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VON NEUMANN INEQUALITY FOR $(B(\mathscr{H})^n)_1$

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0. Introduction.

Let $H^2(\mathsf{D})$ be the Hardy space of analytic functions on the unit disk D, i.e.,

$$H^2(\mathsf{D}) = \left\{ u(\lambda) = \sum_{k=0}^{\infty} \lambda^k a_k;\ a_k \in \mathsf{C},\ \|u\|^2_{H^2(\mathsf{D})} = \sum_{k=0}^{\infty} |a_k|^2 < \infty \right\}.$$

J. von Neumann's well-known inequality [11] on Hilbert space operators asserts that if T is a contraction on a complex Hilbert space $\mathscr{H}$ (i.e., $\|T\| \leqq 1$) and p is an analytic polynomial in $H^2(\mathsf{D})$, then the operator $p(T)$ satisfies the inequality

$$\|p(T)\| \leqq \sup_{|\lambda| \leqq 1} |p(\lambda)| = \sup_{q \in (\mathscr{P}_+)_1} \|pq\|_{H^2(\mathbb{D})}, \tag{0.1}$$

where $(\mathscr{P}_+)_1$ stands for the unit ball of $\mathscr{P}_+ \subset H^2(\mathsf{D})$ and $\mathscr{P}_+$ denote the set of all analytic polynomials in $H^2(\mathsf{D})$.

T. Ando [1] generalized the inequality (0.1) for two commuting contractions. In [10] N. Th. Varopoulos show that this inequality does not generalize to an arbitrary number $n \geqq 3$ of commuting contraction. Moreover, it is shown that, in general, for $n \geqq 3$ and some commuting operators $T_1, \ldots, T_n \in B(\mathscr{H})$ such that

$$\sum_{i=1}^{n} \|T_i\|^2 \leqq 1,$$

the inequality

$$\|p(T_1, \ldots, T_n)\| \leqq K \sup \left\{ |p(\lambda_1, \ldots, \lambda_n)| : \sum_{i=1}^{n} |\lambda_i|^2 \leqq 1 \right\},$$

where $K > 0$ and $p(\lambda_1, \ldots, \lambda_n)$ is any complex polynomial of n variables, is not true.

Concerning the von Neumann inequality see also [9].

Now let us present the results of this paper. For a natural number n let $B(\mathscr{H})^n$

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This paper is part of my Ph. D. thesis, written under the guidance of Prof. Carl Pearcy, at the Texas A & M University denote the set of n-tuples $T = (T_1, \ldots, T_n)$ of elements from $B(\mathscr{H})$ (i.e., the algebra of all bounded operators on the Hilbert space $\mathscr{H}$). We define a Banach space norm on $B(\mathscr{H})^n$ asking that the injective map

$$\pi : B(\mathscr{H})^n \to M_n(B(\mathscr{H}))$$

given by

$$\pi(T)_{1j} = T_j \text{ for } 1 \leqq j \leqq n \text{ and } \pi(T)_{ij} = 0 \text{ for } i > 1,$$

be an isometry. The norm gives $B(\mathscr{H})^n$ the product topology, and for each $T = (T_1, \ldots, T_n) \in B(\mathscr{H})^n$ we have

$$\|T\| = \|\pi(T)\| = \|\sum_{i=1}^{n} T_i T_i^*\|^{\frac{1}{2}}.$$

Let $(B(\mathscr{H})^n)_1$ denote the unit ball of $B(\mathscr{H})^n$, i.e.,

$$(B(\mathscr{H})^n)_1 = \left\{(T_1, \ldots, T_n) \in B(\mathscr{H})^n : \sum_{i=1}^{n} T_i T_i^* \leqq I_{\mathscr{H}}\right\}$$

The main aim of this paper is to extend the von Neumann inequality to $(B(\mathscr{H})^n)_1$, for $n \geqq 2$.

To be more precise, let us consider the full Fock-space [4]

(0.2) $$\mathscr{F}(H_n) = \mathbf{C}\, I \oplus \bigoplus_{m \geqq 1} H_n^{\otimes m},$$

where H_n is an n-dimensional complex Hilbert space with orthonormal basis $e_1, e_2, \ldots, e_n$. We shall denote by $\mathscr{P}$ the set of all $p \in \mathscr{F}(H_n)$ of the form

(0.3) $$p = a_0 + \sum_{\substack{1 \leqq i_1, \ldots, i_k \leqq n \\ 1 \leqq k \leqq m}} a_{i_1 \ldots i_k} e_{i_1} \otimes \ldots \otimes e_{i_k}, \quad m \in \mathbf{N},$$

where $a_0, a_{i_1 \ldots i_k} \in \mathbf{C}$ and the sum contains only a finite number of summands.

The set $\mathscr{P}$ may be viewed as the algebra of the polynomials in n noncommuting indeterminates, with $p \otimes q$, $p, q \in \mathscr{P}$ as multiplication.

Let $p(T_1, \ldots, T_n)$ stand for the operator acting on $\mathscr{H}$, given by

(0.4) $$p(T_1, \ldots, T_n) = a_0 I_{\mathscr{H}} + \sum a_{i_1 \ldots i_k} T_{i_1} \ldots T_{i_k}.$$

Our von Neumann inequality for $(B(\mathscr{H})^n)_1$ asserts that if $(T_1, \ldots, T_n) \in (B(\mathscr{H})^n)_1$ and $p \in \mathscr{P}$, then

(0.5) $$\|p(T_1, \ldots, T_n)\| \leqq \sup_{q \in (\mathscr{P})_1} \|p \otimes q\|_{\mathscr{F}(H_n)},$$

where

$$(\mathscr{P})_1 = \{p \in \mathscr{P} : \|p\|_{\mathscr{F}(H_n)} \leqq 1\}.$$

If $n = 1$, it is easy to see that, we find again (0.1).

Let us remark that, if $n = 2$ and T_1, T_2 are commuting contractions such that $[T_1, T_2]$ is a contraction, the von Neumann inequality (0.5) seems to be sharper than Ando's inequality [1].

For instance, if

$$p(T_1, T_2) = T_1 + T_2,$$

then Ando's inequality shows that

$$\|p(T_1, T_2)\| \leqq 2,$$

while, von Neumann inequality (0.5) (see Corollary 2.2) gives

$$\|p(T_1, T_2)\| \leqq \sqrt{2}.$$

In order to extend (0.5) to a Banach algebra containing $\mathscr{P}$, we need to introduce some Banach algebras $(\mathscr{F}^\infty, \| \ \|_\infty)$ and $(\mathscr{A}, \| \ \|_\infty)$, which may be viewed as a noncommutative analogue of the Hardy space H^∞ and the disk algebra, respectively.

We shall see that if $(T_1, \dots, T_n) \in (B(\mathscr{H})^n)_1$, then the mapping

$$\Psi: \mathscr{P} \to B(\mathscr{H}); \quad \Psi(p) = p(T_1, \dots, T_n)$$

extends to a contractive homomorphism from the noncommutative "disc algebra" $\mathscr{A}$ to $B(\mathscr{H})$.

Also, it is shown that, for a class of elements $(T_1, \dots, T_n) \in (B(\mathscr{H})^n)_1$, there is a functional calculus defined by the mapping

$$f \mapsto f(T_1, \dots, T_n)$$

from the algebra $\mathscr{F}^\infty$ into $B(\mathscr{H})$.

The main tools for proof are some results from dilation theory for the elements of $(B(\mathscr{H})^n)_1$ (see [2, 5, 6, 7]), the Wold decomposition for n-tuple $(V_1, \dots, V_n) \in (B(\mathscr{H})^n)_1$ of isometries with orthogonal final spaces [5, 6, 8], and some facts concerning the Cuntz-algebra $\mathscr{O}_n$ [3].

1. Notation and preliminaries.

Throughout this paper Λ stands for the set $\{1, 2, \dots, n\}$, $n \geqq 2$. For every $k \in \mathbf{N}^* = \{1, 2, \dots\}$, let $F(k, \Lambda)$ be the set of all functions from the set $\{1, 2, \dots, k\}$ to Λ and

$$\mathscr{F} = \bigcup_{k=0}^{\infty} F(k, \Lambda), \text{ where } F(0, \Lambda) \text{ stands for the set } \{0\}. \tag{1.1}$$

A sequence $\mathscr{S} = \{S_\lambda\}_{\lambda \in \Lambda}$ of unilateral shifts on a Hilbert space $\mathscr{H}$ with orthog-

onal final spaces is called a Λ-orthogonal shift if the operator matrix $[S_1, S_2, \ldots]$ is nonunitary, i.e., $\mathscr{L} := \mathscr{H} \ominus (\bigoplus_{\lambda \in \Lambda} S_\lambda \mathscr{H}) \neq \{0\}$.

We need also the following definitions. A subspace $\mathscr{E} \subset \mathscr{H}$ is called cyclic for a sequence $\{A_\lambda\}_{\lambda \in \Lambda}$ of operators on $\mathscr{H}$ if

$$\bigvee_{f \in \mathscr{F}} A_f \mathscr{E} = \mathscr{H},$$

where A_f stands for the product $A_{f(1)} \ldots A_{f(k)}$ if $k \geqq 1$, $f \in F(k, \Lambda)$, and $A_0 := I_{\mathscr{H}}$.

We define the multiplicity of $\{A_\lambda\}_{\lambda \in \Lambda}$ to be the minimum dimension of a cyclic subspace for $\{A_\lambda\}_{\lambda \in \Lambda}$. If $\{B_\lambda\}_{\lambda \in \Lambda}$ is another sequence of operators on a Hilbert space $\mathscr{K}$ and if there exists a unitary operator U mapping $\mathscr{H}$ onto $\mathscr{K}$ such that

$$A_\lambda = U^{-1} B_\lambda U \quad \text{for any } \lambda \in \Lambda,$$

then, we say that $\{A_\lambda\}_{\lambda \in \Lambda}$ is unitarily equivalent to $\{B_\lambda\}_{\lambda \in \Lambda}$.

Let us recall from [6, 7, 8] some results concerning the Λ-orthogonal shifts.

If $\{S_\lambda\}_{\lambda \in \Lambda}$ is a Λ-orthogonal shift on $\mathscr{H}$ then,

$$\mathscr{L} = \bigcap_{\lambda \in \Lambda} \operatorname{Ker} S_\lambda^* \quad \text{and} \quad \mathscr{H} = \bigoplus_{f \in \mathscr{F}} S_f \mathscr{L}$$

Each $h \in \mathscr{H}$ has a unique representation

$$h = \sum_{f \in \mathscr{F}} S_f l_f, \quad l_f \in \mathscr{L},\ f \in \mathscr{F}.$$

In this case $\|h\|^2 = \sum_{f \in \mathscr{F}} \|l_f\|^2$ and $l_f = P_0 S_f^* h$, $f \in \mathscr{F}$,

where

$$P_0 = I_{\mathscr{H}} - \sum_{\lambda \in \Lambda} S_\lambda S_\lambda^*$$

is the projection of $\mathscr{H}$ on $\mathscr{L}$.

Now one can easily prove the following theorem. We omit the proof.

THEOREM 1.1. *If $\mathscr{S} = \{S_\lambda\}_{\lambda \in \Lambda}$ is a Λ-orthogonal shift on $\mathscr{H}$, then $\mathscr{L}$ is cyclic for $\mathscr{S}$ and* $\dim \mathscr{L} \leqq \dim \mathscr{E}$ *for every cyclic subspace $\mathscr{E}$ for $\mathscr{S}$.*

As a corollary, we obtain that the multiplicity of $\mathscr{S}$ is equal to $\dim \mathscr{L}$.

THEOREM 1.2. *Two Λ-orthogonal shifts are unitarily equivalent if and only if they have the same multiplicity.*

PROOF. If $\mathscr{S} = \{S_\lambda\}_{\lambda \in \Lambda} \subset B(\mathscr{H})$ and $\mathscr{S}' = \{S'_\lambda\}_{\lambda \in \Lambda} \subset B(\mathscr{H}')$ are two Λ-orthogonal shifts with the same multiplicity, then $\mathscr{L}$ and $\mathscr{L}'$ have the same dimension. Hence, there is an isometry W which maps $\mathscr{L}$ onto $\mathscr{L}'$. For any $h \in \mathscr{H}$ define

$$Uh = \sum_{f\in\mathcal{F}} S'_f W l_f \quad \text{for } h = \sum_{f\in\mathcal{F}} S_f l_f.$$

Then U is a unitary operator from $\mathcal{H}$ onto $\mathcal{H}'$ and

$$S'_\lambda U = U S_\lambda \quad \text{for any } \lambda \in \Lambda.$$

Thus, $\mathcal{S}$ and $\mathcal{S}'$ are unitarily equivalent.

The converse implication is obvious.

Let $\mathcal{S} = \{S_\lambda\}_{\lambda\in\Lambda}$ be a Λ-orthogonal shift on $\mathcal{H}$ with the multiplicity α, i.e., $\dim \mathcal{L} = \alpha$. If $\{l_i\}_{i\in I}$ is an orthonormal basis of $\mathcal{L}$, then the subspaces

$$\mathcal{M}_i = \bigoplus_{f\in\mathcal{F}} S_f(\mathbf{C} l_i),\ i\in I$$

are orthogonal and reduce each $S_\lambda(\lambda\in\Lambda)$. Hence, it follows that

$$S_\lambda = \bigoplus_{i\in I} S_\lambda|_{\mathcal{M}_i} \quad \text{for any } \lambda\in\Lambda,$$

and for any $i\in I$, $\mathcal{S}_i := \{S_\lambda|_{\mathcal{M}_i}\}_{\lambda\in\Lambda}$ is a Λ-orthogonal shift with the multiplicity 1.

Therefore, $\mathcal{S}$ may be viewed as a direct sum of α copies of a Λ-orthogonal shift of the multiplicity 1.

Let us consider a model Λ-orthogonal shift with multiplicity 1, acting on the full Fock-space $\mathcal{F}(H_n)$, given by (0.2). For each $\lambda\in\Lambda$ we define the isometry S_λ by

$$S_\lambda h = e_\lambda \otimes h \quad \text{for } h\in\mathcal{F}(H_n) \tag{1.2}$$

It is easy to see that $\mathcal{S} = \{S_\lambda\}_{\lambda\in\Lambda}$ is a Λ-orthogonal shift with multiplicity one.

This model will play an important role in our investigation.

Now let us recall the Wold decomposition theorem for sequences of isometries [6].

Let $\mathcal{V} = \{V_\lambda\}_{\lambda\in\Lambda}$ be a sequence of isometries on a Hilbert space $\mathcal{K}$, with orthogonal final spaces.

Then $\mathcal{K}$ decomposes into an orthogonal sum $\mathcal{K} = \mathcal{K}_u \oplus \mathcal{K}_s$ such that $\mathcal{K}_u$ and $\mathcal{K}_s$ reduce each operator $V_\lambda(\lambda\in\Lambda)$ and we have

$(I_{\mathcal{K}} - \sum_{\lambda\in\Lambda} V_\lambda V_\lambda^*)|_{\mathcal{K}_u} = 0$ and $\{V_\lambda|_{\mathcal{K}_s}\}$ is a Λ-orthogonal shift acting on $\mathcal{K}_s$.

This decomposition is uniquely determined; indeed we have:

$$\mathcal{K}_u = \bigcap_{k=0}^{\infty} \Big(\bigoplus_{f\in F(k,\Lambda)} V_f \mathcal{K}\Big) \text{ and } \mathcal{K}_s = \bigoplus_{f\in\mathcal{F}} V_f \mathcal{L},$$

where $\mathcal{L} = \mathcal{K} \ominus (\bigoplus_{\lambda\in\Lambda} V_\lambda \mathcal{K})$.

We recall from [6] that for any sequences $\mathcal{T} = \{T_\lambda\}_{\lambda\in\Lambda}$ of operators on a Hilbert space $\mathscr{H}$ such that $\sum_{\lambda\in\Lambda} T_\lambda T_\lambda^* \leqq I_{\mathscr{H}}$, there exists a minimal isometric dilation $\mathscr{V} = \{V_\lambda\}_{\lambda\in\Lambda}$ on a Hilbert space $\mathscr{K} \supset \mathscr{H}$, which is uniquely determined up to an isomorphism, i.e., the following conditions hold:

(i) $V_\lambda^* V_\lambda = I_{\mathscr{K}}$ for any $\lambda\in\Lambda$,

(ii) $\sum_{\lambda\in\Lambda} V_\lambda V_\lambda^* \leqq I_{\mathscr{K}}$,

(iii) $V_\lambda^* \mathscr{H} \subset \mathscr{H}$ and $V_\lambda^*|_{\mathscr{H}} = T_\lambda^*$ for any $\lambda\in\Lambda$,

(iv) $\mathscr{K} = \bigvee_{f\in\mathscr{F}} V_f \mathscr{H}$.

2. The Von Neumann inequality.

We begin this section by recalling some facts concerning the Cuntz-algebra $\mathcal{O}_n$ and a certain extension of $\mathcal{O}_n$. In [3] the C*-algebra $\mathcal{O}_n$ $(n \geqq 2)$ was defined as the C*-algebra generated by n isometries $V_1, V_2, \ldots, V_n$ such that $\sum_{i=1}^n V_i V_i^* = I$. It was shown that $\mathcal{O}_n$ does not depend, up to canonical isomorphism, on the choice of the generators $V_1, \ldots, V_n$. In other words, if $\hat{V}_1, \ldots, \hat{V}_n$ is a second family of isometries satisfying $\sum_{i=1}^n \hat{V}_i \hat{V}_i^* = I$, then $C^*(\hat{V}_1, \ldots, V_n)$ is canonically isomorphic to $C^*(V_1, \ldots, V_n)$, i.e., the map $\hat{V}_i \to V_i$ extends to an isomorphism from $C^*(\hat{V}_1, \ldots, \hat{V}_n)$ onto $C^*(V_1, \ldots, V_n)$.

Now, let $V_1, \ldots, V_n$ be isometries on a Hilbert space K such that $\sum_{i=1}^n V_i V_i^* \leqq I_{\mathscr{K}}$ (n finite). Then the projection $P = I_{\mathscr{K}} - \sum_{i=1}^n V_i V_i^*$ generates a closed two-sided ideal $\mathscr{J}$ in $C^*(V_1, \ldots, V_n)$ which is isomorphic to the C*-algebra of all compact operators on an infinite-dimensional separable Hilbert space, and contains P as a minimal projection.

We have the short exact sequence

$$0 \to \mathscr{J} \to C^* V_1, \ldots, V_n) \to \mathcal{O}_n \to 0 \tag{2.1}$$

The main result of this paper is the following

THEOREM 2.1. *If* $(T_1, \ldots, T_n) \in (B(\mathscr{H})^n)_1$, $n \geqq 2$ *and* $p \in \mathscr{P}$, *then*

$$\|p(T_1, \ldots, T_n)\| \leqq \sup_{q\in(\mathscr{P})_1} \|p \otimes q\|_{\mathscr{F}(H_n)} \tag{2.2}$$

PROOF. Since $(T_1, \ldots, T_n) \in (B(\mathscr{H})^n)_1$, there is a minimal isometric dilation $(V_1, \ldots, V_n) \in (B(\mathscr{K})^n)_1$, on a Hilbert space $\mathscr{K} \supset \mathscr{H}$, such that

$$V_i^* V_i = I_{\mathscr{K}} \quad , i = 1, 2, \ldots, n \tag{2.3}$$

$$\sum_{i=1}^{n} V_i V_i^* \leqq I_{\mathscr{K}} \tag{2.4}$$

$$V_{i|\mathscr{H}}^* = T_i^*, \qquad i = 1, \ldots, n. \tag{2.5}$$

By (2.5) it follows that

$$p(T_1, \ldots, T_n) = P_{\mathscr{H}}\, p(V_1, \ldots, V_n)|_{\mathscr{H}}, \quad p \in \mathscr{P},$$

where $P_{\mathscr{H}}$ stands for the orthogonal projection of $\mathscr{K}$ on $\mathscr{H}$. Hence we get

$$\|p(T_1, \ldots, T_n)\| \leqq \|p(V_1, \ldots, V_n)\|. \tag{2.6}$$

According to the Wold decomposition for the sequence $V_1, \ldots, V_n$ of isometries, we infer that the Hilbert space $\mathscr{K}$ decomposes into an orthogonal sum

$$\mathscr{K} = \mathscr{K}_u \oplus \mathscr{K}_s \tag{2.7}$$

such that $\mathscr{K}_u$ and $\mathscr{K}_s$ reduce each operator V_i $(i = 1, 2, \ldots, n)$ and we have

$$\sum_{i=1}^{n} W_i W_i^* = I_{\mathscr{K}_u} \tag{2.8}$$

$$\{U_i\}_{i=1}^{n} \text{ is a } \Lambda\text{-orthogonal shift on } \mathscr{K}_s, \tag{2.9}$$

where, for each $i = 1, 2, \ldots, n$, $V_i = W_i \oplus U_i$ is the decomposition of the operator V_i with respect to (2.7).

Therefore we have

$$p(V_1, \ldots, V_n) = p(W_1, \ldots, W) \oplus p(U_1, \ldots, U_n) \tag{2.10}$$

and

$$\|p(V_1, \ldots, V_n)\| = \max\{\|p(W_1, \ldots, W_n)\|, \|p(U_1, \ldots, U_n)\|\}. \tag{2.11}$$

First, let us consider the case where $\mathscr{K}_s \neq \{0\}$. Since $\sum_{i=1}^{n} U_i U_i^* \leqq I_{\mathscr{K}_s}$ we have the following short exact sequence

$$0 \to \mathscr{J} \to C^*(U_1, \ldots, U_n) \to \mathcal{O}_n \to 0 \tag{2.12}$$

where $\mathscr{J}$ denote the closed two-sided ideal in $C^*(U_1, \ldots, U_n)$ generated by the projection

$$P := I_{\mathscr{K}_s} - \sum_{i=1}^{n} U_i U_i^*.$$

If π_s denote the natural quotient map from $B(\mathscr{K}_s)$ to $B(\mathscr{K}_s)/\mathscr{J}$, from (2.12) we deduce

(2.13) $\|\pi_s(p(U_1,\dots,U_n))\| = \|p(\pi_s(U_1),\dots,\pi_s(U_n))\| = \|p(\sigma_1,\dots,\sigma_n)\|,$

where $\sigma_1,\dots,\sigma_n$ is a system of generators for the Cuntz-algebra $\mathcal{O}_n$.

On the other hand, since (2.8) holds we have also that

(2.14) $$\|p(W_1,\dots,W_n)\| = \|p(\sigma_1,\dots,\sigma_n)\|.$$

By (2.11), (2.13), (2.14) and the fact that

$$\|\pi_s(p(U_1,\dots,U_n))\| \leqq \|p(U_1,\dots,U_n)\|$$

we infer that

(2.15) $$\|p(V_1,\dots,V_n)\| = \|p(U_1,\dots,U_n)\|.$$

According to Section 1, if the multiplicity of the Λ-orthogonal shift $\{U_1,\dots,U_n\}$ is α, then the operator $p(U_1,\dots,U_n)$ is unitarily equivalent to the direct sum of α copies of $p(S_1,\dots,S_n)$, where $\{S_1,\dots,S_n\}$ is the model Λ-orthogonal shift with the multiplicity 1, acting on the full Fock space $\mathcal{F}(H_n)$, given by (1.2).

Therefore (2.15) implies

(2.16) $$\|p(V_1,\dots,V_n)\| = p(S_1,\dots,S_n)\|.$$

The second case is $\mathcal{K}_s = \{0\}$.

Since $\sum_{i=1}^{n} V_i V_i^* = I_{\mathcal{K}}$ we have

(2.17) $$\|p(V_1,\dots,V_n)\| = \|p(\sigma_1,\dots,\sigma_n)\|.$$

Considering $\{S_1,\dots,S_n\}$ be the model Λ-orthogonal shift on $\mathcal{F}(H_n)$ we have, as in the first case, the following short exact sequence

$$0 \to \mathcal{J}_0 \to C^*(S_1,\dots,S_n) \to \mathcal{O}_n \to 0.$$

Here $\mathcal{J}_0$ is the closed two-sided ideal in $C^*(S_1,\dots,S_n)$ generated by P_{CI}, which is the orthogonal projection of $\mathcal{F}(H_n)$ on C1.

Consequently, if π_0 denote the quotient map from $B(\mathcal{F}(H_n))$ onto $B(\mathcal{F}(H_n))/\mathcal{J}_0$, then we have

(2.18) $$\begin{aligned}\|p(\sigma_1,\dots,\sigma_n) &= \|p(\pi_0(S_1),\dots,\pi_0(S_n))\| \\ &= \|\pi_0(p(S_1,\dots,S_n))\| \leqq \|p(S_1,\dots,S_n)\|.\end{aligned}$$

The relations (2.17) and (2.18) imply

(2.19) $$\|p(V_1,\dots,V_n)\| \leqq \|p(S_1,\dots,S_n)\|.$$

Now, taking into account (1.2) it is easy to see that

$$p(S_1,\dots,S_n)h = p \otimes h \qquad \text{for any } h \in \mathcal{F}(H_n).$$

Since $\mathscr{P}$ is dense in $\mathscr{F}(H_n)$, it is clear that

$$\|p(S_1,\ldots,S_n)\| = \sup_{q\in(\mathscr{P})_1} \|p\otimes q\|_{\mathscr{F}(H_n)}. \tag{2.20}$$

From (2.6), (2.16), (2.19) and (2.20) the result follows.
The proof is complete.

COROLLARY 2.2. *If* $(T_1,\ldots,T_n)\in(B(\mathscr{H})^n)_1$, $n\geqq 2$ *and* $p\in\mathscr{P}$, *then*

$$\|p(T_1,\ldots,T_n)\| \leqq \|p(S_1,\ldots,S_n)\| = \sup_{q\in(\mathscr{P})_1} \|p\otimes q\|_{\mathscr{F}(H_n)}$$

where $\mathscr{S} = \{S_1,\ldots,S_n\}$ *is the model* Λ*-orthogonal shift on* $\mathscr{F}(H_n)$.

3. Functional calculus for $(T_1,\ldots,T_n)\in(B(\mathscr{H})^n)_1$.

Throughout this section we keep the definitions from the previous sections. Let us note that any element $g\in\mathscr{F}(H_n)$ can be written as follows

$$g = \sum_{f\in\mathscr{F}} a_f e_f \qquad \text{with } a_f\in\mathbf{C} \text{ and} \tag{3.1}$$

$$\|g\|_2^2 = \sum_{f\in\mathscr{F}} |a_f|^2 < \infty,$$

where e_f stands for $e_{f(1)}\otimes\ldots\otimes e_{f(k)}$ if $f\in F(k,\Lambda)$, $k\geqq 1$, and $e_0 = I$.

We make the natural identification of $e_f\otimes I$ with e_f, for any $f\in\mathscr{F}$. If $p\in\mathscr{P}$, then there is $m\in\mathbf{N}$ such that

$$p = \sum_{f\in\mathscr{F}_m} a_f e_f, \quad \text{where } \mathscr{F}_m = \bigcup_{k=0}^{m} F(k,\Lambda).$$

We omit the proof of the following lemma, which is straightforward.

LEMMA 3.1. (i) *If* $g\in\mathscr{F}(H_n)$ *and* $p\in\mathscr{P}$, *then* $g\otimes p\in\mathscr{F}(H_n)$.
(ii) *If* $g_m\in\mathscr{F}(H_n)$ *such that* $\|g_m\|_2\to 0$ *(as* $m\to\infty$*) then* $\|g_m\otimes p\|_2\to 0$ *(as* $m\to\infty$*), for any* $p\in\mathscr{P}$.

Now let us define $\mathscr{F}^\infty$ as being the set of all $g\in\mathscr{F}(H_n)$ for which

$$\|g\|_\infty := \sup_{p\in(\mathscr{P})_1} \|g\otimes p\|_2 < \infty. \tag{3.2}$$

It is easy to see that, if $f\in\mathscr{F}^\infty$ and $g\in\mathscr{F}(H_n)$, then the multiplication defined by

$$f\otimes g := \lim_{n\to\infty} f\otimes p_n \quad (\text{in } \mathscr{F}(H_n)), \tag{3.3}$$

where $p_n\in\mathscr{P}$ and $\|p_n - g\|_2\to 0$, is well-defined and $f\otimes g\in\mathscr{F}(H_n)$.

THEOREM 3.2. *$(\mathscr{F}^\infty, \|\ \|_\infty)$ is a noncommutative Banach algebra.*

PROOF. That $\mathscr{F}^\infty$ is a linear space and that $\|\ \|_\infty$ is a norm is obvious. Let us suppose that $\{g_n\}_{n=1}^\infty$ is a Cauchy sequence in $(\mathscr{F}^\infty, \|\ \|_\infty)$. Since $\|g_n - g_m\|_2 \leqq \|g_n - g_m\|_\infty$ $(n, m \in \mathbf{N})$, the sequence $\{g_n\}_{n=1}^\infty$ is a Cauchy sequence in $\mathscr{F}(H_n)$, so there exists $g \in \mathscr{F}(H_n)$ such that $\|g_n - g\|_2 \to 0$ (as $n \to \infty$).

If N is chosen so that $n, m \geqq N$ implies $\|g_n - g_m\|_\infty < 1$, then, according to Lemma 3.1 and (3.2), for any $p \in \mathscr{P}$ we have

$$\begin{aligned}\|g \otimes p\|_2 &\leqq \|g \otimes p - g_N \otimes p\|_2 + \|g_N \otimes p\|_2 \\ &\leqq \limsup_{n\to\infty} \|g_n - g_N\|_\infty \|p\|_2 + \|g_N\|_\infty \|p\|_2 \\ &\leqq (1 + \|g_N\|_\infty) \|p\|_2\end{aligned}$$

Thus, $g \in \mathscr{F}^\infty$ and it remains to show that $\lim_{n\to\infty} \|g - g_n\|_\infty = 0$.

Given $\varepsilon > 0$, choose N such that $n, m \geqq N$ implies $\|g_n - g_m\|_\infty < \varepsilon$. Then, for any $p \in \mathscr{P}$ and $n, m \geqq N$, we have

$$\|(g - g_n) \otimes p\|_2 \leqq \|(g - g_m) \otimes p\|_2 + \|(g_m - g_n) \otimes p\|_2 \leqq \|(g - g_m) \otimes p\|_2 + \varepsilon \|p\|_2.$$

Since $\lim_{m\to\infty} \|(g - g_m) \otimes p\|_2 = 0$, we have $\|g - g_n\|_\infty < \varepsilon$. Therefore $(\mathscr{F}^\infty, \|\ \|_\infty)$ is a Banach space.

Now let f, g be in $\mathscr{F}^\infty$. According to (3.3) it follows that $f \otimes g \in \mathscr{F}(H_n)$. On the other hand, if $p_n \in \mathscr{P}$ such that $\|p_n - g\|_2 \to 0$, then, for any $p \in \mathscr{P}$ we have

$$\begin{aligned}\|(f \otimes g) \otimes p\|_2 &= \lim_{n\to\infty} \|f \otimes (p_n \otimes p)\|_2 \\ &\leqq \lim_{n\to\infty} \|f\|_\infty \|p_n \otimes p\|_2 \\ &= \|f\|_\infty \|g \otimes p\|_2 \\ &\leqq \|f\|_\infty \|g\|_\infty \|p\|_2,\end{aligned}$$

Hence, it follows that $f \otimes g \in \mathscr{F}^\infty$ and

$$\|f \otimes g\|_\infty \leqq \|f\|_\infty \|g\|_\infty.$$

The proof is complete.

Now let us denote by $\mathscr{A}$ the closure of the polynomials $\mathscr{P}$ in $(\mathscr{F}^\infty, \|\ \|_\infty)$.

COROLLARY 3.3. *$(\mathscr{A}, \|\ \|_\infty)$ is a noncommutative Banach algebra.*

Let $g \in \mathscr{F}(H_n)$ be given by (3.1) and let $\{S_1, \ldots, S_n\}$ be the model Λ-orthogonal shift on $\mathscr{F}(H_n)$. We denote by $g(S_1, \ldots, S_n)$ the formal sum

$$g(S_1, \ldots, S_n) := \sum_{f \in \mathscr{F}} a_f S_f, \tag{3.4}$$

where S_f stands for $S_{f(1)} \ldots S_{f(k)}$ if $f \in F(k, \Lambda)$ and $S_0 = I_{\mathscr{F}(H_n)}$.

THEOREM 3.4. *Let g be in $\mathscr{F}(H_n)$. Then $g(S_1, \ldots, S_n)$ is strongly convergent in $B(\mathscr{F}(\mathscr{H}_n))$ if and only if g belongs to $\mathscr{F}^\infty$.*

PROOF. Let $g \in \mathscr{F}(H_n)$ be given by (3.1) and

$$g_m = \sum_{f \in \mathscr{F}_m} a_f e_f$$

If $g(S_1, \ldots, S_n)$ is strongly convergent in $B(\mathscr{F}(H_n))_1$ then

$$\|g(S_1, \ldots, S_n)p\|_2 \leqq \|g(S_1, \ldots, S_n)\| \; \|p\|_2 \tag{3.5}$$

for any $p \in \mathscr{P}$.

On the other hand, for any $p \in \mathscr{P}$, we have

$$\begin{aligned} g(S_1, \ldots, S_n)p &= \lim_{m \to \infty} g_m(S_1, \ldots, S_n)p = \\ &= \lim_{m \to \infty} g_m \otimes p = g \otimes p. \end{aligned} \tag{3.6}$$

By (3.5) and (3.6) it follows that $g \in \mathscr{F}^\infty$

The converse implication can be easily deduced.

COROLLARY 3.5. *The mapping*

$$f \mapsto f(S_1, \ldots, S_n),$$

from $\mathscr{F}^\infty$ to $B(\mathscr{F}(H_n))$, is an isometric functional calculus.

Let us remark that, according to Section 1, the above corollary remains true if we replace $\{S_1, \ldots, S_n\}$ by a Λ-orthogonal shift of arbitrary multiplicity.

THEOREM 3.6. *If $(T_1, \ldots, T_n) \in (B(\mathscr{H})^n)_1$ such that*

$$\lim_{k \to \infty} \sum_{f \in F(k, \Lambda)} \|T_f^* h\|^2 = 0 \quad \text{for any } h \in \mathscr{H}, \tag{3.7}$$

then, the mapping

$$g \mapsto g(T_1, \ldots, T_n)$$

is an algebra homomorphism of $\mathscr{F}^\infty$ into $B(\mathscr{H})$ with the following properties:

(i) $g(T_1,\ldots,T_n) = T_i$ *if* $g = e_i$, $i = 1,2,\ldots,n$

$= I_{\mathscr{H}}$ *if* $g = I$

(ii) $\|g(T_1,\ldots,T_n)\| \leqq \|g\|_\infty$

(iii) $g(T_1,\ldots,T_n) = P_{\mathscr{H}}\, g(V_1,\ldots,V_n)|_{\mathscr{H}}$, *where* $(V_1,\ldots,V_n)$

is the minimal isometric dilation of $(T_1,\ldots,T_n)$.

PROOF. Since $(T_1,\ldots,T_n)$ satisfies (3.7), its minimal isometric dilation $(V_1,\ldots,V_n)$ is a Λ-orthogonal shift on a Hilbert space $\mathscr{K} \supset \mathscr{H}$ (see [6, 8]).

Therefore,

$$p(T_1,\ldots,T_n) = P_{\mathscr{H}}\, p(V_1,\ldots,V_n)|_{\mathscr{H}} \tag{3.8}$$

for any $p \in \mathscr{P}$.

Taking into account the results so far, it follows that $g(V_1,\ldots,V_n)$ is strongly convergent in $B(\mathscr{K})$ for any $g \in \mathscr{F}^\infty$. Hence, and by (3.8), we deduce that $g(T_1,\ldots,T_n)$ is strongly convergent in $B(\mathscr{H})$.

Since $\|g(V_j,\ldots,V_n)\| = \|g\|_\infty$, the result follows.

REMARK 3.7. If $(T_1,\ldots,T_n) \in (B(\mathscr{H})^n)_r$, $0 < r < 1$, then $(T_1,\ldots,T_n)$ has the property (3.7) (see [6]).

Let $\operatorname{Alg}(S_1,\ldots,S_n)$ denote the smallest closed subalgebra of $B(\mathscr{F}(H_n))$ containing $I, S_1,\ldots,S_n$. This algebra is the closure in the uniform norm of the collection of polynomials in $S_1,\ldots,S_n$, that is,

$$\operatorname{Alg}(S_1,\ldots,S_n) = \operatorname{clos}\left\{ \sum_{f \in \mathscr{F}_m} a_f S_f;\ a_f \in \mathbf{C},\ m \in \mathbf{N} \right\}$$

THEOREM 3.8. *The following equality holds*

$$\operatorname{Alg}(S_1,\ldots,S_n) = \{g(S_1,\ldots,S_n) : g \in \mathscr{A}\}.$$

PROOF. If $A \in \operatorname{Alg}(S_1,\ldots,S_n)$, then there exists a sequence $\{P_m\}_{m=1}^\infty$ of polynomials such that

$$\|A - p_m(S_1,\ldots,S_n)\| \to 0 \ \ (\text{as } m \to \infty). \tag{3.9}$$

Since $\{p_m(S_1,\ldots,S_n)\}_{m=1}^\infty$ is a Cauchy sequence and

$$\|q(S_1,\ldots,S_n)\| = \|q\|_\infty \quad \text{for any } q \in \mathscr{F}^\infty \tag{3.10}$$

it follows that $\{p_m\}_{m=1}^\infty$ is a Cauchy sequence in the norm $\|\ \ \|_\infty$. Thus, there exists $g \in \mathscr{A}$ such that $\|g - p_m\|_\infty \to 0$ (as $m \to \infty$). Again by (3.10) we deduce that

$$\|g(S_1,\ldots,S_n) - p_m(S_1,\ldots,S_n)\| \to 0 \quad (\text{as } m \to \infty).$$

According to (3.9) we have $A = g(S_1,\ldots,S_n)$.

The converse inclusion is simple to deduce.

Now, let us note that, by using the von Neumann inequality (2.2), that is,

$$\|p(T_1,\dots,T_n)\| \leqq \|p\|_\infty,\ (T_1,\dots,T_n)\in(B(\mathscr{H})^n)_1$$

one can easily show the following

THEOREM 3.9. *If* $(T_1,\dots,T_n)\in(B(\mathscr{H})^n)_1$, *then the mapping*

$$\Psi:\mathscr{P}\to B(\mathscr{H}),\ \Psi(p)=p(T_1,\dots,T_n)$$

extends to a contractive homomorphism from the Banach algebra $\mathscr{A}$ *to* $B(\mathscr{H})$.

Finally, let us remark that all the results of this paper hold true, if we replace the set $\Lambda=\{1,2,\dots,n\}, n\geqq 2$, by the set $\Lambda=\{1,2,\dots\}$, in a slightly adapted version.

ADDITION BY THE EDITOR. After this paper was submitted a result similar to the main theorem appeared in a paper of M. Bozeiko, "Positive-definite kernels, length functions on groups and a noncommutative von Neumann inequality", Studia Math. 95 (1989), 107–118, in particular Theorem 8.1.

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