuation* of the mapping $s \mapsto R_s$. One immediately checks that $\square \omega^{\frac{s+2-d}{2}} = s(s-d+2)\omega^{\frac{s-d}{2}}$. So we have, for $\operatorname{Re} s > d$ and $f \in S(\Gamma^+)$, that $\langle \square R_{s+2}, f \rangle = \langle R_{s+2}, \square f \rangle = \langle R_s, f \rangle$, since we are allowed to use the second Green identity (the vertex of the cone is of codimension $\geqq 2$) and the partial derivatives of the powers of ω vanish on $\mathrm{Cl}(X_0^+)$. Thus $\square^k R_{s+2k} = R_s$ ($\operatorname{Re} s > d, k \in \mathbf{Z}_{\geqq 0}$). It is now obvious that $s \mapsto R_s$ can be analytically continued from the domain $\operatorname{Re} s > d$, so that

(3.3) $s \mapsto R_s$ is a complex-analytic mapping $\mathbf{C} \to S'(\Gamma^+)^G$ with the property

(3.4) $$\square^k R_{s+2k} = R_s \quad (s \in \mathbf{C}, k \in \mathbf{Z}_{\geqq 0}).$$

Notice that R_s is a distribution homogeneous of degree $s-d$, that is

$$\langle R_s, f \rangle = t^{s-d} \langle R_s, f_t \rangle, \quad \text{with } f_t(x) = t^d f(tx) \quad (t > 0).$$

So the R_s satisfy the following differential equation expressing their *homogeneity*

(3.5) $$\mathscr{E} R_s = (s-d) R_s \quad (s \in \mathbf{C}).$$

We observe that formula (3.2) directly implies the *normalization*

(3.6) $$\langle R_s, e^{-\tau x_0} \rangle = \tau^{-s} \quad (s \in \mathbf{C}, \tau > 0).$$

But this gives $\langle R_{s+t}, e^{-\tau x_0} \rangle = \langle R_s, e^{-\tau x_0} \rangle \langle R_t, e^{-\tau x_0} \rangle$. On the other hand, we have the following homomorphism property of the Laplace transform, where we denote the convolution of distributions by $*$,

$$\langle R_s, e^{-\tau x_0} \rangle \langle R_t, e^{-\tau x_0} \rangle = \langle R_s * R_t, e^{-\tau x_0} \rangle.$$

Accordingly

$$\langle R_{s+t}, e^{-\tau x_0} \rangle = \langle R_s * R_t, e^{-\tau x_0} \rangle.$$

Now one can use the fact that $(\Gamma^+)^{\text{int}} = \{\tau g \cdot (1, 0, \ldots, 0) \mid \tau > 0, g \in G\}$ to show that if $T \in S'(\Gamma^+)^G$ and $T(e^{-\tau x_0}) = 0$, for all $\tau > 0$, then $T = 0$. So we obtain the *group property* of the Riesz distributions

(3.7) $$R_{s+t} = R_s * R_t \quad (s, t \in \mathbf{C}).$$

1.4. *The space of distributions* $D'(\Gamma^+)^G$. One can proceed as in [M] [GL] [T] to describe all the elements in $D'(\Gamma^+)^G$. For this purpose we should introduce the *restricted mean value function*

$$(M^+ f)(t) = \int_{R^n} \frac{f(x_0', x)}{2x_0'} d^n x, \quad x_0' = \sqrt{|x|^2 + t}, t > 0.$$

Here f varies over $\mathrm{C}_c^\infty(\mathrm{R}^{1,n})$. We denote by γ the function on $(0, \infty)$ which depends on the parity of d as follows:

$$\gamma(t) = \begin{cases} t^{1/2}, & d \text{ odd}, \\ \log \frac{1}{t}, & d \text{ even}. \end{cases}$$

We define $\mathfrak{M}^+$ as the space of all functions h on $(0, \infty)$ that are smooth and of the form

$$h = h_1 + \gamma h_2,$$

where h_i $(i = 1, 2)$ are in $C_c^\infty([0, \infty))$. If $h \in \mathfrak{M}^+$ and if

$$m_j = \frac{h_1^{(j)}(0)}{j!}, \quad l_j = \frac{h_2^{(j)}(0)}{j!},$$

it follows as in [KV, §7.3] that $m_j = m_j(h)$ and $l_j = l_j(h)$ are uniquely determined by h. It is easy to give a topology for $\mathfrak{M}^+$ that is entirely analogous to the one given for the image of the unrestricted mean value map. We can then obtain the following

THEOREM. *Let* $m = \left[\frac{d-2}{2}\right]$. *Then* $f \mapsto M^+ f$ *is a continuous, linear, and surjective mapping*

$$C_c^\infty(\mathsf{R}^{1,n}) \to \mathfrak{M}^+_{[m]},$$

where $\mathfrak{M}^+_{[m]}$ *denotes the subspace of all* $h \in \mathfrak{M}^+$ *for which* $l_j(h) = 0$, *for* $0 \leqq j < m$. *Moreover, the dual map into* $D'(\Gamma^+)^G$, *the space of* G-*invariant distributions on* $\mathsf{R}^{1,n}$ *that are supported by* Γ^+, *is surjective.*

As a consequence we have the following characterization of the R_s (using the formulae for $\mathscr{E}_\omega$ in [KV, §5.5]). We denote by $D'((\Gamma^+)^{\text{int}})^G$ the space of G-invariant distributions on the open solid forward light cone.

COROLLARY. *Let* $s \in \mathsf{C}$. *Every distribution in* $D'((\Gamma^+)^{\text{int}})^G$ *that is homogeneous of degree* $s - d$, *is a scalar multiple of the restriction to* $(\Gamma^+)^{\text{int}}$ *of the Riesz distribution* R_s.

From the formulae in [KV, §5.3] we now obtain, for $f \in C_c^\infty(\mathsf{R}^{1,n})$ and $s > 0$,

$$H_d(s)\langle R_s, f\rangle = \int_0^\infty M^+ f(t) t^{\frac{s-d}{2}}\, dt. \tag{4.1}$$

If we define $\bar{h}$, the Mellin transform of h, for $h \in L^1([0, \infty))$, by

$$\bar{h}\colon \{s \in \mathsf{C} \mid \operatorname{Re} s > 0\} \to \mathsf{C}, \quad \bar{h}(s) = \int_0^\infty h(t) t^s \frac{dt}{t},$$

it follows that $H_d(s)\langle R_s, f\rangle = \overline{M^+ f}\left(\frac{s-d+2}{2}\right)$; and Mellin inversion gives us, for all $t > 0$ and $\sigma > 0$,

$$M^+ f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \langle t^{-(\sigma+i\tau)} H_d(d-2+2(\sigma+i\tau)) R_{d-2+2(\sigma+i\tau)}, f\rangle \, d\tau.$$

This expression for $M^+ f$, in conjunction with the Theorem leads to the following

PROPOSITION. *Every element of $D'(\Gamma^+)^G$ can be written as a generalized superposition over (a subset of the) $s \in \mathbf{C}$ of the Riesz distributions R_s.*

1.5. *Riesz distributions supported on* $\mathrm{Cl}(X_0^+)$. The parameter $s \in \mathbf{C}$ is said to be a *singular value* of the family of parameters if $\operatorname{supp}(R_s) \subset \mathrm{Cl}(X_0^+)$. In this and the next paragraph we shall determine all the singular values s and identify the corresponding R_s.

In view of (4.1) and (1.1) we have, on $\Gamma^+ \setminus \{0\}$

$$R_s = \frac{1}{H_d(s)}\omega^*(t_+^{\frac{s-d}{2}}) = \omega^* \frac{1}{\pi^{\frac{d-2}{2}} 2^{s-1}\Gamma\left(\frac{s}{2}\right)} M_{\frac{s-d+2}{2}} . \tag{5.1}$$

Using (1.2) we get that $\operatorname{supp}(R_s) \subset \mathrm{Cl}(X_0^+)$ if $\frac{1}{2}(s-d+2) \in \mathbf{Z}_{\leqq 0}$, that is

$$\operatorname{supp}(R_s) \subset \mathrm{Cl}(X_0^+) \text{ if } s = d-2-2k \ \ (k \in \mathbf{Z}_{\geqq 0}). \tag{5.2}$$

More precisely, according to (1.2)

$$R_{d-2-2k} = \omega^* \frac{1}{\pi^{\frac{d-2}{2}} 2^{d-3-2k}\Gamma\left(\frac{d-2-2k}{2}\right)} \delta_0^{(k)} . \tag{5.3}$$

Moreover, because of the zeros of $s \mapsto \dfrac{1}{\Gamma\left(\frac{s}{2}\right)}$ we find an additional restriction on the supports

$$\operatorname{supp}(R_s) = \{0\} \Leftrightarrow s = -2l \quad (l \in \mathbf{Z}_{\geqq 0}). \tag{5.4}$$

A further study of the conditions (5.2) and (5.4) requires distinguishing between odd and even d. Before doing so, we first identify R_0 and R_{d-2} with well-known distributions. We have, cf. (3.5),

$$\operatorname{supp}(R_0) \subset \{0\}, \quad R_0 \text{ homogeneous of degree } -d,$$

where $-d$ is the degree of homogeneity of δ. Thus $R_0 = c\delta$, for some constant c.

But (3.6) gives $\langle R_0, e^{-\tau x_0}\rangle = 1$, hence

(5.5) $$R_0 = \delta.$$

Furthermore, on $\Gamma^+ \setminus \{0\}$

$$R_{d-2} = \frac{1}{\pi^{\frac{d-2}{2}} 2^{d-3}\, \Gamma\left(\frac{d-2}{2}\right)} \omega^*(\delta_0), \quad R_{d-2} \text{ homogeneous of degree } -2.$$

On $\Gamma^+ \setminus \{0\}$ we have that $\omega^*(\delta_0) = \alpha_0^+$, the invariant measure on X_0^+. Now R_{d-2} still might have a summand supported on $\{0\}$; this then has the form $P(\Box) = \sum a_j \Box^j$, for some polynomial $P(X) = \sum a_j X^j$, because of the G-invariance of R_{d-2}. But a distribution of the form $P(\Box)$ consists of summands homogeneous of degree $\leqq -d$, whereas R_{d-2} is homogeneous of degree -2. Hence there is no summand in R_{d-2} supported at $\{0\}$, at so we have obtained, in $S'(\Gamma^+)^G$

(5.6) $$R_{d-2} = \frac{1}{\pi^{\frac{d-2}{2}} 2^{d-3}\, \Gamma\left(\frac{d-2}{2}\right)} \alpha_0^+$$

We now suppose that d is *odd*. Then (cf. (5.2) and (5.4)) $d-2-2k$ is always odd, and $-2l$ is even, so the two families of singular values are disjoint, and we obtain

$$R_{d-2-2k} = \Box^k R_{d-2} = \frac{1}{\pi^{\frac{d-2}{2}} 2^{d-3}\, \Gamma\left(\frac{d-2}{2}\right)} \Box^k \alpha_0^+ \quad (k \in \mathbf{Z}_{\geqq 0}),$$

$$R_{-2l} = \Box^l \delta \quad (l \in \mathbf{Z}_{\geqq 0}).$$

It is now clear from [KV, Lemma 6.5] (for $j = 0$) that the R_s for singular s span the space of invariant distributions supported on $\mathrm{Cl}(X_0^+)$.

Let now d be *even*. Then $d-2-2k$ is even, as is $-2l$; so the two families of singular values do overlap, viz. for $k \geqq k_0$, where $d-2-2k_0 = 0$. Enumerating in this case the R_s exactly once for singular values of s, we find

$$R_{d-2-2k} = \frac{1}{\pi^{\frac{d-2}{2}} 2^{d-3}\, \Gamma\left(\frac{d-2}{2}\right)} \Box^k \alpha_0^+ \quad \left(0 \leqq k < \frac{d-2}{2}\right),$$

$$R_{-2l} = \Box^l \delta \quad (l \in \mathbf{Z}_{\geqq 0}).$$

From the relation

$$\square^{\frac{d-2}{2}} R_{d-2} = R_0 = \delta,$$

we obtain (compare with [KV, §7.1])

$$\square^{\frac{d-2}{2}} \alpha_0^+ = \pi^{\frac{d-2}{2}} 2^{d-3} \Gamma\left(\frac{d-2}{2}\right) \delta.$$

These formulae also display the validity of the Huygens principle in a very simple and direct manner. However, because of the double multiplicity of the singular values, one should expect the existence of additional distributions. We shall find that this is indeed the case, and that they are obtained by the Frobenius method of differentiation with respect to the parameter s. We take this up in the next paragraph.

1.6. *Infinitesimal generator of the Riesz family.* We continue to suppose d to be even. In view of (3.3) we have well-defined elements

$$\frac{d}{ds} R_s \in S'(\Gamma^+)^G.$$

From (3.5) and (3.6) we obtain by differentiation

$$(\mathscr{E} - s + d)\frac{d}{ds} R_s = R_s, \quad \left\langle \frac{d}{ds} R_s, e^{-\tau x_0} \right\rangle = -\tau^{-s} \log \tau.$$

In particular, let us put

$$\rho := \frac{d}{ds}\bigg|_{s=0} R_s \in S'(\Gamma^+)^G. \tag{6.1}$$

The distribution ρ is the infinitesimal generator of the group of the Riesz distributions, in a setting of the dual of a Fréchet space. Furthermore ρ is a generalized eigendistribution for $\mathscr{E}$ for the eigenvalue $-d$, and is normalized in a natural fashion, viz.

$$(\mathscr{E} + d)\rho = \delta, \quad \langle \rho, e^{-\tau x_0} \rangle = -\log \tau. \tag{6.2}$$

In view of (5.1) we have, on $\mathrm{R}^{1,n} \setminus (-\Gamma^+)$

$$\rho = \omega^* \left(\frac{1}{\pi^{\frac{d-2}{2}}} \frac{d}{ds}\bigg|_{s=0} \left(\frac{1}{2^{s-1} \Gamma\left(\frac{s}{2}\right)} M_{\frac{s-d+2}{2}} \right) \right)$$

Because $\frac{1}{\Gamma\left(\frac{0}{2}\right)} = 0$, and $\frac{d}{ds}\Big|_{s=0} \frac{1}{\Gamma\left(\frac{s}{2}\right)} = \frac{1}{2}$, we actually have, on $\mathrm{R}^{1,n} \setminus (-\Gamma^+)$

$$\rho = \pi^{-\frac{d-2}{2}} \omega^* (M_{\frac{-d+2}{2}}) = \pi^{-\frac{d-2}{2}} \omega^* \left(\delta_0^{\left(\frac{d-2}{2}\right)}\right). \tag{6.3}$$

This shows

$$\text{(6.4)} \qquad \operatorname{supp}(\rho) \subset \mathrm{Cl}(X_0^+).$$

Expressing (6.2) in terms of $H = \mathscr{E} + \frac{d}{2}$, we find

$$\left(H + \frac{d}{2}\right)\rho = \delta.$$

(Despite the absence of the factor 2, this is in complete agreement with [KV, Lemma 6.8]. Indeed, the τ in that lemma is supported by the full cone X_0, whereas ρ is supported by $\mathrm{Cl}(X_0^+)$. But ρ corresponds to τ under the mapping $s \mapsto s + s^\circ$ of Lemma 6.6, and therefore gives only half the contribution of τ.)

We summarize the results obtained in the following

THEOREM. *The generalized Riesz distribution $\rho \in S'(\Gamma^+)^G$ is uniquely determined by* (6.1)-(6.4) *and generates $\bar{J}(\mathrm{C})$ as a $\mathfrak{a}$-module.*

REMARK 1. An easy computation gives

$$\omega R_s = s(s - d + 2)R_{s+2}, \quad \square \omega R_s = s(s - d + 2)R_s.$$

Accordingly

$$\text{(6.5)} \qquad \square \omega \rho = \frac{d}{ds}\bigg|_{s=0} (s^2 - ds + 2s)R_s = -(d-2)\left(H + \frac{d}{2}\right)\rho.$$

Note that (6.5) is another proof of [KV, Lemma 6.9]. In other words, the γ-invariant of the $\mathfrak{a}$-module $\bar{J}(\mathrm{C})$ equals $\frac{1}{4}(d-2)$. Notice also that this computation is actually independent of the particular identification in (6.3) of the distribution ρ on $\mathrm{R}^{1,n} \backslash (-\Gamma^+)$ as a pullback.

REMARK 2. If $d = 4$, one can give a direct proof that ρ is equal to $\frac{1}{\pi}$ times the space-time expression τ_1 for τ (cf. [KV, §7.2]) on $S_0(\mathrm{R}^{1,3})$, the subspace of the Schwartz space consisting of functions that vanish at 0. That is

$$\rho(f) = \frac{1}{4\pi}\int_{\mathrm{R}^3} f(|x|, x)\frac{dx}{|x|^3} - \frac{1}{4\pi}\int_{\mathrm{R}^3} \frac{\partial f}{\partial x_0}(|x|, x)\frac{dx}{|x|^2} \quad (f \in S_0(\mathrm{R}^{1,3})).$$

In fact, for $f \in S_0(\mathrm{R}^{1,3})$, we are allowed to compute $\frac{dM^+f}{dt}(0)$ by differentiating $M^+f(t)$ (cf. §1.4) under the integral sign; this immediately leads to

$$\frac{dM^+f}{dt}(0) = -\tau_1(f) \quad (f \in S_0(\mathsf{R}^{1,3})).$$

But, taken in conjunction with the one-sided Methée's calculus, this gives that we can write

$$(6.6) \qquad M^+f(t) = h(t) + k(t)t\log t, \quad k(0) = 0, \quad h, k \in C_c^\infty([0,\infty)).$$

On the other hand, if we start from $\langle \rho, f \rangle = \frac{d}{ds}\Big|_{s=0} \langle R_{s+4}, \Box^2 f \rangle$ we get, for $f \in S_0(\mathsf{R}^{1,3})$,

$$\langle \rho, f \rangle = \frac{1}{16\pi} \int_{(\Gamma^+)^{\text{int}}} \Box^2 f(x) \log \omega(x)\, dx.$$

The Methée calculus enables us to rewrite this as

$$\langle \rho, f \rangle = \frac{1}{\pi} \int_0^\infty \log t \left(t^2 \frac{d^4}{dt^4} + 2t \frac{d^3}{dt^3} \right) M^+f(t)\, dt.$$

By a straightforward calculation and using (6.6), we now find

$$\langle \rho, f \rangle = -\frac{1}{\pi} \frac{dM^+f}{dt}(0), \text{ and thus } \rho = \frac{1}{\pi}\tau_1 \text{ on } S_0(\mathsf{R}^{1,3}).$$

2.1 Riesz distributions in the ultra-hyperbolic case. We now study distributions analogous to those of Riesz that can be associated with an ultra-hyperbolic form, i.e. a (normalized) indefinite quadratic form on $\mathsf{R}^{m,n}$ of signature (m, n), with $m > 1$ and $n > 1$:

$$\omega(x) = x_1^2 + \ldots + x_m^2 - x_{m+1}^2 - \ldots - x_{m+n}^2,$$

for $x = (x_1, \ldots, x_m, x_{m+1}, \ldots, x_{m+n}) \in \mathsf{R}^{m,n}$. We shall write G for $SO(m, n)$, the orthogonal group of the form ω. Such distributions have been considered before, for recent contributions, see Orloff [O]. They can be used to find G-invariant fundamental solutions for the associated constant-coefficient ultra-hyperbolic differential operator:

$$\Box = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_m^2} - \frac{\partial^2}{\partial x_{m+1}^2} - \ldots - \frac{\partial^2}{\partial x_{m+n}^2}.$$

Fundamental solutions have been constructed by Gelfand and Shilov [GS, Sect. III. 2.5], Fourès-Bruhat [F] and de Rham [Rh].

Here we shall give a simple and unified approach to these problems, using *two* generically distinct one-parameter families of Riesz distributions, denoted by $(R_s^\pm)_{s\in\mathbb{C}}$.

If m or n is odd, then at least one of these families contains a nonzero multiple of δ; and therefore it also contains a fundamental solution (cf. (3.3), (3.4), (3.5) below) for $\square$, which turns out to coincide exactly with those given by de Rham. But if both m and n are even, then the $R_s^\pm$ degenerate at $s = 0$, that is, $R_0^+ = R_0^- = 0$. As a remedy we differentiate with respect to the parameter s at $s = 0$. This, indeed, will provide nonzero generalized Riesz distributions $\rho^\pm$ on $\mathbb{R}^{m,n}$ associated with ω; actually they are the infinitesimal generators of the families $R_s^\pm$. In the hyperbolic case the counterpart ρ of these $\rho^\pm$ is supported on all of the forward light cone and fails to be homogeneous at 0. In the case at hand the degeneracy of the Riesz distributions $(R_s^\pm)$ causes the $\rho^\pm$ to be supported at 0 and also to be homogeneous at 0. In fact, we have

$$\rho^+ = 2^{1-\frac{d}{2}}\pi(-1)^{\frac{m}{2}-1}\sum_{k=0}^{\frac{m}{2}-1}\binom{\frac{d}{2}-1}{k}\delta; \tag{1}$$

$$\rho^+ + (-1)^{\frac{d}{2}}\rho^- = (-1)^{\frac{m}{2}-1}\pi\delta. \tag{2}$$

As an easy consequence the same fundamental solution (cf. (4.4)) as found by de Rham is easily obtained.

In constrast to the situation above, in the hyperbolic case the distribution ρ is not needed to construct fundamental solutions. In Section 1 we encountered ρ only when finding a generator for the $\mathfrak{a}$-module $\bar{J}(\mathbb{C})$, when $n + 1$ was even.

2.2. *Riesz distributions.* We set $d = m + n$, thus $\mathbb{R}^{m,n} \simeq \mathbb{R}^d$. In contrast to the case of $m = 1$, we have for $m > 1$ that the complement in $\mathbb{R}^{m,n}$ of the cone $X_0 = \{x \in \mathbb{R}^{m,n} \mid \omega(x) = 0\}$ consists of two connected components; we shall denote these by:

$$C^\pm = \{x \in \mathbb{R}^{m,n} \mid \pm\,\omega(x) > 0\}.$$

For $s \in \mathbb{C}$ with $\operatorname{Re} s > d - 2$, we introduce two Riesz distributions, R_s^+ and R_s^-, both in $S'(\mathbb{R}^{m,n})$, defined by:

$$\langle R_s^\pm, f\rangle = \frac{1}{H_d(s)}\int_{C^\pm}(\pm\,\omega(x))^{\frac{s-d}{2}}f(x)\,dx \quad (f \in S(\mathbb{R}^{m,n})). \tag{2.1}$$

Notice that the factor

$$H_d(s) = \pi^{\frac{d-2}{2}}2^{s-1}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s-d+2}{2}\right) \tag{2.2}$$

is independent of m and n separately. We have

$$(2.3)\qquad R_s^{\pm} = \frac{1}{\pi^{\frac{d-2}{2}} 2^{s-1} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s-d+2}{2}\right)} (\pm\omega)^{\frac{s-d}{2}} 1_{C^{\pm}}.$$

It is immediate that the $R_s^{\pm}$ are invariant under G, and that $\operatorname{supp}(R_s^{\pm}) \subset C^{\pm}$. It is obvious from (2.3) that

$$(2.4)\qquad R_s^- = \tilde{R}_s^+.$$

Here $\tilde{R}_s^+$ is the $+$ Riesz distribution associated with $\tilde{\omega}$ which is defined to be the quadratic form $-\omega$ of signature (n, m):

$$\tilde{\omega}(x) = x_{m+1}^2 + \ldots + x_{m+n}^2 - (x_1^2 + \ldots + x_m^2).$$

We now can apply, mutatis mutandis, the arguments in § 1.3. If we keep in mind that $-\Box = \tilde{\Box}$, the differential operator associated with the quadratic form $\tilde{\omega} = -\omega$, we obtain:

$$(2.5)\qquad s \mapsto R_s^{\pm} \text{ is a complex-analytic mapping } \mathbf{C} \to S'(C^{\pm})^G$$

$$(2.6)\qquad \Box^k R_{s+2k}^+ = R_s^+, \quad \Box^k R_{s+2k}^- = (-1)^k R_s^- \quad (s \in \mathbf{C}, k \in \mathbf{Z}_{\geq 0})$$

$$(2.7)\qquad \mathscr{E} R_s^{\pm} = (s-d) R_s^{\pm} \quad (s \in \mathbf{C}).$$

Further we obtain that there exist constants $c_{\pm} \in \mathbf{R}$ such that

$$(2.8)\qquad R_0^{\pm} = c_{\pm}\delta.$$

In order to determine c_+, we compute

$$\langle R_s^+, f\rangle, \quad \text{for } f(x) = e^{-(x_1^2 + \ldots + x_m^2)}.$$

Although this function f is not in $S(\mathbf{R}^{m,n})$, the computation below may be justified exactly as the case of $x \mapsto e^{-x_0}$ in § 1.2. Using bipolar coordinates $(x_1, \ldots, x_m) = u\phi$, $(x_{m+1}, \ldots, x_{m+n}) = v\psi$, it is easy to find that

$$(2.9)\qquad \langle R_s^+, f\rangle = \frac{1}{2^{s-1}} \frac{\pi}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{s}{2} + 1 - \frac{m}{2}\right)} \quad (s \in \mathbf{C}).$$

In other words,

$$(2.10)\qquad R_0^+ = \begin{cases} 2(-1)^{\frac{m-1}{2}}\delta, & \text{for } m \text{ odd,} \\ 0, & \text{for } m \text{ even.} \end{cases}$$

In view of (2.4) we obtain from (2.10)

(2.11)
$$R_0^- = \begin{cases} 2(-1)^{\frac{n-1}{2}}\delta, & \text{for } n \text{ odd}, \\ 0, & \text{for } n \text{ even}. \end{cases}$$

2.3. *Fundamental solutions.* If m or n is odd the results above immediately lead to the same fundamental solution for $\square$ as given in [Rh, pp. 365-6]. We notice that, in view of (2.3)

(3.1)
$$R_d^\pm = \frac{1}{\pi^{\frac{d-2}{2}} 2^{d-1} \Gamma\left(\frac{d}{2}\right)} 1_{C^\pm}.$$

(3.2)
$$R_{d-1}^\pm = \frac{1}{\pi^{\frac{d-1}{2}} 2^{d-2} \Gamma\left(\frac{d-1}{2}\right)} |\omega|^{-\frac{1}{2}} 1_{C^\pm}.$$

We distinguish cases.

m odd and *n odd.* Then $d = m + n$ is even, thus $\square^{\frac{d}{2}} R_d^+ = R_0^+$. In view of (3.1) and (2.10) this implies

(3.3)
$$\square^{\frac{d}{2}} \frac{(-1)^{\frac{m-1}{2}}}{\pi^{\frac{d-2}{2}} 2^d \Gamma\left(\frac{d}{2}\right)} 1_{C^+} = \delta.$$

Notice that the fundamental solutions for $\square^k$, for $1 \leqq k \leqq \dfrac{d-2}{2}$, described here have support in the cone $\partial C^+ = X_0 = \{x \in \mathrm{R}^{m,n} \mid \omega(x) = 0\}$.

m odd and *n even.* Then d is odd, and $\left[\dfrac{d}{2}\right] = \dfrac{d-1}{2}$, thus $\square^{[\frac{d}{2}]} R_{d-1}^+ = R_0^+$. In view of (3.2) and (2.10) this implies

(3.4)
$$\square^{[\frac{d}{2}]} \frac{(-1)^{\frac{m-1}{2}}}{\pi^{\frac{d-1}{2}} 2^{d-1} \Gamma\left(\frac{d-1}{2}\right)} \omega^{-\frac{1}{2}} 1_{C^+} = \delta.$$

The fundamental solutions for $\square^k$, for $1 \leq k \leq \left[\dfrac{d}{2}\right] - 1$, described here have support in the solid cone C^+.

m even and *n odd.* Then d is odd, thus (cf. (2.6)) we get $\square^{[\frac{d}{2}]} R_{d-1}^- = (-1)^{\frac{d-1}{2}} R_0^-$. In view of (3.2), (2.11) and $\dfrac{d-1}{2} + \dfrac{n-1}{2} \equiv \dfrac{m}{2}$ mod 2, we obtain

(3.5) $$\square^{\left[\frac{d}{2}\right]} \frac{(-1)^{\frac{m}{2}}}{\pi^{\frac{d-1}{2}} 2^{d-1} \Gamma\left(\frac{d-1}{2}\right)} |\omega|^{-\frac{1}{2}} 1_{C^-} = \delta.$$

These fundamental solutions for $\square^k$, for $1 \leqq k \leqq \left[\frac{d}{2}\right] - 1$, have support in the solid cone C^-.

m even and *n even*. The method employed above fails in this case, since $R_0^+ = R_0^- = 0$. Therefore we need another approach, which is developped in the next section.

2.4. *Infinitesimal generators* $\rho^\pm$. We assume that m and n are even. Then $(\pm\square)^{\frac{d}{2}} R_{s+d}^\pm = R_s^\pm$ and therefore we get

(4.1) $$\rho^\pm := \frac{d}{ds}\bigg|_{s=0} R_s^\pm = (\pm\square)^{\frac{d}{2}} \frac{d}{ds}\bigg|_{s=0} R_{s+d}^\pm \in S'(C^\pm)^G.$$

We shall also refer to the $\rho^\pm$ as the generalized Riesz distributions. Here (cf. (2.3))

$$R_{s+d}^\pm = \frac{1}{\pi^{\frac{d-2}{2}} 2^{s+d-1} \Gamma\left(\frac{s+d}{2}\right) \Gamma\left(\frac{s+2}{2}\right)} 1_{C^\pm} e^{\frac{s}{2}\log|\omega|}.$$

If we introduce the function Ψ of Gauss, and the constant C, by $\Psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$, and $C = -(\log 2 + \frac{1}{2}\Psi\left(\frac{d}{2}\right) + \frac{1}{2}\Psi(1))$ resp., then we obtain

(4.2) $$\frac{d}{ds}\bigg|_{s=0} R_{s+d}^\pm = CR_d^\pm + \frac{1}{\pi^{\frac{d-2}{2}} 2^d \Gamma\left(\frac{d}{2}\right)} 1_{C^\pm} \log|\omega|.$$

Since $(\pm\square)^{\frac{d}{2}} R_d^\pm = R_0^\pm = 0$, we get from (4.1) and (4.2)

$$\rho^\pm = (\pm\square)^{\frac{d}{2}} \frac{1}{\pi^{\frac{d-2}{2}} 2^d \Gamma\left(\frac{d}{2}\right)} 1_{C^\pm} \log|\omega| .$$

Thus

(4.3) $$\rho^+ + (-1)^{\frac{d}{2}} \rho^- = \square^{\frac{d}{2}} \frac{1}{\pi^{\frac{d-2}{2}} 2^d \Gamma\left(\frac{d}{2}\right)} \log|\omega| .$$

As in Formula (6.3) in §1.6 we have, on $\mathsf{R}^d \setminus \{0\}$,

$$\rho^{\pm} = \pi^{-\frac{d-2}{2}} (\pm\omega)^* (\delta_0^{(\frac{d-2}{2})}).$$

If we write $g^{\vee}(t) = g(-t)$, then we have

$$(-\omega)_* f(t) = \int_{\{-\omega(x)=t\}} f(x)\, d\mu(x) = (\omega_* f)^{\vee}(t).$$

Hence, on $\mathsf{R}^d \setminus \{0\}$

$$\rho^- = \pi^{-\frac{d-2}{2}} \omega^*((\delta_0^{(\frac{d-2}{2})})^{\vee}) = \pi^{-\frac{d-2}{2}} (-1)^{\frac{d-2}{2}} \omega^*(\delta_0^{(\frac{d-2}{2})}) = (-1)^{\frac{d}{2}-1} \rho^+.$$

Thus on $\mathsf{R}^d \setminus \{0\}$ we have $\rho^+ + (-1)^{\frac{d}{2}} \rho^- = \rho^+ - \rho^+ = 0$; in other words, $\rho^+ + (-1)^{\frac{d}{2}}\rho^-$ is supported at 0. Applying $\left.\dfrac{d}{ds}\right|_{s=0}$ to (2.7) we obtain that $\mathscr{E}\rho^{\pm} = R_0^{\pm} - d\rho^{\pm} = -d\rho^{\pm}$. That is, the $\rho^{\pm}$ are distributions homogeneous of degree $-d$, and supported at 0. But therefore they are scalar multiples of δ. From (A.5) in the Appendix below we obtain:

$$\rho^+ + (-1)^{\frac{d}{2}} \rho^- = (-1)^{\frac{m}{2}-1} \pi\delta.$$

Thus we get from (4.3) the following (cf. [Rh, pp. 365-6])

$$\text{(4.4)} \qquad \square^{\frac{d}{2}} \frac{(-1)^{\frac{m}{2}-1}}{\pi^{\frac{d}{2}} 2^d \Gamma\left(\frac{d}{2}\right)} \log|\omega| = \delta.$$

Notice that the fundamental solutions for $\square^k$, for $1 \leqq k \leqq \dfrac{d-2}{2}$, described here have support in all of R^d.

2.5. *The modules* $\bar{J}(\mathsf{C})$. In this section these are defined as the $\mathfrak{a}$-modules of G-invariant distributions on $\mathsf{R}^{m,n}$ with support on the cone X_0. It can be shown that (cf. [KV, Section 2] for the definition of the modules of the type W, V and M infra):

$$\bar{J}(\mathsf{C}) \approx \begin{cases} W\left(\frac{d}{2} - 2 : \frac{1}{2}\frac{d-2}{2}\right), & \text{for } m \text{ and } n \text{ both odd;} \\ V\left(\frac{d}{2} - 2\right) \oplus V\left(-\frac{d}{2}\right), & \text{for } \begin{cases} m \text{ odd and } n \text{ even;} \\ m \text{ even and } n \text{ odd;} \end{cases} \\ M\left(\frac{d}{2} - 2\right) \oplus V\left(-\frac{d}{2}\right), & \text{for } m \text{ and } n \text{ both even.} \end{cases}$$

It is remarkable that, in all cases, $\bar{J}(\mathrm{C})$ is spanned by Riesz distributions with a singular value of the parameter and by generalized Riesz distributions. More precisely, it is spanned by

$$\begin{array}{ll} (R_s^+) \text{ and } \rho^+, & \text{for } m \text{ and } n \text{ both odd;} \\ (R_s^+) \text{ and } \rho^-, & \text{for } m \text{ odd and } n \text{ even;} \\ (R_s^-) \text{ and } \rho^+, & \text{for } m \text{ even and } n \text{ odd;} \\ (R_s^\pm) \text{ and } \rho^\pm, & \text{for } m \text{ and } n \text{ both even,} \end{array}$$

where s runs through the singular values.

We also remark that the analogue of Proposition 1.4 is true in this case.

2.6. *Eigenvalue problem.* Notice that

$$T^\pm(s:\lambda) = \sum_{j=0}^{\infty} \lambda^j R^\pm_{s+2+2j} \in D'(C^\pm)^G$$

defines an entire function of the variables s and $\lambda \in \mathrm{C}$. The results above now immediately lead to a G-invariant fundamental solution to the eigenvalue problem for the ultra-hyperbolic operator, that is, an element $U_\lambda \in D'(\mathrm{R}^{m,n})^G$ satisfying:

$$(\square - \lambda)U_\lambda = \delta \quad (\lambda \in \mathrm{C}).$$

Indeed, U_λ can be taken to be:

$$\begin{array}{ll} \tfrac{1}{2}(-1)^{\frac{m-1}{2}} T^+(0:\lambda) \text{ or } \tfrac{1}{2}(-1)^{\frac{n-1}{2}} T^-(0:\lambda), & m, n \text{ odd;} \\ \tfrac{1}{2}(-1)^{\frac{n-1}{2}} T^-(0:\lambda), & m \text{ even, } n \text{ odd;} \\ \tfrac{1}{2}(-1)^{\frac{m-1}{2}} T^+(0:\lambda), & m \text{ odd, } n \text{ even;} \\ \left.\dfrac{d}{ds}\right|_{s=0} \dfrac{1}{\pi}(-1)^{\frac{m}{2}-1}(T^+(s:\lambda) + (-1)^{\frac{d}{2}} T^-(s:\lambda)), & m, n \text{ even.} \end{array}$$

2.7. REMARK. Finally, the elliptic case $m = 0$ can be treated likewise. There is only one Riesz family (R_s) and $R_0 = \delta$.

APPENDIX: SOME COMPUTATIONS. Here we assume both m and n to be even. For $x \in \mathrm{R}^{m,n}$, we set $\|x\|^2 = x_1^2 + \dots + x_m^2 + x_{m+1}^2 + \dots + x_{m+n}^2$, and, for $s \in \mathrm{C}$ and $\tau > 0$,

$$\text{(A.1)} \qquad I^+(s:\tau) := \langle R_s^+, e^{-\tau^2\|\cdot\|^2}\rangle = \frac{1}{H_d(s)} \int_{C^+} e^{-\tau^2\|x\|^2} \omega(x)^{\frac{s-d}{2}} dx.$$

We shall prove that

$$\text{(A.2)} \qquad I^+(s:\tau) = \pi 2^{2-s-\frac{d}{2}} \tau^{-s} (-1)^{\frac{m}{2}-1} \times$$

$$\times \sum_{k=0}^{\frac{m}{2}-1} \frac{\left(-\frac{s}{2}+\frac{d}{2}-1\right)\left(-\frac{s}{2}+\frac{d}{2}-2\right)\dots\left(-\frac{s}{2}+\frac{d}{2}-k\right)\left(\frac{d}{2}-k-1\right)!}{k!\left(\frac{d}{2}-1-k\right)!} \times$$

$$\times \frac{\left(-\frac{s}{2}+\frac{m}{2}-1-k\right)\dots\left(-\frac{s}{2}+2\right)\left(-\frac{s}{2}+1\right)\frac{s}{2}}{\left(\frac{m}{2}-1-k\right)!\,\Gamma\left(1+\frac{s}{2}\right)}.$$

As in §2.2 we begin by introducing bipolar coordinates, and we get, by means of the substitution $u = \dfrac{z}{\tau^2(1+t)}$,

$$I^+(s:\tau) = \frac{1}{4H_d(s)}|S^{m-1}||S^{n-1}|\,\tau^{-s}\,\Gamma\left(\frac{s}{2}\right)F(s),$$

$$\text{where} \quad F(s) = \int_0^1 \left(\frac{1-t}{1+t}\right)^{\frac{s}{2}-\frac{d}{2}} \frac{t^{\frac{n}{2}-1}}{(1+t)^{\frac{d}{2}}}\,dt.$$

The change of variables $\dfrac{1-t}{1+t} = x$ now gives

$$F(s) = 2^{1-\frac{d}{2}} \int_0^1 x^{\frac{s}{2}-\frac{d}{2}}(1-x)^{\frac{n}{2}-1}(1+x)^{\frac{m}{2}-1}\,dx.$$

We calculate this in terms of the hypergeometric integral

$$F\left(1-\frac{m}{2}, 1-\frac{d}{2}+\frac{s}{2}, 1-\frac{m}{2}+\frac{s}{2}: -1\right),$$

or by expanding $(1+x)^{\frac{m}{2}-1}$, and we obtain

$$F(s) = 2^{1-\frac{d}{2}}\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)\sum_{k=0}^{\frac{m}{2}-1} \frac{\Gamma\left(\frac{s}{2}-\frac{d}{2}+1+k\right)}{\Gamma\left(\frac{m}{2}-k\right)k!\,\Gamma\left(\frac{s}{2}-\frac{m}{2}+1+k\right)};$$

and therefore we find that $I^+(s:\tau)$ is equal to

$$\pi 2^{2-s-\frac{d}{2}}\tau^{-s}\sum_{k=0}^{\frac{m}{2}-1} \frac{\Gamma\left(\frac{s}{2}-\frac{d}{2}+1+k\right)}{\Gamma\left(\frac{s}{2}-\frac{d}{2}+1\right)k!}\,\frac{1}{\Gamma\left(\frac{m}{2}-k\right)\Gamma\left(1+\frac{s}{2}-\frac{m}{2}+k\right)}.$$

Formula (A.2) now follows easily.

Notice that $\frac{s}{2}$ occurs as a factor just once in every summand in (A.2). Therefore $\left.\frac{d}{ds}\right|_{s=0} I^+(s:\tau)$ is easy to compute:

$$\text{(A.3)} \qquad \left\langle \left.\frac{d}{ds}\right|_{s=0} R_s^+, e^{-\tau^2\|\cdot\|^2} \right\rangle = \pi\, 2^{1-\frac{d}{2}}(-1)^{\frac{m}{2}-1} \sum_{k=0}^{\frac{m}{2}-1} \binom{\frac{d}{2}-1}{k}.$$

If we apply (A.3) to $R_s^- = \tilde{R}_s^+$ (cf. (2.4)), we get at once

$$\text{(A.4)} \qquad \left\langle \left.\frac{d}{ds}\right|_{s=0} R_s^-, e^{-\tau^2\|\cdot\|^2} \right\rangle = (-1)^{\frac{d}{2}}\pi\, 2^{1-\frac{d}{2}}(-1)^{\frac{m}{2}-1} \sum_{k=\frac{m}{2}}^{\frac{d}{2}-1} \binom{\frac{d}{2}-1}{k}.$$

From (A.3) and (A.4) now follows

$$\text{(A.5)} \qquad \langle \rho^+ + (-1)^{\frac{d}{2}}\rho^-, e^{-\tau^2\|\cdot\|^2} \rangle = (-1)^{\frac{m}{2}-1}\pi.$$

REFERENCES

[D] J. J. Duistermaat, *M. Riesz's families of operators*, to appear in Nieuw Arch. Wisk.

[F] Y. Fourès-Bruhat, *Solution élémentaire d'équations ultra-hyperboliques*, J. Math. Pures Appl. (9) 35 (1956), 277–288.

[GL] L. Gårding, J. L. Lions, *Functional analysis*, Supplemento al Nuovo Cimento 14 (10) (1959), 9–66.

[GS] I. M. Gelfand, G. E. Shilov, *Generalized Functions*, *Volume* 1, Academic Press Inc., New York London, 1964.

[H] L. Hörmander, *The Analysis of Linear Partial Differential Operators* I, Springer-Verlag, Berlin Heidelberg New York Tokyo, 1983.

[KV] J. A. C. Kolk, V. S. Varadarajan, *Lorentz invariant distributions supported on the forward light cone*, to appear in Compositio Math.

[M] P-D. Methée, *Sur les distributions invariantes dans le groupe des rotations de Lorentz*, Comment. Math. Helv. 28 (1954), 225–269.

[O] J. Orloff, in (J. Carmona, P. Delorme and M. Vergne, eds.), *Non-commutative Harmonic Analysis and Lie Groups*, Proceedings, Marseille-Luminy 1985, Lecture Notes in Math. 1243, Springer-Verlag, Berlin Heidelberg New York London Paris Tokyo, 1987, 198–239.

[Rh] G. de Rham, *Solution élémentaire d'opérateurs différentiels du second ordre*, Ann. Inst. Fourier (Grenoble) 8 (1958), 337–366.

[Ri] M. Riesz, *L'intégrale de Riemann-Liouville et le problème de Cauchy*, Acta Math. 81 (1949), 1–223. = Collected Papers, Springer-Verlag, Berlin Heidelberg New York London Paris Tokyo, 1988, pp. 571–793.

[T] A. Tengstrand, *Distributions invariant under an orthogonal group of arbitrary signature*, Math. Scand. 8 (1960), 201–218.

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