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ORTHOGONALLY PINCHED CURVATURE TENSORS AND APPLICATIONS

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Introduction.

In his paper we study a condition on algebraic curvature tensors which is strictly weaker than imposing bounds on the sectional curvature. We characterize a class of curvature tensors in terms of this condition and the minimal eigenvalue of the Weitzenbock operator. Finally, we give as an application a generalization of part of the "Sphere Theorem" in Riemannian geometry.

DEFINITION (0.1). Let V be a real m-dimensional vector space with inner product $\langle,\rangle$, and let R be a curvature tensor on V (see section one for more details). For P a two-dimensional subspace of V let $K(P)$ denote its sectional curvature. We say that R is orthogonally pinched between Δ and δ, where $\Delta \geqq \delta$ are numbers, if for any two mutually orthogonal two dimensional subspace, P_1, P_2, of V (i.e., $P_2 \subset (P_1)^{\perp}$), one has

$$2\Delta \geqq K(P_1) + K(P_2) \geqq 2\delta. \tag{0.2}$$

Of course this notion only has content if $m \geqq 4$. We shall shortly give examples of curvature tensors R which are orthogonally pinched as in (0.2), but whose sectional curvatures are not pinched between Δ and δ (i.e. there are planes P for which $K(P) > \Delta$ and planes P for which $K(P) < \delta$). In this sense requiring R to be orthogonally pinched between Δ and δ is strictly weaker than requiring the sectional curvature of R to be pinched between Δ and δ (this latter certainly implies R is orthogonally pinched between Δ and δ).

The "Sphere Theorem" (cf. [CE]) classifies compact simply connected Riemannian manifolds whose sectional curvature K, is globally pinched between 1 and 1/4, by showing that such manifolds are globally symmetric. Using either the classification of symmetric space [H], or [GG], [GS], or [GWZ], one deduces the isometry type of such spaces. As an application of our study of

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orthogonally pinched curvature tensors, we prove an extension of part of the Sphere Theorem:

0.3 THEOREM. *Let (M, g) be a compact orientable Riemannian manifold of dimension $\geqq 5$ and suppose $b_2(M) \neq 0$, where $b_2(M)$ is the second (real) Betti number of M. Suppose that for some nonnegative function Δ on M, for each $p \in M$, $R(p)$ is orthogonally pinched between $\Delta(p)$ and $\Delta(p)/4$, where $R(p)$ is the curvature tensor of g at p. Then Δ is a constant function. If M is odd dimensional, $\Delta \equiv 0$ and M is flat. If M is $2n$ dimensional $(n \geqq 3)$ then either $\Delta \equiv 0$ and M is flat or $\Delta \equiv c > 0$ and then (M, g) is biholomorphically isometric to $(\mathbf{C}P^n, c \cdot g_{\mathrm{can}})$ where g_{can} is the Fubini-Study metric.*

If one starts by assuming Δ is a positive constant and (M, g) has its sectional curvature pinched between Δ and $\Delta/4$, then this theorem can be deduced from the Sphere theorem, as mentioned above, by the process of elimination. If one allows Δ to be variable but still requires sectional curvature pinched (pointwise) between $\Delta(p)$ and $\Delta(p)/4$, this theorem can be deduced from Theorem (1.3) of [S]. As in [S], our proof of Theorem (0.3) shows directly, that under the assumptions there, one has, in the $2n$ dimensional case, $R(p) = \Delta(p)R_{\mathbf{C}P^n}$, where $R_{\mathbf{C}P^n}$ is the canonical curvature tensor for $\mathbf{C}P^n$.

The orthogonal complex structure used to define $R_{\mathbf{C}P^n}$ comes from an eigenvector X, of R_2, the Weitzenbock operator of R (see section one). The orthogonal "quarter" pinching assumption guarantees that X is parallel, making M Kähler (or flat). The remainder of the proof proceeds as in the proof of Theorem (1.3), [S].

The bulk of the proof of Theorem (0.3) is to prove that Theorem (1.4) of [S] holds under the assumption that R is orthogonally pinched between Δ and δ. Theorem (1.4) of [S] is a characterization of algebraic curvature tensors on $2n$ dimensional vector spaces, whose sectional curvature is pinched between Δ and δ and whose Weitzenbock operator has minimum eigenvalue $(n-1)4/3(4\delta - \Delta)$.

We will show in section one that under assumption (0.2), the minimum possible eigenvalue of the Weitzenbock operator is (still) $(n-1)4/3(4\delta - \Delta)$ and we get the following analog of Theorem (1.4), [S].

(0.4) THEOREM. (i) (Existence) *Let J be an orthogonal complex structure on V^{2n}, $n \geqq 2$, and define*

$$(*) \qquad \tilde{R} = 4/3(\Delta - \delta)R_{\mathbf{C}P^n} + 1/3(4\delta - \Delta)Id.$$

Then the sectional curvature, $\tilde{K}$, of $\tilde{R}$, satisfies (the sharp bounds) $\Delta \geqq \tilde{K} \geqq \delta$ (hence $\tilde{R}$ is orthogonally pinched between Δ and δ) and the minimal eigenvalue of the Weitzenbock operator, $\tilde{R}_2$, of $\tilde{R}$, equals $(n-1)4/3(4\delta - \Delta)$.

(ii) (Uniqueness) *If R is any curvature tensor on V^{2n}, $n \geqq 3$, which is orthogonally pinched between Δ and δ, and whose Weitzenbock operator has*

$(n-1)4/3(4\delta - \Delta)$ as an eigenvalue (necessarily minimal), then there is an orthogonal complex structure J on V^{2n} such that $R = \tilde{R}$.

The proof of the uniqueness portion of theorem (0.4) definitely breaks down in case $n = 2\,(\dim V = 4)$. One can see the reason for the breakdown even in the following example.

(0.5) EXAMPLE. Let R be orthogonally pinched between c and c on V^m, where $m \geqq 5$. Then $K \equiv c$ ("orthogonally constant implies constant for dimensions $\geqq 5$").

PROOF. First assume $m \geqq 6$. Let $P_1 \subset V$ be any two plane, and let P_2, P_3 be two planes such that P_1, P_2, P_3 are mutually orthogonal. Then $2c = K(P_1) + K(P_2) = K(P_1) + K(P_3) = K(P_2) + K(P_3)$, so subtracting the last term from the sum of the first two yields $K(P_1) = c$. A slightly different argument also shows that constant orthogonal pinching implies constant if dimension $V = 5$. Suppose R is orthogonally pinched between c and c on a 5 dimensional vector space. Let P be a two plane in V with orthonormal basis e_1, e_2. Let e_3, e_4, e_5 be an orthonormal basis for $P^\perp$ in V. Letting K_{ij} = the sectional curvature of the plane spanned e_i and e_j, we have: $2c = K_{12} + K_{34} = K_{12} + K_{45}$ (so $K_{34} = K_{45}$); $2c = K_{13} + K_{45} = K_{13} + K_{25}$ (so $K_{45} = K_{25}$); $2c = K_{34} + K_{25} = K_{34} + K_{12}$ (so $K_{12} = K_{25} = K_{45} = K_{34}$); now the firstequation gives $K_{12} = c$.

Note that these arguments break down in the four dimensional case.

Some information concerning the four dimensional case is still available, and this is discussed at the end of section one. We now give examples of curvature tensors which are orthogonally pinched between Δ and δ but whose sectional curvatures are not pinched between Δ and δ.

(0.6) EXAMPLE A). Let $V_1 = V_3 = \mathsf{R}^2$, V_2 = any real vector space of dimension $\geqq 2$. Let the curvature tensor, R_i, on V_i have constant sectioal curvature k_i, $i = 1,2,3$, where $k_1 > k_2 \geqq 0 > k_3$. Let $V = V_1 \oplus V_2 \oplus V_3$, $R = R_1 \oplus R_2 \oplus R_3$ (with the obvious meaning), and K = the sectional curvature of R. If P_2 is a two plane in V_2, then $k_1 + k_2 = K(V_1,0,0) + K(0,P_2,0) = \max(K(P) + K(Q))$ over all mutually orthogonal two planes, P, $Q \subset V$, and $k_3 = K(0,0,V_3) + K(Q)$ (where Q is the two plane spanned by $\{(\vec{e}_1,0,0),(0,\vec{e}_2,0)\}$ where $\vec{e}_i$ is a unit vector in V_i) $= \min(K(P) + K(Q))$ over all mutually orthogonal two planes in V. Then R is orthogonally pinched between $(k_1 + k_2)/2$ and $k_3/2$, but $K(V_1,0,0) = k_1 > (k_1 + k_2)/2$ and $K(0,0,V_3) = k_3 < k_3/2$.

EXAMPLE B). Let $V_1 = \mathsf{R}^n$ $(n \geqq 4)$, $V_2 = \mathsf{R}$, $V = V_1 \oplus V_2$. Give V_1 the curvature tensor with constant sectional curvature 1, and V the curvature tensor of the direct sum (so that $K(P) = 0$ if P is spanned by v_1, v_2, where $v_i \in V_i$). Then one has $1 \geqq K(P) \geqq 0$ *for any* plane $P \subset V$ (and the bounds are sharp), while

$2 \geqq K(P) + K(Q) \geqq 1$ if P and Q are any two *orthogonal* two planes in V (again the bounds are sharp). Note that this is just the situation on $S^n \times S^1$ at each point.

The Sphere Theorem has been generalized in [GS] and [GG], to allow $K \geqq 1$ and $d(M) \geqq \pi/2$. Additionally, in [MM], the notion of a curvature tensor being positive on totally isotropic two planes was introduced and it is shown in that paper that a compact simply connected Riemannian manifold M whose curvature is positive on totally isotropic two planes is homeomorphic to a sphere.

A remarkable corollary to this result is the following: if there is a strictly positive function Δ on M, such that the sectional curvature K of M satisfies $\Delta(p) \geqq K(p) > \Delta(p)/4$, then M is homeomorphic to a sphere. The main point here is that "pointwise strict quarter pinching" implies positive on totally isotropic two planes. We will show in section one ((1.21)) that if a curvature tensor on V^{2n}, $n \geqq 2$, is orthogonally pinched between Δ and $\Delta/4$, then it is nonnegative on totally isotropic two planes (and strictly positive if the orthogonal pinching is strict). Thus "quarter" orthogonal pinching lies somewhere between quarter pinching and positive on totally isotropic two planes.

A natural question stemming from these remarks is the following: can the orthogonal pinching assumption in Theorem (0.3) be weakened to nonnegative on totally isotropic two planes? Since any metric with nonnegative curvature operator automatically has nonnegative curvature on totally isotropic two planes, it is clear that some additional assumption, besides nonnegative curvature on totally isotropic two planes, would have to be made in order to get some result analogous to Theorem 0.3. This will be discussed in [S_1].

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Section 1. Proof of Theorem (0.4).

Our notation for an algebraic curvature tensor, R, on a vector space V with inner product $\langle , \rangle$, its sectional curvature K, and Weitzenbock operator R_2, are exactly as in [S], section 1. In particular we will assume V is $2n$ or $2n + 1$ dimensional (initially we will allow $n \geqq 2$), and if $\{e_1, \ldots, e_{2n}\}$ (e_{2n+1}) is an orthonormal basis for V, then R_{ijkl} means $\langle R(e_i, e_j)e_k, e_l \rangle$. and *distinct* letters (i, j, k, l) stands for *distinct* numbers. Also, K_{ij} means the sectional curvature of the plane spanned by e_i and e_j, while $K_{i\pm j, k\pm l}$ stands for the sectional curvature of the plane spanned by $\frac{e_i \pm e_j}{\sqrt{2}}, \frac{e_k \pm e_l}{\sqrt{2}}$. The properties of $\tilde{R}$ in Theorem (0.4) (i) are given in Proposition (2.17) and (2.21) a–d in [S].

We start off by showing that [S] Proposition (2.8) and (2.9) hold for orthogonal pinching.

From now on we assume that R is orthogonally pinched between Δ and δ.

PROPOSITION (1.1) a) $|R_{ijkl}| \leqq \frac{2}{3}(\Delta - \delta)$. $R_{ijkl} = \frac{2}{3}(\Delta - \delta)(-\frac{2}{3}(\Delta - \delta))$ iff $K_{i+l,j+k} + K_{i-l,j-k} = 2\Delta(2\delta) = K_{j-l,i+k} + K_{j+l,i-k}$ and $K_{i-l,j+k} + K_{i+l,j-k} = 2\delta(2\Delta) = K_{j+l,i+k} + K_{j-l,i-k}$.

b) *The minimum eigenvalue, r, of the Weitzenböck operator, R_2, satisfies $r \geqq (n-1)\frac{4}{3}(4\delta - \Delta)$ if dimension $V = 2n$. If dimension $V = 2n+1$, then $r \geqq \frac{2}{3}((8n-5)\delta - 2\Delta(n-1))$.*

PROOF. a) This follows from the decomposition of R_{ijkl} given in [S] 2.8A) and 2.8B), together with the fact that the plane spanned by $\left\{\frac{e_i + e_l}{\sqrt{2}}, \frac{e_j + e_k}{\sqrt{2}}\right\}$ and the plane spanned by $\left\{\frac{e_i - e_l}{\sqrt{2}}, \frac{e_j - e_k}{\sqrt{2}}\right\}$, and so on, are mutually orthogonal. Similar remarks handle the remaining curvature terms in R_{ijkl}.

b) The proof of this follows just as in Proposition (2.9) [S], up to the number (2.12). We now show how to use the orthogonal pinching assumption to go from (2.12) to (2.12a) in the $2n$ dimensional case:

$$(1.2) \quad \sum_{j=3}^{2n} (K_{j1} + K_{j2}) = \sum_{l=1}^{n-1} (K_{(2l+1),1} + K_{(2l+2),2} + K_{(2l+2),1} + K_{(2l+1),2})$$

by the orthogonal pinching assumption, one has

$$K_{(2l+1),1} + K_{(2l+2),2} \geqq 2\delta, K_{(2l+2),1} + K_{(2l+1),2} \geqq 2\delta$$

since the planes spanned by $\{e_{2l+1}, e_1\}$ and $\{e_{2l+2}, e_2\}$, are mutually orthogonal. Thus the term in (1.2) is $\geqq 4\delta(n-1)$, and the proof of (1.1) b) now proceeds exactly as in Proposition (2.9) of [S], using part a) of this proposition between (2.12) b) and (2.12c) of [S].

To get the conclusion in the $2n+1$ dimensional case, use the following regrouping of terms:

$$(1.2.1) \quad \sum_{j=3}^{2n+1} (K_{j1} + K_{j2}) = K_{31} + K_{32} + K_{41} + K_{42} + K_{51} + K_{52}$$
$$+ \sum_{j=6}^{2n+1} (K_{j1} + K_{j2})$$
$$= (K_{13} + K_{42}) + (K_{15} + K_{23}) + (K_{14} + K_{25})$$
$$+ \sum_{l=3}^{n} ([K_{2l,1} + K_{2l+1,2}] + [K_{2l,2} + K_{2l+1,1}])$$
$$\geqq 6\delta + (n-2)4\delta = 2\delta(2n-1)$$

now proceed as in the $2n$ dimensional case.

From now on we will assume V is $2n$-dimensional.

REMARK. If $r = (n-1)\frac{4}{3}(4\delta - \Delta)$ then just as in [S], *we have* $|R_{2i-1,2i,2,1}| = \frac{2}{3}(\Delta - \delta)$ for $i = 2,\ldots,n$, and we can take $1 = \lambda_1 = \lambda_2 = \ldots = \lambda_n$. (Note that this requires $\Delta > \delta$, which we assume from now on).

COROLLARY (1.3). *If* $r = (n-1)\frac{4}{3}(4\delta - \Delta)$, *then for* i,j *distinct, between* 1 *and* n, *we have*

$$\text{(1.3)}\quad \text{a)}\quad R_{2i-1,2i,2j,2j-1} = \tfrac{2}{3}(\Delta - \delta)$$

and

$$\text{b)}\quad \sum_{\substack{l \neq j \\ 1 \leq l \leq n}} (K_{2l-1,2j-1} + K_{2l,2j}) + (K_{2l,2j-1} + K_{2l-1,2j}) = 4\delta(n-1)$$

PROOF. This follows as in Corollary (2.13), [S], where one can, in a similar grouping to (1.2), write

$$\text{(1.4)}\quad (n-1)\tfrac{4}{3}(4\delta - \Delta) = r = \sum_{\substack{l \neq 2j \\ l \neq 2j-1 \\ 1 \leq l \leq 2n}} (K_{l,2j-1} + K_{l,2j}) - 2 \sum_{\substack{i \neq j \\ 1 \leq i \leq n}} R_{2i-1,2i,2j,2j-1}$$

$$\text{(1.5)}\quad = \sum_{\substack{l \neq j \\ 1 \leq l \leq n}} (K_{2l-1,2j-1} + K_{2l,2j}) + (K_{2l,2j-1} + K_{2l-1,2j}) - 2 \sum_{\substack{i \neq j \\ 1 \leq i \leq n}} R_{2i-1,2i,2j,2j-1}$$

$$\geq 4\delta(n-1) - \tfrac{4}{3}(n-1)(\Delta - \delta) = (n-1)\tfrac{4}{3}(4\delta - \Delta).$$

From now on we assume $r = (n-1)\frac{4}{3}(4\delta - \Delta)$.

PROPOSITION (1.6) a) For $i \neq j$ between 1 and n, we have

$$2\Delta = K_{(2i-1)+(2j-1),2i+2j} + K_{(2i-1)-(2j-1),2i-2j}$$

$$2\Delta = K_{2i-(2j-1),(2i-1)+2j} + K_{(2i)+(2j-1),(2i-1)-2j}$$

(1.6) (b) *If* $n \geq 3$, *then*

$$\delta = K_{2i,2j} = K_{2i-1,2j-1} = K_{2i,2j-1} = K_{2i-1,2j}$$

PROOF. a) This follows from (1.3) a) and Proposition (1.1) a).

b) The proof of (1.6) b) consists of examining the terms in (1.3) b), and various rearrangements of those terms, and using the fact that R is orthogonally pinched. We first show that the sectional curvatures appearing in (1.6b) are equal, and then compute their common value to be δ.

Since $n \geqq 3$, there are distinct numbers l_1, l_2, and j between 1 and n. Thus, the following sum appears in (1.5):

(1.7) $$(K_{2l_1-1,2j-1} + K_{2l_1,2j}) + (K_{2l_1,2j-1} + K_{2l_1-1,2j}) + (K_{2l_2-1,2j-1} + K_{2l_2,2j}) + (K_{2l_2,2j-1} + K_{2l_2-1,2j}).$$

Since R is orthogonally pinched, the sum in (1.7) is $\geqq 8\delta$. (the terms are grouped to make the orthogonal planes clear). Since $r = (n-1)\frac{4}{3}(4\delta - \Delta)$, the sum in (1.7) must actually *equal* 8δ. This means that each of the four sums enclosed in parentheses in (1.7) must actually equal 2δ. However, the terms in (1.7) can be rearranged in such a way that one still has sums of sectional curvatures of orthogonal planes. We now list the results of these observations: (1.7) equals 8δ and R orthogonally pinched implies each of (1.8), (1.9), and (1.10) are true:

(1.8)

a) $K_{2l_1-1,2j-1} + K_{2l_1,2j} = 2\delta$

b) $K_{2l_1,2j-1} + K_{2l_1-1,2j} = 2\delta$

c) $K_{2l_2-1,2j-1} + K_{2l_2,2j} = 2\delta$

d) $K_{2l_2,2j-1} + K_{2l_2-1,2j} = 2\delta$

(1.9)

a) $K_{2l_1-1,2j-1} + K_{2i_2,2j} = 2\delta$

b) $K_{2l_1,2j} + K_{2l_2-1,2j-1} = 2\delta$

c) $K_{2l_1,2j-1} + K_{2l_2-1,2j} = 2\delta$

d) $K_{2l_1-1,2j} + K_{2l_2,2j-1} = 2\delta$

(1.10)

a) $K_{2l_1-1,2j-1} + K_{2l_2-1,2j} = 2\delta$

b) $K_{2l_1,2j} + K_{2l_2,2j-1} = 2\delta$

c) $K_{2l_1,2j-1} + K_{2l_2,2j} = 2\delta$

d) $K_{2l_1-1,2j} + K_{2l_2-1,2j-1} = 2\delta$

We now conclude that various sectional curvatures are equal by equating entries from (1.8), (1.9), and (1.10). We first write the equality of the sectional curvatures and then parenthetically show which equations from (1.8), (1.9) or (1.10) are used: $K_{2l_1,2j} = K_{2l_2,2j}$. ((1.8a) = (1.9a)); $K_{2l_1,2j} = K_{2l_1-1,2j}$ ((1.9b) = (1.10d)); $K_{2l_1-1,2j} = K_{2l_2-1,2j}$ ((1.8d) = (1.9d)). Summarizing so far, we get:

(1.11) $$K_{2l_1,2j} = K_{2l_2,2j} = K_{2l_1-1,2j} = K_{2l_2-1,2j}$$

Similarly, we have: $K_{2l_1-1,2j-1} = K_{2l_2-1,2j-1}$((1.8a) = 1.9b)); $K_{2l_2,2j-1} = K_{2l_1,2j-1}$((1.8d) = (1.9c)); $K_{2l_2-1,2j-1} = K_{2l_1,2j-1}$((1.8c) = (1.10c)).

Summarizing, we now have

$$(1.12) \qquad K_{2l_1-1,2j-1} = K_{2l_2-1,2j-1} = K_{2l_1,2j-1} = K_{2l_2,2j-1}.$$

Now (1.11) and (1.12) must be true for any distinct j, l_1, l_2 between 1 and n. Thus, for example if $a \neq b \in \{1,\dots,n\}$ and $c \neq d \in \{1,\dots,n\}$, $c \neq b$, then (1.11) implies $K_{2a,2b} = K_{2c,2b} = K_{2c,2d}$. Thus $K_{2a,2b} = K_{2c,2d}$ if $b \neq c$, but also $K_{2a,2b} = K_{2b,2d}$ by (1.11), so $K_{2a,2b} = K_{2c,2d}$ in all cases. Similarly, (1.12) implies $K_{2a-1,2b-1} = K_{2c-1,2d-1}$. Briefly, we can use (1.11) to show all terms of the form "$K_{\text{even},\text{even}}$" are equal, and (1.12) to show all terms of the form "$K_{\text{odd},\text{odd}}$" are equal. Finally, using the $K_{2l_1-1,2j}$ appearing in (1.11) and the $K_{2l_2,2j-1}$, in (1.12), with l_2 replaced by j and j replaced by l_1, we conclude $K_{2l_1-1,2j} =$ "$K_{\text{even},\text{even}}$" (from (1.11)) and $K_{2l_1-1,2j} =$ "$K_{\text{odd},\text{odd}}$".

We have now shown that all the sectional curvature terms appearing in 1.6b) are equal (note that we do not yet know about terms of the form $K_{2i-1,2i}$). Now one can use just 1.8a) to conclude that all these terms equal δ.

PROPOSITION (1.13). *If $n \geq 3$ then we have*

$$(1.13) \qquad K_{2i-1,2i} = \Delta \quad i = 1,\dots,n.$$

PROOF. Id a, b, c, d represent any four orthonormal vectors in V, then a straightforward computation shows:

$$(1.14) \quad K_{a+b,c+d} + K_{a-b,c-d} = \tfrac{1}{2}\{K_{ac} + K_{ad} + K_{bc} + K_{bd} + 2R_{acdb} + 2R_{adcb}\}$$

Use (1.14) together with the first equation in (1.6a), ($a = e_{2i-1}, b = e_{2j-1}, c = e_{2i}$, $d = e_{2j}$), together with (1.6b) and (1.3a) to get:

$$(1.15) \quad 4\Delta = K_{2i-1,2i} + K_{2j-1,2j} + 2\delta + \tfrac{4}{3}(\Delta - \delta) + 2R_{2i-1,2j,2i,2j-1}.$$

Use (1.14) together with the second equation in (1.6a) ($a = e_{2i}$, $b = -e_{2j-1}$, $c = e_{2i-1}, d = e_{2j}$), together with (1.6b) and (1.3a) to get:

$$(1.16) \quad 4\Delta = K_{2i-1,2i} + K_{2j-1,2j} + 2\delta + \tfrac{4}{3}(\Delta - \delta) - 2R_{2i,2j,2i-1,2j-1}.$$

Adding (1.15) and (1.16), using the Bianchi identity on the "R" terms (which yields another $R_{2i-1,2i,2j,2j-1} = \frac{2}{3}(\Delta - \delta)$ by (1.3a) again), yields

$$(1.17) \qquad 2\Delta = K_{2i-1,2i} + K_{2j-1,2j}.$$

Just as in Example (0.5) in the introduction, this yelds 1.13a).

(1.18) REMARK. Starting with Corollary (1.3), the indices have referred to an orthonormal basis $e_1,\dots,e_{2n}$ for which $X = \sum_{i=1}^{n} e_{2i-1} \wedge e_{2i}$, where X is the eigenvector of R_2 with eigenvalue $r = \frac{4}{3}(n-1)(4\delta - \Delta)$. This X yields an orthogonal complex structure J, by $J(e_{2i-1}) = -e_{2i}$, $J^2 = -\mathrm{Id}$, so in this case we have $X = -\sum_{i=1}^{n} e_{2i-1} \wedge J(e_{2i-1})$. But now it is straightforward to verify that if $\{f_i, Jf_i\}$, $i = 1 \dots n$ is *any* orthonormal basis of V "adapted to J", then $X = -\sum_{i=1}^{n} f_i \wedge Jf_i$. Therefore the above results, starting with Corollary (1.3) are in fact valid for *any* such orthonormal basis. That is, the orthonormal basis $e_1, \dots, e_{2n}$, may be replaced by $f_1, -Jf_1, \dots, f_n, -Jf_n$ in all of the cited results, and the conclusions still hold for this latter basis.

Using the above remark, we can now complete the

PROOF OF THEOREM (0.4) (ii). Let V and R be as in Theorem (0.4) (ii). Let P be any two dimensional subspace of V. Without loss of generality, we may assume that P has an orthonormal basis of the form $\{f_1, aJf_1 + bf_2\}$, where f_1, Jf_1, f_2 are orthonormal vectors, $a^2 + b^2 = 1$. Now

$$K(P) = a^2K(Jf_1, f_1) + b^2K(f_1, f_2) + 2ab\langle R(Jf_1, f_1)f_1, f_2\rangle. \tag{1.19}$$

From (1.13) and remark (1.18), we have $K(Jf_1, f_1) = \Delta$. From (1.6b) and remark (1.18), we have $K(f_1, f_2) = \delta$. We will now show that $\langle R(Jf_1, f_1)f_1, f_2\rangle = 0$ Let P_1 be the plane spanned by f_3, Jf_3, where f_3, Jf_3 are any adapted orthonormal vectors perpendicular to f_1, Jf_1, f_2, Jf_2. Then we have $K(f_3, Jf_3) = \Delta$, as above. Let $P_2(\theta)$ be the plane spanned by $\{\cos\theta Jf_1 + \sin\theta f_2, f_1\}$. Note that P_1 and $P_2(\theta)$ are mutually orthogonally planes. Letting $f(\theta) = K(P_1) + K(P_2(\theta))$, we see that a $f(0) = 2\Delta$ is a maximum for $f(\theta)$. Thus $f'(0) = 0$ and this yields $\langle R(Jf_1, f_1)f_1, f_2\rangle = 0$. Now (1.19) yields

$$K(P) = a^2\Delta + b^2\delta. \tag{1.20}$$

But (1.20) is exactly the result obtained from computing the sectional curvature, $\tilde{K}(P)$, of P with respect to the curvature tensor $\tilde{R}$ (using $J = -\sum f_i \wedge Jf_i$ as in remark (1.18)). Therefore R and $\tilde{R}$ have the same sectional curvatures and must therefore be identical.

We now examine the consequences of the assumptions: $\dim V = 4$ $(n = 2)$, R is orthogonally pinched between Δ and δ, and $r = \frac{4}{3}(4\delta - \Delta)$. We still find an orthonormal basis $e_1, \dots, e_4$ of V such that $X = e_{12} + e_{34}$, $R_2X = \frac{4}{3}(4\delta - \Delta)X$, from Corollary (1.3). We can also still conclude $R_{1243} = \frac{2}{3}(\Delta - \delta)$, ((1.3a)), $K_{13} + K_{24} = K_{23} + K_{14} = 2\delta$ ((1.3b)), $K_{12} + K_{34} = 2\Delta$ (1.17), and $R_{1423} = \frac{1}{3}(\Delta - \delta) = R_{4213}$. Using these facts allows one to determine the values of sums of various sectional curvatures, but does not seem to pin down R exactly.

In [MM], Micallef and Moore introduce the notion of nonnegative (positive) curvature on totally isotropic two planes. For an algebraic curvature tensor R to

satisfy this condition is equivalent to the following condition ([MM], p. 203):
(1.21) For all orthonormal e_i, e_j, e_k, e_l,

$$K_{ij} + K_{kl} + K_{il} + K_{kj} + 2R_{ikjl} \geqq 0 (> 0).$$

From Proposition (1.1a), we see that if R is orthogonally pinched between Δ and $\Delta/4(\Delta/4 + \varepsilon, \varepsilon > 0)$, then R is indeed nonnegative (positive) on totally isotropic two planes.

Section 2. Proof of Theorem (0.3).

We just indicate the proof, since it is so similar to that of Theorem (1.3), [S].

At any point where $\Delta(p) = 0$, one trivially has (from example (0.5)), $R(p) = \Delta(p) \cdot R_{CP^n}$. Since $b_2(M) \neq 0$, there is a harmonic two form, X, on M. Since $R(p)$ is orthogonally pinched between $\Delta(p)$ and $\Delta(p)/4$, Proposition (1.1b) guarantees that $R_2(p)$ is nonnegative definite. The Bochner method now guarantees that X is parallel and $R_2 X \equiv 0$. In the odd dimensional case, this implies $\Delta \equiv 0$. At any point where $\Delta(p) \neq 0$, the above statement means that $R_2(p)$ *has* 0 as an eigenvalue, hence $R_2(p) = \Delta(p) R_{CP^n}$, from Theorem (0.4) ii). Now proceed as in the rest of the proof of Theorem (1.3) [S].

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