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QUASIADDITIVITY OF RIESZ CAPACITY

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Introduction.

Let $0 < \alpha < n$ and $k_\alpha(x) = |x|^{\alpha - n}$ the Riesz kernel on R^n. Define the Riesz capacity by

$$R_{\alpha,p}(E) = \begin{cases} \inf\{\|f\|_p^p : k_\alpha * f(x) \geqq 1 \text{ on } E, f \geqq 0\} & \text{if } 1 < p < \infty, \\ \inf\{\|\mu\| : k_\alpha * \mu(x) \geqq 1 \text{ on } E, \mu \geqq 0\} & \text{if } p = 1. \end{cases}$$

In view of [7] we see that $R_{\alpha,1}(E)$ is equal to the usual (outer) α-capacity $C_\alpha(E)$. It is obvious that $R_{\alpha,p}$ is countably subadditive, i.e.

$$R_{\alpha,p}(E) \leqq \sum_k R_{\alpha,p}(E_k)$$

with $E = \bigcup_k E_k$. The main purpose of this paper is to investigate for what decompositions the inequality

$$R_{\alpha,p}(E) \geqq N \sum_k R_{\alpha,p}(E_k)$$

holds with some positive constant N. We refer to this inequality as "quasiadditivity". Quasiadditivity for decompositions into spherical shells has been considered by Landkof [9, Lemma 5.5 on p. 304] and Adams [1, Theorem 7.5]. In the case of Green energy (for the definition see Section 5), quasiadditivity for the Whitney decomposition (cf. [14, p. 16]) of a half space is discussed in Essén [5].

We shall show that the Whitney decomposition associated with a certain closed set has quasiadditivity.

DEFINITION. Let F be a closed set having no interior points. Put $\delta(x) = \operatorname{dist}(x, F)$ and let m_β be the measure defined by

$$m_\beta(E) = \int_E \delta(x)^{-\beta}\, dx.$$

We associate the least number $d = d(F)$ for which

$$m_\beta(C(x,r)) \leqq N_\beta r^{n-\beta} \tag{1.1}$$

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holds for all $x \in F$ and $r > 0$ with a positive constant N_β, whenever $0 < \beta < n - d$.

The constant $d(F)$ is related to the dimension of F. In fact, if L is an m-dimensional affine subspace in R^n, then $d(L) = m$. We can easily see that if F is an m-dimensional compact Lipschitz manifold, then $d(F) = m$. By definition if $F_1 \subset F_2$, then $d(F_1) \leqq d(F_2)$. The Hausdorff dimension of F is not greater than $d(F)$. They are, in general, different; if $F = \{0\} \cup \bigcup_{j=1}^{\infty} \{(j^{-1}, 0, \ldots, 0)\}$, then $d(F) > 0$ and yet the Hausdorff dimension of F is equal to 0.

Our main result is

THEOREM 1. *Let $1 \leqq p < \infty$ and suppose $\alpha p + d(F) < n$. Let $\{Q_k\}$ be the Whitney decomposition of $\mathsf{R}^n \setminus F$. Then, for any set $E \subset \mathsf{R}^n$,*

$$R_{\alpha,p}(E) \geqq N \sum_k R_{\alpha,p}(E_k)$$

holds with $E_k = E \cap Q_k$ for some positive constant N.

Let us note that $R_{\alpha,p}(F) = 0$ since the Hausdorff dimension of F is not greater than $d(F) < n - \alpha p$ (see [10, Theorem 21]). Since $d(\{0\}) = 0$, we see that Theorem 1 is a generalization of the aforementioned results of Landkof and Adams. Our proof is completely different; it relies on the following comparison between the Riesz capacity $R_{\alpha,p}$ and the measure $m_{\alpha p}$.

THEOREM 2. *Let $1 \leqq p < \infty$. Suppose $\alpha p + d(F) < n$. If E is measurable, then*

$$m_{\alpha p}(E) \leqq N R_{\alpha,p}(E)$$

for some positive constant N.

The plan of this paper is as follows. In Section 2 we shall prove Theorem 1 assuming Theorem 2. Theorem 2 will, in turn, be proved in Section 3 as a corollary to a certain weighted norm inequality. Section 4 will be devoted to applications of Theorems 1 and 2. We shall deal with sets E for which $m_{\alpha,p}(E)$ and $R_{\alpha,p}(E)$ are comparable. We shall observe that α-thin sets are characterized by Wiener type conditions associated with the Whitney decomposition. In Section 5, we shall study quasiadditivity of Green energy in connection with the notion of minimal thinness. We shall characterize minimally thin sets in terms of ordinary capacity (cf. [5] and [6, Section 1]). Also we shall observe that [4, Theorem 1 and 2] follows from our method.

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2. Proof of Theorem 1.

By the symbol N we denote an absolute positive constant whose value is unimportant and may change from line to line. We shall say that two positive functions f and g are comparable, written $f \approx g$, if and only if there exists a constant N such that $N^{-1}g \leqq f \leqq Ng$. By $C(x,r)$ we denote the closed ball with center at x and radius r. For the Whitney decomposition $\{Q_k\}$ of $\mathsf{R}^n \setminus F$, we write r_k for the side-length of Q_k. Note that dist $(Q_k, F) \approx r_k$. By $\tilde{Q}_k$ we denote the double of Q_k.

LEMMA 1. *Let $\beta + d(F) < n$. Then*

$$0 < r < \delta(x)/2 \ \Rightarrow\ m_\beta(C(x,r)) \approx r^n \delta(x)^{-\beta}, \tag{2.1}$$

$$r \geqq \delta(x)/2 \ \Rightarrow\ m_\beta(C(x,r)) \approx r^{n-\beta}. \tag{2.2}$$

In particular, (1.1) *holds for all $x \in \mathsf{R}^n$ and $r > 0$; the measure m_β is a doubling measure. Let $\alpha p + d(F) < n$. Then for a Whitney cube Q_k*

$$R_{\alpha,p}(Q_k) \approx m_{\alpha p}(Q_k) \approx r_k^{n-\alpha p}. \tag{2.3}$$

PROOF. By definition (2.1) is obvious. Let $r \geqq \delta(x)/2$. Then we find $x_0 \in F$ such that $C(x,r) \subset C(x_0, 3r)$. Hence by (1.1) we have $m_\beta(C(x,r)) \leqq N r^{n-\beta}$. Let us prove the opposite inequality. Since $\delta(y) \leqq r + \delta(x) \leqq 3r$ *for all* $y \in C(x,r)$, it follows that $m_\beta(C(x,r)) \geqq (3r)^{-\beta} \int_{C(x,r)} dx \geqq N r^{n-\beta}$. Thus (2.2) is proved. It is well known that $R_{\alpha,p}(C(x,r)) = N r^{n-\alpha p}$ for $\alpha p < n$. Hence (2.3) follows.

Let us put

$$\tilde{R}_{\alpha,p}(E) = \sum_k R_{\alpha,p}(E_k) \text{ with } E_k = E \cap Q_k.$$

We need to prove $\tilde{R}_{\alpha,p}(E) \approx R_{\alpha,p}(E)$. Let us begin with comparing $\tilde{R}_{\alpha,p}$ with a Hausdorff type outer measure. For $\beta > 0$ define the Hausdorff type outer measure H_β by

$$H_\beta(E) = \inf\left\{\sum_i r_i^\beta : E \subset \bigcup_i C(z_i, r_i), z_i \in F\right\}.$$

One should note that a point x has positive H_β measure unless it lies on F. In fact, $H_\beta(\{x\}) = \delta(x)^\beta$.

LEMMA 2. *Let $\alpha p + d(F) < n$. Then*

$$\tilde{R}_{\alpha,p}(E) \leqq N H_{n-\alpha p}(E).$$

PROOF. Let us prove first

$$\tilde{R}_{\alpha,p}(C(z,r)) \leqq N r^{n-\alpha p} \text{ for } z \in F. \tag{2.4}$$

Observe that if Q_k meets $C(z,r)$, then $Q_k \subset C(z, Nr)$. Hence

$$\tilde{R}_{\alpha,p}(C(z,r)) \leqq \sum_{Q_k \cap C(z,r) \neq \emptyset} R_{\alpha,p}(Q_k) \leqq N \sum_{Q_k \cap C(z,r) \neq \emptyset} m_{\alpha p}(Q_k)$$
$$\leqq N m_{\alpha p}(C(z, Nr)) \leqq N r^{n-\alpha p}$$

by (2.3) and Lemma 1. Thus (2.4) follows.

Take an arbitrary positive number ε. By definition we can find $z_i \in F$ and $r_i > 0$ such that

$$E \subset \bigcup_i C(z_i, r_i),$$

$$\sum_i r_i^{n-\alpha p} \leqq H_{n-\alpha p}(E) + \varepsilon.$$

By (2.4)

$$\tilde{R}_{\alpha,p}(E) \leqq \sum_i \tilde{R}_{\alpha,p}(C(z_i, r_i)) \leqq N \sum_i r_i^{n-\alpha p} \leqq N(H_{n-\alpha p}(E) + \varepsilon).$$

Since ε is arbitrary, we have the desired inequality. The lemma is proved.

PROOF OF THEOREM 1. Let us prove the inequality only for $1 < p < \infty$. The case $p = 1$ is similar. It suffices to prove that $\tilde{R}_{\alpha,p}(E) \leqq N R_{\alpha,p}(E)$ for $R_{\alpha,p}(E) < \infty$. Take an arbitrary positive number ε. We can find a nonnegative function f such that $k_\alpha * f \geqq 1$ on E and $\|f\|_p^p < R_{\alpha,p}(E) + \varepsilon$. We split $k_\alpha * f(x)$ into

$$I(x) = \int_{\tilde{Q}_k} k_\alpha(x-y) f(y)\, dy, \quad \text{for } x \in Q_k,$$

$$J(x) = \int_{\mathbf{R}^n \setminus \tilde{Q}_k} k_\alpha(x-y) f(y)\, dy, \quad \text{for } x \in Q_k.$$

Put $E' = \{x : I(x) \geqq \frac{1}{2}\}$ and $E'' = \{x : J(x) \geqq \frac{1}{2}\}$. Then $E \subset E' \cup E''$. Since the multiplicity of $\tilde{Q}_k$ is bounded by a constant depending only on the dimension, it follows that

$$\tilde{R}_{\alpha,p}(E') = \sum_k R_{\alpha,p}(E' \cap Q_k) \leqq 2^p \sum_k \int_{\tilde{Q}_k} f^p\, dx \leqq N \|f\|_p^p \leqq N(R_{\alpha,p}(E) + \varepsilon). \tag{2.5}$$

By an elementary calculation we see that if $E'' \cap Q_k \neq \emptyset$, then $J(x) \geqq N$ on Q_k. Hence $k_\alpha * f(x) \geqq J(x) \geqq N$ on

$$\tilde{E}'' = \bigcup_{Q_k \cap E'' \neq \emptyset} Q_k,$$

whence $R_{\alpha,p}(\tilde{E}'') \leqq N \|f\|_p^p$. Since $H_{n-\alpha p}(Q_k) \approx r_k^{n-\alpha p}$, it follows (2.3) and Theorem 2

$$H_{n-\alpha p}(E'') \leqq H_{n-\alpha p}(\tilde{E}'') \leqq \sum_{Q_k \cap E'' \neq \emptyset} H_{n-\alpha p}(Q_k) \leqq N \sum_{Q_k \cap E'' \neq \emptyset} r_k^{n-\alpha p}$$
$$\leqq N m_{\alpha p}(\tilde{E}'') \leqq N R_{\alpha,p}(\tilde{E}'') \leqq N \|f\|_p^p \leqq N(R_{\alpha,p}(E) + \varepsilon).$$

Hence $\tilde{R}_{\alpha,p}(E'') \leqq N(R_{\alpha,p}(E) + \varepsilon)$ by Lemma 2. This, together with (2.5), completes the proof, since ε is arbitrary.

3. Proof of Theorem 2.

We shall show Theorem 2 as a corollary to a certain weighted norm inequality. For future reference we shall state the result in a slightly more general form. Let $K_\alpha(x, y) = |x - y|^{\alpha - n}$. Define

$$T_\alpha f(x) = \int_{\mathbf{R}^n} K_\alpha(x, y) f(y)\, dm_\alpha(y).$$

Let us prove

THEOREM 3. *Let* $\alpha + d(F) < n$.

(i) *Let* $1 < p < \infty$. *Then* $\|T_\alpha f\|_{p,\alpha} \leqq N \|f\|_{p,\alpha}$, *where* $\|f\|_{p,\alpha} = (\int |f|^p\, dm_\alpha)^{1/p}$. *Moreover, if* w *satisfies the Muckenhoupt* A_p *condition with respect to* m_α, i.e.

$$(A_p) \qquad \sup_Q \left(\frac{1}{m_\alpha(Q)} \int_Q w\, dm_\alpha\right)\left(\frac{1}{m_\alpha(Q)} \int_Q w^{1/1-p}\, dm_\alpha\right)^{p-1} < \infty,$$

then

$$\int_{\mathbf{R}^n} |T_\alpha f|^p\, w\, dm_\alpha \leqq N \int_{\mathbf{R}^n} |f|^p\, w\, dm_\alpha.$$

(ii) *If* $\lambda > 0$, *then*

$$m_\alpha\left(\left\{x \in \mathbf{R}^n : \int_{\mathbf{R}^n} K_\alpha(x, y)\, d\mu(y) > \lambda\right\}\right) \leqq N\, \|\mu\|/\lambda.$$

First we prove that Theorem 2 follows from Theorem 3.

PROOF OF THEOREM 2. Suppose $p = 1$. Then the conclusion readily follows from Theorem 3 (ii). Suppose $1 < p < \infty$. In view of Lemma 1, we see that the weight $w(x) = \delta(x)^{(1-p)\alpha}$ satisfies (A_p). Hence, Theorem 3 (i) implies that

$$\int_{\mathbf{R}^n} |k_\alpha * g|^p\, dm_{\alpha p}(x) \leqq N \int_{\mathbf{R}^n} |g|^p\, dx,$$

where we put $g(x) = f(x)\delta(x)^{-\alpha}$. This immediately yields Theorem 2.

Now let us prove Theorem 3. Although the proof is carried out in a standard way (cf. [3]), we give it for the completeness. In the rest of this section we let

$$\alpha + d(F) < n.$$

First we note

LEMMA 3. *Let* $1 < p < \min\left\{\frac{n}{n-\alpha}, \frac{n-\alpha}{d(F)}\right\}$ *and let* $1/p + 1/q = 1$. *If* Q *is a cube in* R^n, *then*

$$\int_Q |T_\alpha f(x)|\, dm_\alpha(x) \leqq N \|f\|_{q,\alpha}\, m_\alpha(Q)^{1/p}.$$

PROOF. Let us prove first

$$\int_{\mathsf{R}^n} K_\alpha(x, y)^p\, dm_\alpha(y) \leqq N \delta(x)^{(\alpha - n)(p-1)}. \tag{3.1}$$

Since $p < n/(n - \alpha)$, it follows that

$$\int_{C(x,\delta(x)/2)} K_\alpha(x, y)^p\, dm_\alpha(y) \leqq N \delta(x)^{-\alpha} \int_{C(x,\delta(x)/2)} |x - y|^{(\alpha - n)p} dy \leqq N \delta(x)^{(\alpha - n)(p-1)}.$$

In view of Lemma 1, we have

$$\begin{aligned}\int_{\mathsf{R}^n \setminus C(x,\delta(x)/2)} K_\alpha(x, y)^p\, dm_\alpha(y) &= \sum_{j=0}^{\infty} \int_{2^{j-1}\delta(x) < |x-y| \leqq 2^j \delta(x)} |x - y|^{(\alpha - n)p}\, dm_\alpha(y) \\ &\leqq N \sum_{j=0}^{\infty} (2^{j-1}\delta(x))^{(\alpha - n)p} (2^j \delta(x))^{n-\alpha} \leqq N \delta(x)^{(\alpha-n)(p-1)},\end{aligned}$$

since $(\alpha - n)(p - 1) < 0$. Thus (3.1) holds. Hölder's inequality and (3.1) yield

$$|T_\alpha f(x)| \leqq \|f\|_{q,\alpha} \left(\int_Q K_\alpha(x, y)^p\, dm_\alpha(y)\right)^{1/p} \leqq N \|f\|_{q,\alpha}\, \delta(x)^{(\alpha - n)/q}. \tag{3.2}$$

Observe that $p < (n - \alpha)/d(F)$ implies that $\alpha + (n - \alpha)/q + d(F) < n$. Hence Lemma 1 yields that

$$m_{\alpha + (n-\alpha)/q}(Q) \approx N m_\alpha(Q)^{1/p}.$$

This, together with (3.2), completes the proof.

Let $\mathcal{M}_\alpha f(x)$ be the maximal function defined by

$$\mathcal{M}_\alpha f(x) = \sup_Q \frac{1}{m_\alpha(Q)} \int_Q |f|\, dm_\alpha,$$

where the supremum is taken over all cubes containing x. Observe

$$\sup_{x,x' \in C(x_0,1)} \int_{\mathsf{R}^n \setminus C(x_0,2)} |K_\alpha(x, y) - K_\alpha(x', y)| f(y)\, dm_\alpha(y) \leqq N \mathcal{M}_\alpha f(x_0). \tag{3.3}$$

As a result we have the following

LEMMA 4. *Let Q be a cube and $\tilde{Q}$ the double of Q. Then*

$$\sup_{x,x'\in Q}\int_{y\notin\tilde{Q}}|K_\alpha(x,y)-K_\alpha(x',y)|\,dm_\alpha(y)\leqq N.$$

We observe that if $\gamma=(p+1)^{-1}$, then $p=\dfrac{1}{\gamma}-1$ and

$$1<p<\min\left\{\frac{n}{n-\alpha},\frac{n-\alpha}{d(F)}\right\}\Leftrightarrow\max\left\{\frac{n-\alpha}{2n-\alpha},\frac{d(F)}{n-\alpha+d(F)}\right\}<\gamma<\tfrac{1}{2}.$$

LEMMA 5. *Let* $\max\left\{\dfrac{n-\alpha}{2n-\alpha},\dfrac{d(F)}{n-\alpha+d(F)}\right\}<\gamma<\frac{1}{2}$. *Let Q be a cube and $\tilde{Q}$ the double of Q. If $f\geqq 0$, $\operatorname{supp} f\subset\tilde{Q}$ and $\|f\|_{1,\alpha}\leqq\varepsilon m_\alpha(\tilde{Q})$ with $0<\varepsilon<1$, then*

$$m_\alpha(\{x\in Q:T_\alpha f(x)>1\})\leqq N\varepsilon^{1-\gamma}m_\alpha(Q).$$

PROOF. By the Caldéron-Zygmund lemma (e.g. [14, p. 17]) we have a family of mutually disjoint cubes Q_j such that

(i) $f(x)\leqq\varepsilon^\gamma$ a.e. on $\mathsf{R}^n\setminus\Omega$ with $\Omega=\cup_j Q_j$;
(ii) For each cube Q_j

$$\varepsilon^\gamma<\frac{1}{m_\alpha(Q_j)}\int_{Q_j}f\,dm_\alpha(x)\leqq N\varepsilon^\gamma.$$

Let

$$g(x)=\begin{cases}f(x) & \text{on } \mathsf{R}^n\setminus\Omega,\\ \dfrac{1}{m_\alpha(Q_j)}\displaystyle\int_{Q_j}f\,dm_\alpha(x) & \text{on } Q_j.\end{cases}$$

and $b=f-g$. Obviously, $\|g\|_{1,\alpha}\leqq\|f\|_{1,\alpha}$ and $\|b\|_{1,\alpha}\leqq 2\|f\|_{1,\alpha}$. Let $p=\dfrac{1}{\gamma}-1$ and $1/p+1/q=1$. Since $0\leqq g\leqq N\varepsilon^\gamma$, it follows that

$$\|g\|_{q,\alpha}^q\leqq N\varepsilon^{\gamma(q-1)}\|f\|_{1,\alpha}\leqq N\varepsilon^{\gamma(q-1)+1}m_\alpha(Q).$$

We have from Lemma 3

$$m_\alpha(\{x\in Q:T_\alpha g(x)\geqq 1/2\})\leqq N\|g\|_{q,\alpha}m_\alpha(Q)^{1/p}\leqq N\varepsilon^{1-\gamma}m_\alpha(Q). \tag{3.4}$$

Let y_j be the center of Q_j and $\tilde{Q}_j$ the double of Q_j. We put $\tilde{\Omega}=\cup_j\tilde{Q}_j$. It follows from Lemma 4 and the symmetry of K_α that

$$\int_{\mathsf{R}^n\setminus\tilde{\Omega}}|T_\alpha b|\,dm_\alpha\leqq\sum_j\int_{x\notin\tilde{Q}_j}\int_{y\in Q_j}|K_\alpha(x,y)-K_\alpha(x,y_j)|\,|b(y)|\,dm_\alpha(y)\,dm_\alpha(x)$$

$$\leqq N \sum_j \int_{y \in Q_j} |b(y)| \, dm_\alpha(y) \leqq N \|f\|_{1,\alpha}.$$

Therefore

$$m_\alpha(\{x \in \mathsf{R}^n : |Tb(x)| > 1/2\}) \leqq N(\|f\|_{1,\alpha} + \varepsilon^{1-\gamma} m_\alpha(Q)) \leqq N \varepsilon^{1-\gamma} m_\alpha(Q),$$

since

$$m_\alpha(\tilde{\Omega}) \leqq N \sum_j m_\alpha(Q_j) \leqq \frac{N}{\varepsilon^\gamma} \sum_j \int_{Q_j} f \, dm_\alpha(x) \leqq \frac{N}{\varepsilon^\gamma} \|f\|_{1,\alpha} \leqq N \varepsilon^{1-\gamma} m_\alpha(Q).$$

This, together with (3.4), implies

$$m_\alpha(\{x \in Q : T_\alpha f(x) > 1\}) \leqq N \varepsilon^{1-\gamma} m_\alpha(Q).$$

The proof is complete.

LEMMA 6. *Let* $\max\left\{\dfrac{n-\alpha}{2n-\alpha}, \dfrac{d(F)}{n-\alpha+d(F)}\right\} < \gamma < \frac{1}{2}$. *Then there is a positive constant B such that if $\lambda > 0$, $0 < \varepsilon < 1$, $f \geqq 0$, and a cube Q has a point x' satisfying $T_\alpha f(x') \leqq \lambda$, then*

$$m_\alpha(\{x \in Q : T_\alpha f(x) > B\lambda, \mathcal{M}_\alpha f(x) \leqq \varepsilon\lambda\}) \leqq N \varepsilon^{1-\gamma} m_\alpha(Q).$$

PROOF. We may assume that there is a point x_0 in Q such that $\mathcal{M}_\alpha f(x_0) \leqq \varepsilon\lambda$. Let $\tilde{Q}$ be the double of Q. In view of (3.3), we have for $x \in Q$

$$\int_{\mathsf{R}^n \setminus \tilde{Q}} K_\alpha(x, y) f(y) \, dm_\alpha(y) \leqq \int_{\mathsf{R}^n \setminus \tilde{Q}} |K_\alpha(x, y) - K_\alpha(x', y)| \, f(y) \, dm_\alpha(y) + T_\alpha f(x')$$
$$\leqq N\varepsilon\lambda + \lambda \leqq N_1 \lambda.$$

Let $h = f/\lambda$ on $\tilde{Q}$ and $h = 0$ elsewhere. Since $\|h\|_{1,\alpha} \leqq \varepsilon m_\alpha(\tilde{Q})$, it follows from Lemma 5 that

$$m_\alpha(\{x \in Q : T_\alpha h(x) > 1\}) \leqq N \varepsilon^{1-\gamma} m_\alpha(Q).$$

Let $B = N_1 + 1$. Then

$$m_\alpha(\{x \in Q : T_\alpha f(x) > B\lambda\}) \leqq N \varepsilon^{1-\gamma} m_\alpha(Q).$$

The lemma follows.

PROOF OF THEOREM 3. Suppose that w satisfies (A_p). In the same way as in [3, Theorem I], we see that

$$\int_{\mathsf{R}^n} (\mathcal{M}_\alpha f)^p \, w \, dm_\alpha \leqq N \int_{\mathsf{R}^n} |f|^p \, w \, dm_\alpha. \tag{3.5}$$

Hence it is sufficient to show that

$$\int_{\mathbb{R}^n} |T_\alpha f|^p \, w \, dm_\alpha \leqq N \int_{\mathbb{R}^n} (\mathcal{M}_\alpha f)^p \, w \, dm_\alpha. \tag{3.6}$$

In the proof of (3.6) we may assume that f is nonnegative, bounded and has compact support. Since $0 \leqq T_\alpha f(x) \leqq N|x|^{\alpha-n} \leqq N\mathcal{M}_\alpha f(x)$ as $|x| \to \infty$, it follows from (3.5) that

$$\int_{\mathbb{R}^n} (T_\alpha f)^p \, w \, dm_\alpha < \infty.$$

Let $\lambda > 0$ and let $\{Q_j\}$ be the Whitney decomposition of the set $\{T_\alpha f > \lambda\}$. Observe that there is a constant $N_2 > 1$ such that the cube Q_j^* with the same center as Q_j but expanded N_2 times meets the sets $\{T_\alpha f \leqq \lambda\}$. Hence it follows from Lemma 6 that if γ is as in Lemma 6, then

$$m_\alpha(\{x \in Q_j : T_\alpha f(x) > B\lambda, \mathcal{M}_\alpha f(x) \leqq \varepsilon\lambda\}) \leqq N\varepsilon^{1-\gamma} m_\alpha(Q_j^*) \leqq N\varepsilon^{1-\gamma} m_\alpha(Q_j)$$

for $0 < \varepsilon < 1$. It is well known ([3, Lemma 3]) that w satisfies (A_∞), that is, there exists $\delta > 0$ such that for given any cube Q and any measurable set $E \subset Q$

$$\frac{w(E)}{w(Q)} \leqq N \left(\frac{m_\alpha(E)}{m_\alpha(Q)} \right)^\delta,$$

where $w(E) = \int_E w \, dm_\alpha$. Hence

$$w(\{x \in Q_j : T_\alpha f(x) > B\lambda, \mathcal{M}_\alpha f(x) \leqq \varepsilon\lambda\}) \leqq N\varepsilon^{\delta(1-\gamma)} w(Q_j).$$

Summing over j, we obtain

$$w(\{T_\alpha f > B\lambda, \mathcal{M}_\alpha f \leqq \varepsilon\lambda\}) \leqq N\varepsilon^{\delta(1-\gamma)} w(\{T_\alpha f > \lambda\}),$$

which implies that

$$\int_{\mathbb{R}^n} (T_\alpha f)^p \, w \, dm_\alpha \leqq N(\varepsilon) \int_{\mathbb{R}^n} (\mathcal{M}_\alpha f)^p \, w \, dm_\alpha + N_3 \varepsilon^{\delta(1-\gamma)} \int_{\mathbb{R}^n} (T_\alpha f)^p \, w \, dm_\alpha.$$

Letting $\varepsilon > 0$ be so small that $N_3 \varepsilon^{\delta(1-\gamma)} < 1/2$, we obtain (3.6). Thus (i) follows.

For the proof of (ii) in the case when the measure μ is absolutely continuous, we refer to [14, pp. 31–35]. This additional assumption, however, can be easily dropped as follows: Without loss of generality, we may assume that μ is a nonnegative finite measure. We can find a sequence of nonnegative functions f_j converging to μ vaguely such that $\|f_j\|_{1,\alpha} \leqq \|\mu\|$. Since $K_\alpha \geqq 0$ is lower semicontinuous, it follows that if $x_j \to x$, then

$$\liminf_{j\to\infty} \int K_\alpha(x_j, y) f_j(y) \, dm_\alpha(y) \geqq \int K_\alpha(x, y) \, d\mu(y)$$

(see e.g. [10, Lemma 1]). Take a compact set E in $\{x : \int K_\alpha(x, y) \, d\mu(y) > \lambda\}$.

Then there is j such that the open set $\{x : \int K_\alpha(x,y) f_j(y)\, dm_\alpha(y) > \lambda\}$ includes E. Hence $m_\alpha(E) \leqq N \|f_j\|_{1,\alpha}/\lambda \leqq N \|\mu\|/\lambda$. Since E is arbitrary, we obtain (ii). The proof is complete.

4. Applications.

First we observe that for some sets E the quantities $m_{\alpha p}(E)$ and $R_{\alpha,p}(E)$ are comparable. The following corollary immediately follows from (2.3) and Theorem 2.

COROLLARY 1. *Let $\alpha p + d(F) < n$. For a set E we let*

$$\tilde{E} = \bigcup_{Q_k \cap E \neq \emptyset} Q_k.$$

Then

$$m_{\alpha p}(\tilde{E}) \approx R_{\alpha,p}(\tilde{E}) \approx \tilde{R}_{\alpha,p}(\tilde{E}) \approx H_{n-\alpha p}(\tilde{E}) \approx \sum_{Q_k \cap E \neq \emptyset} r_k^{n-\alpha p}.$$

In other words, for a union U of Whitney cubes, the quantities $m_{\alpha p}(U)$, $R_{\alpha,p}(U)$, $\tilde{R}_{\alpha,p}(U)$ and $H_{n-\alpha p}(U)$ are all comparable.

Theorem 1 and Corollary 1 give an estimate of the Riesz capacity of a rectangle. Let us observe that [1, Theorem 5.2 (i)] follows.

COROLLARY 2. *Let $0 < a_1 \leqq a_2 \leqq \cdots \leqq a_n$ and set $a = (a_1, \ldots, a_n)$ with $S(x,a) = \{y \in \mathsf{R}^n : |y_j - x_j| \leqq a_j, j = 1, \ldots, n\}$. Suppose $1 \leqq i \leqq n$ and $i - 1 < \alpha p < i$. Then*

$$R_{\alpha,p}(S(x,a)) \approx a_i^{i-\alpha p} a_{i+1} \cdots a_n.$$

PROOF. Let F be the affine subspace $\{x \in \mathsf{R}^n : x_{i+1} = \ldots = x_n = 0\}$ and let $\{Q_k\}$ be the Whitney decomposition of $\mathsf{R}^n \setminus F$. Then $d(F) = n - i$ and $\delta(x) = \sqrt{x_1^2 + \ldots + x_i^2}$. Let

$$x_0 = (\underbrace{0, \ldots, 0}_{i-1}, 4a_i, \underbrace{0, \ldots, 0}_{n-i})$$

and we need only consider $S = S(x_0, a)$, since $R_{\alpha,p}(\cdot)$ is translation invariant. Observe that $\delta(x) \approx a_i$ for $x \in S$. Hence if a Whitney cube Q_k meets S, then the side-length of Q_k is comparable to a_i; the number of those Whitney cubes is comparable to

$$\frac{a_{i+1}}{a_i} \cdots \frac{a_n}{a_i}. \tag{4.1}$$

Therefore Corollary 1 yields that

$$R_{\alpha,p}(S) \leqq N a_i^{n-\alpha p} \frac{a_{i+1}}{a_i} \cdots \frac{a_n}{a_i} = N a_i^{i-\alpha p} a_{i+1} \cdots a_n.$$

Thus the upper estimate follows. For the lower estimate we use the inequality $i - 1 < \alpha p$. Observe that an $n - i + 1$ dimensional cube has positive capacity. More precisely,

$$R_{\alpha,p}(\underbrace{\{0\} \times \cdots \times \{0\}}_{i-1} \times \underbrace{[-r,r] \times \cdots \times [-r,r]}_{n-i+1}) = N r^{n-\alpha p} \tag{4.2}$$

by the homogeneity. Let $S^* = S(x_0, 2a)$. Then $R_{\alpha,p}(S^*) = 2^{n-\alpha p} R_{\alpha,p}(S)$. If Q_k meets S, then $Q_k \cap S^*$ includes an $n - i + 1$ dimensional cube of side-length greater that $N a_i$. Hence (4.1), (4.2) and Theorem 1 yield that

$$R_{\alpha,p}(S) = 2^{\alpha p - n} R_{\alpha,p}(S^*) \geqq N a_i^{n-\alpha p} \frac{a_{i+1}}{a_i} \cdots \frac{a_n}{a_i} = N a_i^{i-\alpha p} a_{i+1} \cdots a_n.$$

Thus the lower estimate follows. The proof is complete.

Next we shall consider α-thin sets. Let us define the notion of α-thinness as follows.

Definition. A set E is called α-thin at ξ if

$$\sum_{j=1}^{\infty} 2^{j(n-\alpha)} C_\alpha(E \cap I_j(\xi)) < \infty,$$

where $I_j(\xi) = C(\xi, 2^{1-j}) \setminus C(\xi, 2^{-j})$.

Observe that E is α-thin at ξ if and only if there is a potential $k_\alpha * \mu \not\equiv \infty$ such that

$$\lim_{\alpha \to \xi, x \in E} \frac{k_\alpha * \mu(x)}{k_\alpha(x - \xi)} = \infty$$

(*cf.* [2, Theorem IX, 7] and [11, Theorem A]). By the aid of Theorems 1 and 2 we obtain the following corollaries immediately.

Corollary 3. *Let $\alpha + d(F) < n$ and let $\xi \in F$. Suppose E is a bounded set. Then E is α-thin at ξ if and only if*

$$\sum_k \frac{C_\alpha(E_k)}{\rho_k(\xi)^{n-\alpha}} < \infty,$$

where $E_k = E \cap Q_k$ and $\rho_k(\xi) = \operatorname{dist}(\xi, Q_k)$.

Corollary 4. *Let $\alpha + d(F) < n$ and let $\xi \in F$. Suppose E is a bounded measurable set. If E is α-thin at ξ, then*

$$\int_E \frac{\delta(x)^{-\alpha}}{|x-\xi|^{n-\alpha}}\,dx < \infty.$$

Corollary 4 may be considered to be a counterpart of [4, Theorem 2]. Let us state a result corresponding to [4, Theorem 1]. A sequence $\{x_j\}$ is said to be separated if there is a positive constant ε such that

$$|x_j - x_k| \geqq \varepsilon\delta(x_j) \quad \text{for } j \neq k.$$

It is easy to see that $\{x_j\}$ is separated if and only if the number of points x_j lying in Q_k is bounded by a positive constant independent of k.

COROLLARY 5. *Let $\alpha + d(F) < n$ and let $\xi \in F$. Suppose E is a bounded set. Let $\tilde{E} = \bigcup_{Q_k \cap E \neq \emptyset} Q_k$. Then the following statements are equivalent:*

(i) *$\tilde{E}$ is α-thin at ξ.*

(ii) $\displaystyle\int_{\tilde{E}} \frac{\delta(x)^{-\alpha}}{|x-\xi|^{n-\alpha}}\,dx < \infty.$

(iii) *E does not contain a separated sequence $\{x_j\}$ convergent to ξ such that*

$$\sum_j \left(\frac{\delta(x_j)}{|x_j - \xi|}\right)^{n-\alpha} = \infty.$$

(iv) $\displaystyle\sum_{Q_k \cap E \neq \emptyset} \left(\frac{r_k}{\rho_k(\xi)}\right)^{n-\alpha} < \infty.$

(v) *There is a measure μ supported on F such that $k_\alpha * \mu \not\equiv \infty$ and*

$$\lim_{x \to \xi, x \in E} \frac{k_\alpha * \mu(x)}{k_\alpha(x-\xi)} = \infty.$$

5. Quasiadditivity of Green energy and minimal thinness.

Let D be the half space

$$D = \{(x_1, \ldots, x_n) : x_n > 0\}$$

with the boundary $\partial D = \{(x_1, \ldots, x_n) : x_n = 0\}$. For x and y in D define the Green function

$$G(x,y) = \begin{cases} \log(|x-\bar{y}|/|x-y|) & \text{if } n = 2, \\ |x-y|^{2-n} - |x-\bar{y}|^{2-n} & \text{if } n \geqq 3, \end{cases}$$

where $\bar{y} = (y_1, \ldots, y_{n-1}, -y_n)$ is the reflection of $y = (y_1, \ldots, y_{n-1}, y_n)$. We write

$$G\mu(x) = \int_D G(x,y)\,d\mu(y).$$

Let $\delta(x) = \operatorname{dist}(x, \partial D)$. Given $E \subset D$, suppose that there exists a measure λ_E whose Green potential is $G\lambda_E = \hat{R}^E_{\delta(x)}$, where $\hat{R}^E_{\delta(x)}$ is the regularized reduced function of $\delta(x)$ on E. Let $\gamma(E) = \int G\lambda_E(x)d\lambda_E(x)$ and call it the Green energy of E.

In view of [7], we can give an alternative definition of the Green energy. Let

$$K(x, y) = \frac{G(x, y)}{\delta(x)\delta(y)}$$

and denote by the same symbol its continuous extension on $\bar{D} \times \bar{D}$. Let

$$T\mu(x) = \int_{D \cup \partial D} K(x, y)d\mu(y)$$

and write $Tf(x)$ for $T\mu(x)$ if $d\mu = fdx$. Then we have

$$\gamma(E) = \inf\{\|\mu\| : T\mu(x) \geqq 1 \text{ on } E\}.$$

Note that the kernel K is comparable to the Naïm's Θ kernel (cf. [12]). The symmetric kernel $K(x, y)$ is homogeneous of degree $-n$ and has singularity $\log\dfrac{1}{|x-y|}$ for $n = 2$ and $|x - y|^{2-n}$ for $n \geqq 3$. Hence

$$\int_D K(x, y)^p dy \leqq N\delta(x)^{-n(p-1)}$$

for $1 < p < n/(n-2)$ (cf. (3.1)). We can, therefore, regard K as the restriction of a standard kernel satisfying (H) in [8, p. 49] and can obtain

THEOREM 4. *Let m be the n-dimensional Lebesgue measure.*
(i) *Let $1 < p < \infty$. Then $\|Tf\|_p \leqq N\|f\|_p$.*
(ii) *If $\lambda > 0$, then $m(\{x \in D : |T\mu(x)| > \lambda\}) \leqq N\|\mu\|/\lambda$.*
(iii) *If E is a measurable subset of D, then $m(E) \leqq N\gamma(E)$.*

REMARK. Theorem 4 (iii) was first proved by Dahlberg [4] with the aid of sharp estimates for the characteristic constants of sets on the unit sphere. Sjögren [13] introduced the notion of convolution sets and provided an alternative proof. According to the referee, E. M. Stein gave around 1980 a proof of the weak type inequality in (ii) using singular integrals (private communication to P. Sjögren). However, this proof has not been published.

REMARK. In the same way as in Section 3 ($F = \partial D, \alpha = 0$), we can prove Theorem 4. In fact, we have the following inequality: Let $\dfrac{n-2}{2(n-1)} < \gamma < \frac{1}{2}$. Then there is a positive constant B such that if $\lambda > 0$, $0 < \varepsilon < 1$, $f \geqq 0$, and a cube $Q \subset D$ has a point x' satisfying $Tf(x') \leqq \lambda$, then

$$m(\{x \in Q : Tf(x) > B\lambda, \mathcal{M}f(x) \leqq \varepsilon\lambda\}) \leqq N\varepsilon^{1-\gamma} m(Q),$$

where $\mathcal{M}f(x)$ is the usual maximal function defined by

$$\mathcal{M}f(x) = \sup_Q \frac{1}{m(Q)} \int_Q |f|\, dy$$

with the supremum taken over all cubes $Q \subset D$ containing x. This approach is applicable to a general $C^{1,\alpha}$-domain D. Necessary estimates for the Green kernel were given by Widman [15]. Thus the results below will hold for a $C^{1,\alpha}$-domain D. In particular, [4, Theorems 1 and 2] follows.

Let $\{Q_k\}$ be the Whitney decomposition of D. By $c(E)$ we denote the logarithmic capacity of E if $n = 2$, and the Newtonian capacity of E if $n \geqq 3$. If a set E is included in a Whitney cube Q_k, then the Green energy of E can be estimated by the ordinary capacity $c(E)$ ([5, Lemma 3]).

LEMMA 7. *Let Q_k be a Whitney cube in D and let E be a subset of Q_k. Then*

$$\gamma(E) \approx \begin{cases} t_k^2(\log 4t_k/c(E)))^{-1} & \text{if } n = 2, \\ t_k^2 c(E) & \text{if } n \geqq 3, \end{cases}$$

where $t_k = \operatorname{dist}(Q_k, \partial D)$.

Using Theorem 4 (iii), Lemma 7 and the homogeneity $\gamma(rE) = r^n \gamma(E)$, we can prove the following theorem in the same way as in Section 2.

THEOREM 5. *Let $\{Q_k\}$ be the Whitney decomposition of D. For $E \subset D$ we set $\Gamma(E) = \Sigma_k \gamma(E \cap Q_k)$. Then $\Gamma(E) \approx \gamma(E)$.*

As an immediate consequence, we have

COROLLARY 6. *Let $\xi \in \partial D$. Suppose E is a bounded set of D. Then E is minimally thin at ξ in D if and only if*

$$\sum_k t_k^2 \frac{(\log(4t_k/c(E_k)))^{-1}}{\rho_k(\xi)^2} < \infty, \quad \text{if } n = 2,$$

$$\sum_k t_k^2 \frac{c(E_k)}{\rho_k(\xi)^n} < \infty, \quad \text{if } n \geqq 3,$$

where $E_k = E \cap Q_k$ and $\rho_k(\xi) = \operatorname{dist}(\xi, Q_k)$.

REMARK. Slightly weaker forms of Theorem 5 and Corollary 6 were first given by Essén [5]. In [6, Section 1], he has given a proof of Corollary 6 based on the weak L^1-estimates of Sjögren [13].

COROLLARY 7. *Let $\xi \in \partial D$. Suppose E is a bounded measurable set. If E is minimally thin at ξ, then*

$$\int_E \frac{dx}{|x-\xi|^n} < \infty.$$

Let $\tilde{E} = \bigcup_{Q_k \cap E \neq \emptyset} Q_k$ as in Corollary 5. We infer from Harnack inequality that $\tilde{E}$ is not minimally thin at ξ if and only if E determines the point measure at ξ in the notation of [4]. By $P(\xi, x)$ we denote the Poisson kernel for D at $\xi \in \partial D$ and $x \in D$. We write $P(\mu, x)$ for the Poisson integral

$$\int_{\partial D} P(\xi, x)\, d\mu(\xi).$$

Observe that if μ is a measure supported by ∂D, then $\delta(x)\, T\mu(x)$ coincides with the Poisson integral $P(\mu, x)$ up to a positive multiplicative constant. Hence we have

COROLLARY 8 ([4, Theorem 1]). *Let $\xi \in \partial D$. Suppose E is a bounded set. Let $\tilde{E} = \bigcup_{Q_k \cap E \neq \emptyset} Q_k$. Then the following statements are equivalent:*

(i) *E does not determine the point measure at ξ.*

(ii) *$\tilde{E}$ is minimally thin at ξ.*

(iii) $\displaystyle\int_{\tilde{E}} \frac{dx}{|x-\xi|^n} < \infty.$

(iv) *E does not contain a separated sequence $\{x_j\}$ convergent to ξ such that*

$$\sum_j \left(\frac{\delta(x_j)}{|x_j - \xi|} \right)^n = \infty.$$

(v)
$$\sum_{Q_k \cap E \neq \emptyset} \left(\frac{r_k}{\rho_k(\xi)} \right)^n < \infty.$$

(vi) *There is a measure μ supported on ∂D such that $P(\mu, \cdot) \not\equiv \infty$ and*

$$\lim_{x \to \xi, x \in E} \frac{P(\mu, x)}{P(\xi, x)} = \infty.$$

ADDED IN PROOF. On the occasion of ICPT91 Professor Maz'ya informed that his paper On *Beurling's theorem for the minimum principle for positive harmonic functions,* (in Russian), Zapiski Naucnyh Seminarov LOMI 30 (1972), 76–90, was precedent to [4].

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