MATH. SCAND. 69 (1991), 45–56

OSCILLATORY INTEGRALS WITH POLYNOMIAL PHASE

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§1. Introduction.

Let $\mathscr{P}_N$ be the space of real-valued polynomials on R of degree at most N. This paper is concerned with uniform estimates for integrals of the form

$$\int_a^b e^{ip(x)}\psi(x)dx,\ p \in \mathscr{P}_N$$

when the weight $\psi(x)$ is a power of a derivative of p. Here is an easy example: since

$$\left|\int_a^b e^{ip(x)}p'(x)dx\right| \leqq 2$$

for any continuously differentiable p such that p' has constant sign on $[a,b]$, it follows that

$$\left|\int_a^b e^{ip(x)}\,|p'(x)|\,dx\right| \leqq 2N \text{ if } p \in \mathscr{P}_N \text{ and } a < b. \tag{1}$$

The form of this estimate is prototypical for our results, Theorems 1 and 2 below.

THEOREM 1. *If N and n are positive integers, there is $C(N,n)$ such that*

$$\left|\int_a^b e^{ip(x)}\,|p^{(n)}(x)|^{1/n}\,dx\right| \leqq C(N,n) \text{ if } p \in \mathscr{P}_N \text{ and } a < b.$$

Received September 6, 1990.

THEOREM 2. *If N is a positive integer and $n = 1$ or 2, there is $C(N, n)$ such that*

$$\left|\int_a^b e^{ip(x)}\,|p^{(n)}(x)|^{1/n+is}\,dx\right| \leqq C(N,n)(1+|s|)^{1/n} \text{ if } p\in\mathscr{P}_N,\ a<b, \text{ and } s\in\mathsf{R}.$$

COMMENTS:

(a) If $N = n$, our results are direct consequences of van der Corput's lemma, which is the case $\psi(x) \equiv 1$ of the following result.

LEMMA 0. ([S], p. 311) *For $a \leqq x \leqq b$ assume that $\varphi(x)$ and $\psi(x)$ are smooth, that $\varphi(x)$ is real-valued, and that for some positive integer n we have $|\varphi^{(n)}(x)| \geqq 1$. If $n = 1$ assume additionally that $\varphi'(x)$ is monotonic. Then*

$$\left|\int_a^b e^{ir\varphi(x)}\psi(x)dx\right| \leqq C(n)\,|r|^{-1/n}\left[|\psi(b)| + \int_a^b |\psi'(x)|\,dx\right] \text{ for } r\in\mathsf{R}.$$

(b) These results are vaguely analogous to those of [C] concerning multidimensional oscillatory integrals damped with a power of the curvature. The proof of Theorem 1 depends on an idea present in that paper.

(c) Our interest in results like these stems from the problem of embedding certain measures on curves in R^k into analytic families of distributions. Here is an example in the case $k = 2$: suppose $p\in\mathscr{P}_N$ and $a < b$. Following [D] we define a measure $d\sigma$ by

$$\int \varphi\, d\sigma = \int_a^b \varphi(x, p(x))\,|p''(x)|^{1/3}\,dx$$

and an analytic family of distributions $d\sigma_z$ by

$$\int \varphi\, d\sigma_z = \frac{\pi^{(z-1)/2}}{\Gamma(z/2)}\int_a^b\int_{-\infty}^{\infty} \varphi(x,y)\left[\frac{|y-p(x)|}{|p''(x)|^{1/3}}\right]^{-1+z} dy\,dx.$$

Here $\varphi\in C_0^\infty(\mathsf{R}^2)$, say. If $z = -\frac{1}{2} + i\gamma$, then Theorem 2 in the case $n = 2$ combines with the calculation of $\hat{\sigma}_z$ in [D] to show that $\|\hat{\sigma}_z\|_\infty \leqq C(z, N)$. Thus the proof of Theorem 1 in [D] yields the inequality

$$\left|\int_{\mathsf{R}^2} (f_1 * f_2)\,d\sigma\right| \leqq C(N)\,\|f_1\|_{3/2}\|f_2\|_{3/2}$$

for $f_1, f_2\in L^{3/2}(\mathsf{R}^2)$.

(d) There is no finite C such that the inequality

$$\left|\int_a^b e^{ip(x)}\,|p''(x)|^{1/2}\,dx\right| \leqq C$$

holds for, say, all twice continuously differentiable functions p with p' and p'' of constant sign on $[a, b]$. (Take $p(x) = \log\log x$.) Thus there is no proof of Theorem 1 analogous to the proof of (1) given above.

(e) Much work on oscillatory integrals is, like Lemma 0, concerned with the decay as $r \to \infty$ of integrals

$$\int e^{ir\phi(x)}\,\psi(x)dx.$$

Theorems 1 and 2 can be cast in this form simply by replacing p with rp and then factoring $|r|^{1/n}$ from the integral. For example, Theorem 1 yields the estimate

$$\left|\int_a^b e^{irp(x)}\,|p^{(n)}(x)|^{1/n}dx\right| \leqq r^{-1/n}C(N,n) \text{ if } p \in \mathscr{P}(N),\ a < b, \text{ and } r > 0.$$

Letting $a = 0$, $b = 1$, and $p(x) = x^N$ shows that such an estimate cannot be substantially improved.

(f) We conjecture that Theorem 2 is true for any $n \in \mathbf{N}$.

§2. Proof of Theorem 1.

Theorem 1 is a consequence of two elementary lemmas, the first of which we give in a little more generality than we require.

LEMMA 1. *Suppose ψ is a real-valued continuously differentiable function on a closed interval I such that ψ and ψ' are of constant sign on I. Then*

$$\left|\int_I e^{irx}\,\psi(x)dx\right| \leqq 5 \sup\left\{\left|\int_J \psi\right| : J \text{ is a subinterval of } I \text{ with length} \leqq \frac{1}{|r|}\right\}.$$

PROOF. Write $I = [a, b]$. Without loss of generality we may assume that $1/|r| \leqq b - a$ and that ψ, $\psi' \geqq 0$ on I. An integration by parts shows that

$$\left|\int_a^{b-1/|r|} e^{irx}\psi(x)dx\right| \leqq \psi(b - 1/|r|)\left|\int_a^{b-1/|r|} e^{irt}dt\right| + \left|\int_a^{b-1/|r|}\int_a^x e^{irt}dt\,\psi'(x)dx\right|.$$

Since ψ' is nonnegative on I and because

$$\left|\int_a^x e^{irt}dt\right| \leqq 2/|r|,$$

this is dominated by

$$\frac{2}{|r|}\psi(b - 1/|r|) + \frac{2}{|r|}[\psi(b - 1/|r|) - \psi(a)].$$

But $\psi(a) \geqq 0$ so the last sum does not exceed

$$\frac{4}{|r|}\psi(b - 1/|r|).$$

Now the fact that ψ is increasing on I gives

$$\left|\int_a^{b-1/|r|} e^{irx}\psi(x)dx\right| \leqq 4 \int_{b-1/|r|}^{b} \psi(x)dx.$$

The estimate

$$\left|\int_{b-1/|r|}^{b} e^{irx}\psi(x)dx\right| \leqq \int_{b-1/|r|}^{b} \psi(x)dx$$

thus completes the proof.

The next result is analogous to Lemma 4.2 of [C].

LEMMA 2. *There is a positive constant $C(N,n)$ such that*

$$\int_a^b |p^{(n)}(t)|^{1/n}dt \leqq C(N,n)\,\|p\|_{L^\infty(a,b)}^{1/n} \text{ for } p\in\mathscr{P}_N \text{ and } a < b.$$

PROOF. Since linear operators on finite-dimensional normed spaces are bounded, there is $C(N,n)$ such that

$$\|p^{(n)}\|_{L^\infty(0,1)} \leqq C(N,n)\|p\|_{L^\infty(0,1)} \text{ for } p\in\mathscr{P}_N.$$

Thus

$$\int_0^1 |p^{(n)}(t)|^{1/n}dt \leqq C(N,n)\|p\|_{L^\infty(0,1)}^{1/n} \text{ for } p\in\mathscr{P}_N.$$

A linear change of variable completes the proof.

It is enough to prove Theorem 1 when p' is of constant sign on $[a, b]$. Then there is a positive function $\psi(x)$ $(= |p^{(n)}(p^{-1}(x))/p'(p^{-1}(x))|)$ and an interval $\tilde{I}$ such that

$$\int_a^b f(p(t))\,|p^{(n)}(t)|^{1/n}\,dt = \int_{\tilde{I}} f\psi$$

for all reasonable functions f on R. A computation shows that (since $p \in \mathscr{P}_N$) there is some $M = M(N, n)$ such that ψ' can have at most M zeroes on $\tilde{I}$. Thus it is enough to show that

$$\left|\int_I e^{ix}\psi(x)dx\right| \leqq C(N, n)$$

if ψ' is of constant sign on the subinterval I of $\tilde{I}$. For such an I, Lemma 1 gives

$$\left|\int_I e^{ix}\psi(x)dx\right| \leqq 5\sup\left\{\int_J \psi(x)\mathrm{dx}\colon J \subseteq I, \text{length } (J) \leqq 1\right\} =$$

$$5\sup\left\{\int_e^f |p^{(n)}(t)|^{1/n}dt\colon\ a \leqq e < f \leqq b, |p(e) - p(f)| \leqq 1\right\}.$$

Now if $a \leqq e < f \leqq b$, the monotonicity of p on $[a, b]$ shows that $\|p - p(f)\|_{L^\infty(e,f)} \leqq 1$. Thus Lemma 2, applied to the polynomial $p(x) - p(f)$, yields

$$\int_e^f |p^{(n)}(t)|^{1/n}\,dt \leqq C(N, n).$$

This completes the proof of Theorem 1.

§3. Proof of Theorem 2.

Theorem 2 depends on a technical lemma.

LEMMA 3. *Fix N. There are positive constants $K = K(N)$ and $L = L(N)$ such that if*

$$r(x) = \prod_{j=1}^{J_1}(x - a_j)\prod_{j=J_1+1}^{J_2}[(x - a_j)^2 + b_j] \doteq \prod_{j=1}^{J_2} g_j(x)$$

is a monic polynomial of degree not exceeding N *with the* a_j*'s distinct and each* $b_j > 0$, *then there exists a collection* $\{I_l\}_{l=1}^{L_1}$ *of pairwise disjoint subintervals of* $\mathbf{R}$ *with* $L_1 \leqq L$ *satisfying*

$$\int_{\mathbf{R}\sim\cup I_l} \left|\frac{r'}{r}\right| \leqq K$$

and such that for each l *there are* $C = C(l) \in (0, \infty)$, $j = j(l) \in \{1, 2, \ldots, J_2\}$, *and a nonnegative integer* $t = t(l)$ *with*

$$\frac{C}{K}|x - a_j|^t \leqq |r(x)| \leqq KC|x - a_j|^t, \; x \in I_l$$

and

$$\frac{1}{K|x - a_j|} \leqq \left|\frac{r'(x)}{r(x)}\right| \leqq \frac{K}{|x - a_j|}, x \in I_l.$$

PROOF. Given r we write

$$\frac{r'}{r} = \sum_{j=1}^{J_2} f_j,$$

where each $f_j(x)$ is either

$$\frac{1}{x - a_j}$$

(in which case we will say that f_j is of type I) or

$$\frac{2(x - a_j)}{(x - a_j)^2 + b_j}$$

(type II). The proof is a consequence of the three observations, Steps I-III, below. In what follows K and L will denote constants, not necessarily the same at each occurence, depending only on N.

Step I. There is L such that given r, $\mathbf{R}$ can be written as the disjoint union of at most L subintervals I_l with the property that for each I_l there is a $j(l)$ with

$$|f_j(x)| \leqq |f_{j(l)}(x)| \text{ if } x \in I_l, 1 \leqq j \leqq J_2.$$

PROOF OF STEP I. This is a consequence of the facts that there are at most N functions f_j and that each equation $|f_{j_1}(x)| = |f_{j_2}(x)|$ $(j_1 \neq j_2)$ can have at most six solutions.

Step II. There exist K and L such that the following holds: given an interval I and an index j_0 such that

$$|f_j(x)| \leqq |f_{j_0}(x)| \text{ for } x \in I, 1 \leqq j \leqq J_2,$$

there is a subset $\tilde{I}$ of I with

$$\int_{\tilde{I}} \left|\frac{r'}{r}\right| \leqq K, \tag{2}$$

with $I \sim \tilde{I}$ the disjoint union of at most L intervals, and such that

$$\frac{1}{4|x - a_{j_0}|} \leqq \left|\frac{r'(x)}{r(x)}\right| \leqq \frac{2N}{|x - a_{j_0}|} \quad \text{if } x \in I \sim \tilde{I}. \tag{3}$$

PROOF OF STEP II. For ease of notation assume $j_0 = 1$. Define

$$T = \{x \in I\colon \text{ for each } j \neq 1 \text{ either } |f_j(x)| \leqq \frac{|f_1(x)|}{2N} \text{ or } f_j(x) \cdot f_1(x) \geqq 0\}.$$

Since

$$\frac{r'(x)}{r(x)} = \sum_{j=1}^{J_2} f_j(x) \text{ and } |f_j(x)| \leqq |f_1(x)| \leqq \frac{2}{|x - a_1|} \quad \text{if } x \in I,$$

we have

$$\frac{|f_1(x)|}{2} \leqq \left|\frac{r'(x)}{r(x)}\right| \leqq N|f_1(x)| \leqq \frac{2N}{|x - a_1|} \quad \text{if } x \in T.$$

If f_1 is of type I, define $\tilde{I}$ by $I \sim \tilde{I} = T$, while if f_1 is of type II, set $I \sim \tilde{I} = T \sim (a_1 - \sqrt{b_1}, a_1 + \sqrt{b_1})$. Reasoning similar to that used to establish Step I shows that there is L (depending only on N) such that $I \sim \tilde{I}$ is the disjoint union of at most L intervals. If f_1 is of type II, then

$$\frac{1}{2|x - a_1|} \leqq |f_1(x)| \quad \text{if } x \notin (a_1 - \sqrt{b_1}, a_1 + \sqrt{b_1}),$$

and so (3) holds whether f_1 is of type I or II. We will complete Step II by showing that

$$\int_{I \sim T} \left|\frac{r'}{r}\right| \leqq K.$$

With the calculation

$$\int_{a_1 - \sqrt{b_1}}^{a_1 + \sqrt{b_1}} |f_1| = 2 \ln 2$$

if f_1 is of type II and the fact that then

$$\tilde{I} \subset (I \sim T) \cup (a_1 - \sqrt{b_1}, a_1 + \sqrt{b_1}),$$

(2) will follow from

$$\left|\frac{r'(x)}{r(x)}\right| \leqq N\,|f_1(x)| \text{ if } x \in I.$$

Now $I \sim T \subseteq \bigcup U_j$, where

$$U_j = \left\{x \in I : |f_j(x)| > \frac{|f_1(x)|}{2N} \text{ and } f_j(x)\cdot f_1(x) < 0\right\}.$$

Define $\tilde{U}_j$ to be

$$U_j \sim \bigcup_i (a_i - \sqrt{b_i}, a_i + \sqrt{b_i})$$

where the union is over $\{i \in \{1,j\}$: f_i is of type II$\}$. Since

$$\int_{a_i - \sqrt{b_i}}^{a_i + \sqrt{b_i}} |f_i| = 2 \ln 2$$

if f_i is of type II, since

$$\left|\frac{r'}{r}\right| \leqq N\,|f_1| \leqq 2N^2\,|f_j|$$

on U_j, and since

$$|f_1(x) \leqq \frac{2}{|x - a_1|},$$

it suffices to show that

(4) $$\int_{\tilde{U}_j} \frac{dx}{|x - a_1|} \leqq K.$$

If $x \in \tilde{U}_j$ then

(5) $$\frac{1}{2\,|x - a_1|} \leqq |f_1(x)| \leqq 2N\,|f_j(x)| \leqq \frac{4N}{|x - a_j|}.$$

Assume for the moment that $a_1 < a_j$. Since $f_j(x)\cdot f_1(x) < 0$ on $\tilde{U}_j$, $\tilde{U}_j$ is contained in (a_1, a_j). Now if $x \in \tilde{U}_j$ (5) implies that $a_j - x \leqq 8N(x - a_1)$, so $a_j + 8N a_1 \leqq (8N + 1)x$, and finally $(a_j - a_1)/(8N + 1) \leqq x - a_1$. Since also $x - a_1 \leqq a_j - a_1$

if $x \in \tilde{U}_j$, (4) is true with $K = \ln(8N + 1)$. If $a_j < a_1$, (4) follows similarly. Thus the proof of Step II is complete.

Step III. There are K and L such that the following holds: suppose given an interval I and an index j_0 such that

$$|f_j(x)| \leqq |f_{j_0}(x)| \quad \text{for} \quad x \in I, 1 \leqq j \leqq J_2.$$

Then I can be written as the union of at most L disjoint intervals I_l such that for each l there are $C = C(l) \in (0, \infty)$ and $t = t(l) \in \mathbf{N}$ with

$$\frac{C}{K}|x - a_{j_0}|^t \leqq |r(x)| \leqq KC|x - a_{j_0}|^t, x \in I_l.$$

PROOF OF STEP III. Assume $j_0 = 1$. With the g_j as in the statement of Lemma 3 and since $J_2 \leqq N$, it is enough to show the following: there exist absolute constants P and B such that given g_j we can write I as the union of at most P subintervals I_p and on each I_p either

(6) $$\text{there is } C \in (0, \infty) \text{ with } \frac{C}{B} \leqq |g_j(x)| \leqq BC, x \in I_p,$$

or

(7) $$\frac{|x - a_1|}{B} \leqq |g_j(x)| \leqq B|x - a_1|, x \in I_p,$$

or

$$\frac{(x - a_1)^2}{B} \leqq g_j(x) \leqq B(x - a_1)^2, x \in I_p.$$

The proof of the next lemma is elementary.

LEMMA. *Suppose* $x, a_1, a_j \in \mathbf{R}$ *and* $|x - a_1| \leqq 4|x - a_j|$.
(a) *If* $|a_1 - a_j|/2 \leqq |x - a_1|$, *then* $|x - a_1|/4 \leqq |x - a_j| \leqq 3|x - a_1|$.
(b) *If* $|a_1 - a_j|/2 > |x - a_1|$, *then* $|a_j - a_1|/2 \leqq |x - a_j| \leqq 3|a_j - a_1|/2$.

Now if f_j is of type I, then $|f_j| \leqq |f_1|$ on I implies $|x - a_1| \leqq 2|x - a_j|$ if $x \in I$. Thus the Lemma and the fact that $g_j(x) = x - a_j$ give subintervals of I on which (6) or (7) hold. If f_j is of type II, then in the interval $(a_j - \sqrt{b_j}, a_j + \sqrt{b_j})$ we have $b_j \leqq g_j \leqq 2b_j$. If $x \in I \sim (a_j - \sqrt{b_j}, a_j + \sqrt{b_j})$ then

$$\frac{1}{2|x - a_j|} \leqq |f_j(x)| \leqq |f_1(x) \leqq \frac{2}{|x - a_1|}$$

and so $|x - a_1| \leqq 4|x - a_j|$. Since then $(x - a_j)^2 \leqq g_j(x) \leqq 2(x - a_j)^2$, the lemma gives

$$\frac{(x-a_1)^2}{16} \leqq (x-a_j)^2 \leqq g_j(x) \leqq 2(x-a_j)^2 \leqq 18(x-a_1)^2 \text{ if } \frac{|a_1 - a_j|}{2} \leqq |x - a_1|,$$

while

$$\frac{(a_1-a_j)^2}{4} \leqq (x-a_j)^2 \leqq g_j(x) \leqq 2(x-a_j)^2 \leqq \frac{9}{2}(a_1-a_j)^2 \text{ if } \frac{|a_1 - a_j|}{2} > |x - a_1|.$$

This completes the proofs of Step III and Lemma 3.

We begin the proof of Theorem 2 with some reductions: first, a scaling argument shows that we may assume $p'(x)$ to be monic. Then an approximation argument shows that it is enough to prove Theorem 2 under the additional assumption that $r(x) \doteq p'(x)$ meets the other hypotheses of Lemma 3. Finally, it will suffice to show that, for such $p \in \mathscr{P}_N$, the conclusion of Theorem 2 holds if p', p'', and

$$\left|\frac{p''}{(p')^2}\right| - \frac{1}{4(1+|s|)}$$

are of constant sign on $I \doteq (a, b)$.

Case I. $\left|\dfrac{p''}{(p')^2}\right| \leqq \dfrac{1}{4(1+|s|)}$ on I.

After making the change of variable $u = p(x)$ we have to estimate an integral of the form

$$\int_J e^{i(u + ns \log|p'(p^{-1}(u))|)} \left|\frac{p^{(n)}(p^{-1}(u))}{p'(p^{-1}(u))^n}\right|^{1/n + is} du$$

where the derivative

$$1 + \frac{nsp''(p^{-1}(u))}{p'(p^{-1}(u))^2}$$

of the phase function has absolute value exceeding $\frac{1}{2}$ on J. If $n = 1$ an appeal to van der Corput's lemma (see Comment (a) at the beginning of the paper) will now suffice. If $n = 2$, let $C(N)$ stand for a constant depending only on N and note that

$$\int_J \left|\frac{d}{du}\left|\frac{p''(p^{-1}(u))}{p'(p^{-1}(u))^2}\right|^{1/2 + is}\right| du =$$

$$|1/2 + is| \int_J \left| \frac{d}{du} \left| \frac{p''(p^{-1}(u))}{p'(p^{-1}(u))^2} \right|^{1/2} \right| du \leqq$$

$$C(N)|1/2 + is| \sup_{u \in J} \left| \frac{p''(p^{-1}(u))}{p'(p^{-1}(u))^2} \right|^{1/2} \leqq$$

$$C(N)(1 + |s|)^{1/2}.$$

Here the first inequality follows from the fact that, since $p \in \mathscr{P}_N$,

$$\frac{d}{du} \left| \frac{p''(p^{-1}(u))}{p'(p^{-1}(u))^2} \right|^{1/2}$$

will have at most $C(N)$ sign changes on J. (The second inequality is a consequence of the inequality which defines Case I.) Now Case I follows from Lemma 0.

Case II. $\dfrac{1}{4(1 + |s|)} \leqq \left| \dfrac{p''}{(p')^2} \right|$ on I.

Take $r = p'$ in Lemma 3 and let the intervals I_l be as in that lemma, so that

$$\int_{\mathbf{R} \sim \cup I_l} \left| \frac{p''}{p'} \right| \leqq K. \tag{8}$$

Put $I' = I \sim \cup I_l$ and $I'' = I \cap \cup I_l$. Then

$$\int_{I'} |p'| \leqq 4(1 + |s|) \int_{I'} \left| \frac{p''}{p'} \right| \leqq 4K(1 + |s|),$$

and

$$\int_{I'} |p''|^{1/2} \leqq 2(1 + |s|)^{1/2} \int_{I'} \left| \frac{p''}{p'} \right| \leqq 2K(1 + |s|)^{1/2},$$

both by the Case II assumption and (8).

For the integrals over I'' it is enough to estimate an integral of $|p'|$ or $|p''|^{1/2}$ over one of the intervals $I \cap I_l$. It follows from Lemma 3 that there are $a \in \mathbf{R}$, $C \in (0, \infty)$, and a nonnegative integer t such that for $x \in I_l$ we have

$$\frac{C}{K} |x - a|^t \leqq |p'(x)| \leqq CK|x - a|^t,$$

$$\frac{1}{K|x-a|} \leqq \left|\frac{p''(x)}{p'(x)}\right| \leqq \frac{K}{|x-a|},$$

and so

$$\left|\frac{p''(x)}{p'(x)^2}\right| \leqq \frac{K^2}{C|x-a|^{t+1}},$$

and

$$|p''(x)| \leqq CK^2|x-a|^{t-1}.$$

Thus

$$\int_{I \cap I_l} |p'| \leqq CK \int_{\{1/4(1+|s|) \leqq K^2/C|x-a|^{t+1}\}} |x-a|^t \, dx \leqq \frac{8K^3(1+|s|)}{t+1}$$

and

$$\int_{I \cap I_l} |p''|^{1/2} \leqq C^{1/2}K \int_{\{1/4(1+|s|) \leqq K^2/C|x-a|^{t+1}\}} |x-a|^{(t-1)/2} \, dx \leqq \frac{8K^2(1+|s|)^{1/2}}{t+1}$$

This completes the proof of Theorem 2.

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