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ON RATIONAL SOLUTIONS OF YANG-BAXTER EQUATION FOR $\mathfrak{sl}(n)$

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Abstract.

In 1984 Drinfeld conjectured that any rational solution $X(u,v)$ of the classical Yang-Baxter equation (CYBE)′ with X taking values in a simple complex Lie algebra $\mathfrak{g}$ is equivalent to one of the form $X(u,v) = C_2/(u-v) + r(u,v)$, where C_2 is the quadratic Casimir element, r is a polynmial in u, v, and $\deg_u r = \deg_v r \leqq 1$. In this paper I will prove this conjecture for $\mathfrak{g} = \mathfrak{sl}(n)$ and reduce the problem of listing "nontrivial" (i.e. nonequivalent to $C_2/(u-v)$ + const) solutions of CYBE to classification of certain quasi-Frobenius subalgebras of $\mathfrak{g}$. There are given all "nontrivial" rational solutions for $\mathfrak{sl}(2)$, $\mathfrak{sl}(3)$, $\mathfrak{sl}(4)$ and several series of examples in general case.

Introduction.

In what follows let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra over the field $\mathbf{C}$ of complex numbers, $X\colon \mathbf{C} \to \mathfrak{g} \otimes \mathfrak{g}$ a function. Solutions of the classical Yang-Baxter equation

$$\text{CYBE}\quad [X^{12}(u_1-u_2), X^{13}(u_1-u_3)] + [X^{12}(u_1-u_2), X^{23}(u_2-u_3)] + {} + [X^{13}(u_1-u_3), X^{23}(u_2-u_3)] = 0$$

where for $X = \sum a_i \otimes b_i \in \mathfrak{g} \otimes \mathfrak{g}$ we set $X^{12} = X \otimes 1$, $X^{13} = \sum a_i \otimes 1 \otimes b_i$, etc. are considered modulo equivalence relations

1) $X \sim cX$, for $c \in \mathbf{C}\backslash\{0\}$;
2) $X(u-v) \sim (\phi(u) \otimes \phi(v))X(u-v)$, where $\phi(u) \in \operatorname{Aut}(\mathfrak{g})$.

A solution X is called *nondegenerate* if it satisfies any of the following equivalent (as proved in [BD1]) conditions

A) the determinant of the matrix formed by the coordinates of the tensor $X(u)$ does not vanish identically;

B) $X(u)$ has at least one pole and there is no proper subalgebra $\mathfrak{h} \subset \mathfrak{g}$ such that $X(u) \in \mathfrak{h} \otimes \mathfrak{h}$ for all u;

C) $X(u)$ has a 1st order pole at $u = 0$ and $\operatorname{Res} X(u) = C_2$, the quadratic Casimir element.

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Belavin and Drinfeld [BD1] proved that the poles of a nondegenerate solution of CYBE form a discrete subgroup $\Gamma \subset \mathbf{C}$ and listed all solution for rk $\Gamma = 2$ (elliptic) and rk $\Gamma = 1$ (trigonometric). For rk $\Gamma = 0$ (rational solution) they gave several series of examples associated with Frobenius subalgebras of $\mathfrak{g}$ and provided with arguments in favour of that there are too many rational solutions to try to list them, see also a more detailed and physicists-oriented exposition [BD2].

About a year later, however, Drinfeld has found all solutions for $\mathfrak{sl}(2)$ and made the following.

CONJECTURE. (Drinfeld, 1984). *If* $X(u,v)$ *is a rational solution of* CYBE′ (*see below*), i.e.
$X(u,v) = C_2/(u-v) + r(u,v)$, *where* r *is a polynomial in* u, v *then* $\deg_u r = \deg_v r \leqq 1$.

It seemed that there was some hope after all.

In this paper I will prove this conjecture and reduce the problem of listing "nontrivial", i.e. nonequivalent to $C_2/(u-v)$ + const, solutions of CYBE′ to classification of so-called isotropic orders of $\mathfrak{g} = \mathfrak{sl}(n)$. They, in turn, are related with quasi-Frobenius subalgebras in $\mathfrak{sl}(n)$.

The problem of finding rational solutions has been raised in [BD1]. The problem we will solve here looks somewhat different: we will consider functions $X\colon \mathbf{C}^2 \to \mathfrak{g} \otimes \mathfrak{g}$ such that

$$\text{(CYBE)}' \quad [X^{12}(u_1,u_2), X^{13}(u_1,u_3)] + [X^{12}(u_1,u_2), X^{23}(u_2,u_3)] + \\ + [X^{13}(u_1,u_3), X^{23}(u_2,u_3)] = 0 \qquad X^{12}(u,v) = -X^{21}(v,u)$$

and a solution will be called *rational* if it is of the form (here $\mathfrak{g}[u] = \mathfrak{g} \otimes \mathbf{C}[u]$):

$$X = +C_2/(u-v) + r(u,v), \quad \text{where} \quad r(u,v) \in \mathfrak{g}[u] \otimes \mathfrak{g}[v]$$

As is shown in [BD3], any solution $X(u,v)$ of (CYBE′) is gauge equivalent to a one which depends on $u - v$ and it is subject to a direct verification that the gauge transformation does not lead out of the class of rational solutions.

CONVENTION. In the main text we list all solutions up to "trivial", constant, one without specifically mentioning this anymore.

The constant solutions, however, are of independent interest as demonstrated for example by Gurevich [G1], [G2] and therefore we list and discuss them in Chapter 2.

REMARKS. 1) As has been verified by V. Drinfeld and A. Panov (unpublished) for simple compact Lie algebras over $\mathbf{R}$ there exists only the trivial maximal order, hence there are no nonconstant solutions.

2) The results of this paper and its approach seem to be of interest from superalgebra point of view when the solutions of CYBE are only trigonometric (listed in [LS]) and rational ([L]). Here, one more reason for interest from supermanifold point of view is provided by Sauvage Lemma. It describes bunddles over P^1 in superizable terms, whereas there is no explicit description of bundles over $\mathsf{P}^{1,n}$ even for $n = 1$.

3) The results of this paper were announced in [$S1$]–[$S4$].

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1. Rational solutions of CYBE ↔ isotropic orders.

We need some notations. Denote by $\mathfrak{g}^{(1)} = \mathfrak{g} \otimes \mathsf{C}[u^{-1}, u]$ the loop algebra (indeed, set $u = \exp(i\phi)$, where ϕ is the angle parameter on the circle, then $\mathfrak{g}^{(1)}$ is the space of loops – maps of the circle to $\mathfrak{g}$ – expandable into finite Fourier series) with the nondegenerate ad-invariant inner product $(x, y) = \operatorname{Res}\operatorname{tr}(adx\, ady)$.

Set $\mathfrak{D} = \mathsf{C}[[u^{-1}]]$, the ring of formal power series in $u^{-1}, K = \mathsf{C}((u^{-1}))$, the field of quotients of $\mathfrak{D}$.

Set $\mathfrak{g}[u] = \mathfrak{g} \otimes \mathsf{C}[u]$, $\mathfrak{g}[[u^{-1}]] = \mathfrak{g} \otimes \mathfrak{D}$, $\mathfrak{g}((u^{-1})) = \mathfrak{g} \otimes K$.

The inner product on $\mathfrak{g}^{(1)}$ is naturally extended to $\mathfrak{g}((u^{-1}))$.

1.1. THEOREM. *There is a natural one-to-one correspondence between rational solutions of* CYBE′ *and subspaces* $W \subset \mathfrak{g}((u^{-1}))$ *such that*

1) *W is an subalgebra in $\mathfrak{g}((u^{-1}))$ such that $W \supset u^{-N}\mathfrak{g}[[u^{-1}]]$ for some $N > 0$;*

2) $W \otimes \mathfrak{g}[u] = \mathfrak{g}((u^{-1}))$;

3) *W is Lagrangian subspace with respect to the inner product of* $\mathfrak{g}((u^{-1}))$, i.e., $W = W^{\perp}$.

A C-subalgebra $W \subset \mathfrak{g}((u^{-1}))$ such that $u^K\mathfrak{g}[[u^{-1}]] \supset W \supset u^{-N}\mathfrak{g}[[u^{-1}]]$ for some K, N is called an *order* in $\mathfrak{g}((u^{-1}))$. It follows from 1) and 3) that W from Theorem 1.1 is an order.

REMARK. The term “order” is suggested by Drinfeld due to the likeness of valuation fields K and Q_p and because for Q_p similar objects are called orders, cf. [BSh]. An order in a finite dimensional algebra over Q_p is a Z_p-subalgebra which is open and compact.

1.2. THEOREM. *Let X_1 and X_2 be rational solutions of* CYBE′, *W_1 and W_2 the corresponding orders in $\mathfrak{g}((u^{-1}))$. Then for a polynomial $\sigma(u)$:* $\mathsf{C} \to \operatorname{Aut}\mathfrak{g}$:

$$X_1 = (\sigma(u) \otimes \sigma(v))(X_2) \leftrightarrow W_1 = \sigma(u)W_2$$

2. Description of orders in $\mathfrak{sl}(n)$.

2.1. THEOREM. *Any order in* $\mathfrak{sl}(n;K)$ *is contained in* $g^{-1}\mathfrak{sl}(n;\mathfrak{O})g$ *for some* $g\in \mathrm{GL}(n;K)$.

REMARK. Thus, any maximal order in $\mathfrak{sl}(n,K)$ is an order $g^{-1}\mathfrak{sl}(n;\mathfrak{O})g$ for some of the form $g\in \mathrm{GL}(n;K)$.

2.2 SAUVAGE LEMMA ([AI]). *The diagonal matrices* $\mathrm{diag}(u^{m_1},\dots,u^{m_n})$, *where* $m_i\in \mathbb{Z}$ *for all* i, $m_1\leqq\dots\leqq m_n$, *represent all double cosets* $\mathrm{GL}(u;\mathfrak{O})\backslash\mathrm{GL}(n;K)/\mathrm{GL}(n;\mathbb{C}[u])$.

REMARK. An equivalent formulation: any vector bundle of rank n on $\mathbb{P}^1$ is of the form $\mathfrak{O}(m_1)\oplus\dots\oplus\mathfrak{O}(m_n)$.

2.3. COROLLARY. *Let* $W\subset g^{-1}\mathfrak{sl}(n;\mathfrak{O})g$ *correspond to a rational solution of the* CYBE′. *Then up to a gauge equivalence* $g=d_k$, *where* $d_k=\mathrm{diag}((1,\dots,1,u,\dots,u)$ (k-many 1's), $1\leqq k\leqq n/2$.

If $W\subset d_k^{-1}\mathfrak{sl}(n;\mathfrak{O})d_k$ then the order W will be said to be of *class k*.

2.4. THEOREM. 1) *A rational solution of* CYBE′ *is gauge equivalent to a rational solution*

$$X(u,v)=\frac{c_2}{u-v}+r(u,v),\ \text{where}\ \deg_u r=\deg_v r\leqq 1$$

2) *If an order* W *corresponds to a rational solution and* $W\subset\mathfrak{sl}(n;\mathfrak{O})$ *then the corresponding solution is a constant one.*

3. Orders $\leftrightarrow$ pairs = (Lie algebra of a group locally transitively acting on a Grassmanian, its quasi-Frobenius subalgebra).

A Lie algebra F is called a *Frobenius* one if the skew-symmetric bilinear form B_f on it given by the formula $B_f(x,y)=f([x,y])$ for $f\in F^*$ and $x,y\in F$ is nondegenerate for some $f\neq 0$.

A Lie algebra F is called a *quasi-Frobenius* one if there is a nondegenerate 2-cocycle $B\in C^2(F)$. Since every B_f is a 2-cocycle, a Frobenius Lie algebra is quasi-Frobenius.

Let P_k be the parabolic subalgebra of $\mathfrak{sl}(n)$ corresponding to the kth simple root, i.e., is generated by all root vectors corresponding to simple roots $\alpha_1,\dots,\alpha_{n-1}$ and their opposite except $-\alpha_k$; let p_k^t be the parabolic subalgebra generated by all the roots except α_k.

3.1. THEOREM. *Let W be an order of class k corresponding to a rational solution of* CYBE′. *Then there is a one-to-one correspondence* $W\leftrightarrow(L,B)$, *where* $L\subset\mathfrak{sl}(n)$ *is a subalgebra such that*

(*) $\left[\begin{array}{l} L + P_k = \mathfrak{sl}(n) \textit{ (note that this is not a direct sum) and } B \textit{ is a 2-cocycle on} \\ L \textit{ nondegenerate on } L \cap P_k \textit{ (therefore } L \cap P_k \textit{ is quasi Frobenius).} \end{array}\right.$

Let us reformulate the condition $L + P_k = \mathfrak{sl}(n)$ from Therem 3.1 in a sometimes more convenient form. Let L be Lie algebra, $G(L)$ be the group generated by $\exp(\operatorname{ad} x)$ for $x \in L$. Clearly, $G(L)$ is connected.

3.2. THEOREM. $L + P_k = \mathfrak{sl}(n)$ *iff*

(*′) $\left[\begin{array}{l} G(L) \textit{ acts locally transitively on the Grassmann manifold } G_k^n \textit{ and the plane} \\ \textit{generated by the first } k \textit{ of basis vectors of } \mathbf{C}^n \textit{ is a generic point whose} \\ \textit{stationary subgroup is } G(L \cap P_k). \end{array}\right.$

Still another reformulation is as follows.

THEOREM 3.3. $L + P_k = \mathfrak{sl}(n)$ *iff*

(*″) $\left[\begin{array}{l} L \oplus \mathfrak{gl}(k) \textit{ acts transitively (i.e. there is a vector } x_0 \in \mathbf{C}^n \otimes \mathbf{C}^k \textit{ such that} \\ (L \oplus \mathfrak{gl}(k)(x_0) = \mathbf{C}^n \otimes \mathbf{C}^k) \textit{ on } \mathbf{C}^n \otimes \mathbf{C}^k \textit{ and } \sum_{i \leq k} e_i \otimes e_i \textit{ is a generic point} \\ \textit{whose stationary subgroup's Lie algebra is isomorphic to } L \cap P_k. \end{array}\right.$

Now, let $L_{r_1, r_2, n} = P^t_{n-r_1} \cap P_{r_2}$.

3.4. COROLLARY. 1) $L_{r_1, r_2, n}$ *is Frobenius* $\Leftrightarrow (r_1 + r_2, n) = 1$.

2) $\mathfrak{gl}(r_1) \oplus \mathfrak{sl}(r_2) \oplus \mathfrak{gl}(k)$ *acts locally transitively on* $\mathbf{C}^{r_1 + r_2} \otimes \mathbf{C}^k$ *and the Lie algebra of the stationary subgroup of a generic point is Frobenius* $\Leftrightarrow (r_1 + r_2, k) = 1$.

4. Gauge equivalence and cohomology.

4.1. LEMMA. *Let* (L_1, B_1) *and* (L_2, B_2) *determine solutions of the class* k. *Let* $(\operatorname{Ad} X)L_1 = L_2$ *and* $B_2(\operatorname{Ad} X(a_1), \operatorname{Ad} X(a_2)) = B_1(a_1, a_2)$ *for all* $a_1, a_2 \in L_1$ *and some* $X \in G(P_k)$.

Then the corresponding solutions are gauge equivalent.

4.2. LEMMA. *Let* N *be a Lie algebra,* L_1, L_2, P *its subalgebras such that* $L_1 + P = L_2 + P = N$ *and* $L_1 = XL_2$ *for some* $X \in \operatorname{Ad} G(N)$. *Then* $L_1 = RL_2$ *for some* $R \in G(P)$.

PROPOSITION. 1) *Let* (L, B_1) *and* (L, B_2) *determine solutions from the same class and* B_1 *be cohomologic to* B_2. *Then the solutions are gauge equivalent.*

2) *Let* $N(L) \subset SL(n)$ *be the normalizer of* L *and* $B_2(\operatorname{Ad} X(a_1), \operatorname{Ad} X(a_2)) = B_1(a_1, a_2)$ *for all* $a_1, a_2 \in L$ *and some* $X \in \operatorname{Ad} N(L)$. *Let* (L, B_1) *and* (L, B_2) *determine solutions from the same class. Then the solutions are gauge equivalent.*

REMARK. $G(L)$ acts trivially on $H^2(L)$ (see [F]).

4.3. Lemma. *Let $L + P_k = \mathfrak{sl}(n)$, $L \cap P_k$ a Frobenius Lie algebra. Then in each class from $H^2(L)$ there exists a cocycle nondegenerate on $L \cap P_k$.*

Proposition. *Let L satisfy conditions of Lemma and $H^2(L) = 0$. Then there is precisely 1 solution of class k with given L.*

4.4. Lemma. *Let (L, B) determine a solution of class k in $\mathfrak{sl}(n)$ and there exists an L-invariant r-dimensional subspace in $\mathbf{C}^n$. Then the solution is gauge equivalent to a solution of class $|k + r - n|$ if $|k + r - n| \leqq n/2$ and to that of class $n - |k + r - n|$ if $|k + r - n| > n/2$.*

Proposition. *Let L be solvable. Then the corresponding solution, if any, is gauge equivalent to a constant one.*

5. Constructing solutions.

5.1. *Preliminaries* (cf. [E1], [E2], [Sp]).

5.1.1. Lemma (Duality principle or "castling"). *If for an $L \oplus \mathfrak{sl}(W)$-action on $V \otimes W$ there exists a generic point with stationary subalgebra $\mathfrak{h}$ then for the $L \oplus \mathfrak{sl}(W')$-action on $V^* \otimes W'$, where* $\dim W' = \dim V - \dim W$, *there exists a generic point with stationary subalgebra $\mathfrak{h}'$ isomorphic to $\mathfrak{h}$.*

We will say that a triple (L_1, V_1, φ_1) and (L_2, V_2, φ_2), where φ_i is a representation of L_i in the space V_i, are obtained from each other by *castling* if there exists a triple (L, V, φ) and $n \in \mathbf{N}$, $n < \dim V = m$, such that

$$(L_1, V_1, \varphi_1) = (L \oplus \mathfrak{sl}(n), V \otimes \mathbf{C}^n, \varphi \oplus \Lambda_1)$$

$$(L_2, V_2, \varphi_2) = (L \oplus \mathfrak{sl}(m - n), V^* \otimes \mathbf{C}^{m-n}, \varphi^* \otimes \Lambda_1)$$

A triple (L, V, φ) will be called *reduced* if there is no triple (L', V', φ') with $\dim V' < \dim V$ obtained by castling from (L, V, φ) (in notations [E2]).

5.1.2. Lemma ([E2]). *Let L be reductive, φ its locally transitive representation and either φ is irreducible or $[L, L]$ is simple. Let in either case the stationary algebra,* i.e. *the Lie algebra of the stationary group of a generic point be Frobenius. Then all such reduced triples are given in Table* 1, *where T^m is the Lie algebra of m-dimensional torus which in cases* 2, 3, 6, 7 *acts in each direct summand with scalar operators.*

5.2. *Results.*

Let (L, B) satisfy conditions of Theorem 3.1 and L be an irreducible subalgebra. Then L is semisimple and $H^2(L) = 0$. Thus, $L \cap P_k$ is a Frobenius algebra.

TABLE 1

	L	V	φ	Additional conditions
1	$\mathrm{gl}(n)$	$\oplus_{i=1}^{m} \mathbf{C}^n$	$\Lambda_1 \oplus \ldots \oplus \Lambda_1$	n is divisible by m
2	$T^m \oplus \mathrm{sl}(n)$	$\oplus_{i=1}^{m} \mathbf{C}^n$	$\Lambda_1 \oplus \ldots \oplus \Lambda_1$	$m > 2$, $n - 1$ is divisible by m
3	$T^{n+1} \oplus \mathrm{sl}(n)$	$\oplus_{i=1}^{n+1} \mathbf{C}^n$	$\oplus_{i=1}^{n+1} \Lambda_1$	
4	$\mathrm{sl}(n) \oplus \mathrm{gl}(m)$	$\mathbf{C}^n \otimes \mathbf{C}^m$	$\Lambda_1 \otimes \Lambda_1$	$(m, n) = 1$, $m/2 \geqq n \geqq 2$
5	$\mathrm{gl}(2)$	$\mathbf{C}^4$	$3\Lambda_1$	
6	$T^2 \oplus \mathrm{sl}(2)$	$\mathbf{C}^3 \oplus \mathbf{C}^2$	$2\Lambda_1 \oplus \Lambda_1$	
7	$T^4 \oplus \mathrm{sl}(3)$	$\mathbf{C}^3 \oplus \mathbf{C}^3 \oplus \mathbf{C}^3 \oplus \mathbf{C}^3$	$\Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1^*$	
8	$\mathrm{sl}(3) \oplus \mathrm{gl}(2)$	$\mathbf{C}^6 \otimes \mathbf{C}^2$	$2\Lambda_1 \otimes \Lambda_1$	
9	$\mathrm{sl}(5) \oplus \mathrm{gl}(2)$	$\mathbf{C}^{10} \otimes \mathbf{C}^2$	$\Lambda_2 \otimes \Lambda_1$	
10	$\mathrm{sl}(5) \oplus \mathrm{gl}(4)$	$\mathbf{C}^{10} \otimes \mathbf{C}^4$	$\Lambda_2 \otimes \Lambda_1$	

THEOREM. *The reduced triples* 1 (m = 1).4.5.8.9.10 *from Lemma* 5.1.2 *and those obtained from them by castling exhaust all irreducible subalgebras* $L \oplus \mathfrak{gl}(k)$ *satisfying the following conditions:*

1) $L \oplus \mathfrak{gl}(k)$ *acts transitively on* $\mathbf{C}^n \otimes \mathbf{C}^k$;
2) *the stationary subalgebra of generic point is a Frobenius one.*

REMARK. Now we can use Theorem 3.3 to find "irreducible" solutions.

PROPOSITION. *The following Tables* 2–7 *list all nonconstant solutions* $X(u, v) = \dfrac{c_2}{u - v} + r(u, v)$ *of* CYBE′ *for* $\mathfrak{sl}(n)$, $n = 2, 3, 4$.

Here "all" means "all up to gauge equivalence" and "nonconstant" means "not equivalent to constant".

TABLE 2

$\mathfrak{sl}(2)$

r	L	$H_2(L)$	φ	$r(u, v)$
1	$\mathfrak{sl}(2)$	0	Λ_1	$\begin{bmatrix} 0 & -1/2 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} u - \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes \begin{bmatrix} 0 & -1/2 \\ 0 & 0 \end{bmatrix} v$

TABLE 3

$\mathfrak{sl}(3)$

r	L	$H_2(L)$	φ	$r(u, v)$
1	$\mathfrak{sl}(2)$	0	Λ_1	$\begin{bmatrix} 1/3 & & \\ & 1/3 & \\ & & -2/3 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} v - \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1/3 & & \\ & 1/3 & \\ & & -2/3 \end{bmatrix} u + {}$ $+ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} v - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} u + {}$ $+ \begin{bmatrix} -1/3 & & \\ & 2/3 & \\ & & -2/3 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} -1/3 & & \\ & 1/3 & \\ & & -2/3 \end{bmatrix} + {}$ $+ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
1	P_1^t	0	id	$+ \begin{bmatrix} -1/3 & & \\ & 2/3 & \\ & & 1/3 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} -2/3 & & \\ & 1/3 & \\ & & 1/3 \end{bmatrix} + {}$ $+ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1/3 & & \\ & 1/3 & \\ & & -2/3 \end{bmatrix} - \begin{bmatrix} 1/3 & & \\ & 1/3 & \\ & & -2/3 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + {}$ $+ \begin{bmatrix} 1/3 & & \\ & 1/3 & \\ & & -2/3 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} u - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1/3 & & \\ & 1/3 & \\ & & -2/3 \end{bmatrix} +$

TABLE 4

$\mathfrak{sl}(4)$

r	L (L is an irreducible subalgebra)	φ	$H_2(L)$
1	$\mathfrak{sl}(4)$	Λ_1	0
1	$\mathfrak{sl}(2)$	$3\Lambda_1$	0

TABLE 5

$\mathfrak{sl}(4)$

r	L (L preserves the only 1-dimensional spase)	φ	$H_2(L)$
1	$L = \mathfrak{sl}(3) \hat{\oplus} \mathbf{C}^3$	id	0
1	$L = \mathfrak{o}(3) \hat{\oplus} \mathbf{C}^3 = \begin{bmatrix} t & a & b & 0 \\ -a & t & c & 0 \\ -b & -c & t & 0 \\ * & * & * & -3t \end{bmatrix} a, b, c, t, * \in \mathbf{C}$	id	0

TABLE 6

$\mathfrak{sl}(4)$

r	L (L preserves the only 2-dimensional space)	φ	$H_2(L)$
1	$L = \mathfrak{sl}(2)$	$\Lambda_1 \oplus \Lambda_1$	0
1	$L = \mathfrak{gl}(2) \oplus \mathfrak{sl}(2)$	$\Lambda_1 \oplus \Lambda_1$	0
1	$L = P_2 t$	id	0
1	$L = \quad T^{-1} \begin{bmatrix} A & 0 \\ \begin{matrix} 0 & x \\ -x & 0 \end{matrix} & -A^t \end{bmatrix} T \colon A \in \mathfrak{gl}(2), T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$	id	0
1	$L = \begin{bmatrix} A & 0 \\ \begin{matrix} u & x \\ x & v \end{matrix} & -A^t \end{bmatrix} \colon A \in \mathfrak{gl}(2)$	id	0
1	$L = \begin{bmatrix} A & 0 \\ B & -A^t \end{bmatrix} \colon A \in \mathfrak{gl}(2), B \in \mathfrak{gl}(2)$	id	0

TABLE 7

$\mathfrak{sl}(4)$

r	L (L preserves a $(1,2)$-flag)	φ	$H_2(L)$
1	$L = \left[\begin{array}{cc\|cc} a & b & \multicolumn{2}{c}{0} \\ c & d & & \\ \hline x & y & p & 0 \\ z & u & 0 & q \end{array}\right]$	id	C
1	$L = \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ x & y & 0 & 0 \\ z & u & q & p \end{bmatrix}$	id	C
1	$C_\lambda^2 = \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ x & y & \lambda p & 0 \\ z & u & q & (\lambda - 1)p \end{bmatrix}$ $\lambda \neq 3/4$	id	0, if $\lambda \neq 1/4$, C, if $\lambda = 1/4$

6. Proofs.

6.1. PROOF OF THEOREM 1.1. Let $\mathfrak{g}$ be a simple Lie algebra, $V = \mathfrak{g}[u]$. Then $V^* = u^{-1}\mathfrak{g}[[u^{-1}]]$ and if $f \in V^*$, $a \in V$ then

$$f(a) = \operatorname{Res} \operatorname{tr} af$$

Now, let $\{I_m\}$ be an orthonormal basis of $\mathfrak{g}$ with respect to the inner product $(a, b) = \operatorname{tr} ab$, then $\{I_m u^k\colon k \in N\}$ is a basis of V. Denote the space of all open (with respect to the topology of K) maps $V^* \to V$.

$$\operatorname{Hom}_{\text{cont}}(V^*, V) = \{f\colon V^* \to V\colon \ker f \supset u^{-N}V^* \text{ for some } N \geqq 0\}$$

a) Let us construct an isomorphism $\Phi\colon V \otimes V \to \operatorname{Hom}_{\text{cont}}(V^*, V)$. Set

$$\Phi(x \otimes y)(f) = (f(y), x) \text{ for any } x, y \in V \text{ and } f \in V^*$$

The inverse map Φ^{-1} is defined as follows:

$$\Phi^{-1}(F) = \sum_{m,n} F(f_{mk}) \otimes e_{mk}, \text{ where } e_{mk} = I_m u^k, f_{mk} = I_m u^{-k-1}, F \in \operatorname{Hom}_{\text{cont}}(V^*, V).$$

b) There is a natural bijection $\operatorname{Hom}_{\text{cont}}(V^*, V) \to \{W \subset \mathfrak{g}((u^{-1}))\colon W \oplus V = \mathfrak{g}((u^{-1}))$ and $W \supset u^{-N}V^*$ for some $N \geqq 0\}$. Indeed, let $F \in \operatorname{Hom}_{\text{cont}}(V^*, V)$. Set $W(F) = \{f + F(f)\colon f \in V^*\} \subset \mathfrak{g}((u^{-1}))$.

c) LEMMA. *Let $r^{12}(u, v) \in V \otimes V$. Then $r^{12}(u, v) = -r^{21}(v, u)$ if and only if $W(\Phi(r^{12}))$ is Lagrangian with respect to the inner product induces on $\mathfrak{g}((u^{-1}))$ from $\mathfrak{g}[u^{-1}, u]$ and $W = W^\perp$.*

Proof is straightforward.

d) Now, let $C_2/(u-v) + r(u,v)$ be a rational solution of CYBE. Let us show that $W(\Phi(r))$ is a subalgebra in $\mathfrak{g}((u^{-1}))$.

Denote: $\rho = \Phi(r)$. We have to show that $[f + \rho(f), g + \rho(g)] \subset W$ for any $f, g \in V^*$.

By lemma c) it suffices to show that $([f + \rho(f), g + \rho(g)], h + \rho(h)) = 0$ for $f = I_m u^{-j-1}$, $g = I_n u^{-i-1}$, $h = I_1 u^{-k-1}$. Indeed, the isomorphism $V \otimes V \simeq \mathrm{Hom}_{\mathrm{cont}}(V^*, V)$ implies the formula

$$(*) \qquad r(u,v) = \sum_{m,k} \rho(I_m u^{-k-1}) \otimes I_m v^k$$

Since both $C_2/(u-v) + r(u,v)$ and $C_2(u-v)$ are solutions of CYBE,

$$[r^{12}, r^{13}] + \left[r^{12} + r^{13}, \frac{c^{23}}{u_2 - u_3} \right] + (\text{cyclic perm. of indices } 1,2,3) = 0$$

Formula (*) implies

$$[r^{12}, r^{13}] = \sum_{i,j,m,l} [\rho(I_l u_1^{-i-1}), \rho(I_m u_1^{j-1})] \otimes I_l u_2^i \otimes I_m u_3^j \text{ and}$$

$$\left[r^{12} + r^{13}, \frac{c_2^{23}}{u_2 - u_3} \right] = \sum_{k,l} \rho(I_l u_1^{-k-1}) \otimes \left[I_l u_2^k \otimes 1 + 1 \otimes I_l u_3^k, \frac{c_2^{23}}{u_2 - u_3} \right]$$

As shown in [BD1] $[a \otimes 1 + 1 \otimes a, c_2] = 0$ for any $a \in \mathfrak{g}$ and therefore

$$\left[r^{12} + r^{13}, \frac{c_2^{23}}{u_2 - u_3} \right] = \sum_{k,l} \rho(I_l u_1^{-k-1}) \otimes [I_l \otimes 1, c_2] \frac{u_2^k - u_3^k}{u_2 - u_3}$$

Let us calculate

$$(I_l \otimes 1, c_2] = \sum_m [I_l, I_m] \otimes I_m = \sum_{m,n} ([I_l, I_m], I_n) I_n \otimes I_m = - \sum_{m,n} ([I_n, I_m], I_l) \cdot I_n \otimes I_m.$$

Therefore

$$\left[r^{12} + r^{13}, \frac{c_2^{23}}{u_2 - u_3} \right] = - \sum_{l,m,n,i+j=k-1} \rho(I_l u_1^{-i-j-2}) \otimes I_n u_2^i$$

$$\otimes I_m u_3^j ([I_n, I_m], I_l) = -\sum \rho(\ )[I_n, I_m], I_l) \cdot I_l u_1^{i-j-2})$$

$$\otimes I_n u_2^i \otimes I_m u_3^j = - \sum_{m,n,i,j} \rho([I_n u_1^{-i-1}, I_m u_1^{-j-1}]) \otimes I_n u_2^i \otimes I_m u_3^j$$

Thus

$$[r^{12}, r^{23}] + \left[r^{12} + r^{13}, \frac{c_2^{23}}{u_2 - u_3} \right] = \sum_{i,j,l,m} ([\rho(I_l u_1^{-i-1}), \rho(I_m u_1^{-j-1})]$$

$$-\rho([I_l u_1^{-i-1}, I_m u_1^{-j-1}]) \otimes I_l u_2^i \otimes I_m u_3^j = \sum_{i,j,k,l,m,n} ([\rho(I_l u_1^{-i-1}, \rho(I_m u_1^{-j-1})]$$

$$-\rho([I_l u_1^{-i-1} I_m u_1^{-j-1}]), I_n u_1^{-k-1}) I_n u_1^k \otimes I_l u_2^i \otimes I_m u_3^j$$

In the last transformation we have made use of the formula $x = \sum_{n,k}(x, I_n u^{-k-1}) I_n u^k$.

Set $f = I_l u^{-i-1}$; $g = I_m u^{-j-1}$; $h = I_k u^{-k-1}$. Having similarly written the remaining summands of CYBE we get

$$0 = [r^{12}, r^{23}] + \left[r^{12} + r^{13}, \frac{c_2^{23}}{u_2 - u_3} \right] + \{\text{cycle}\} = \sum a_{ijk}^{lmn} I_l u_1^i \otimes I_m u_2^j \otimes I_n u_3^k,$$

where

$$a_{ijk}^{lmn} = ([\rho(f), \rho(g)] - \rho([f, g]), h) + ([\rho(g), \rho(h)] - \rho([g, h], f)$$

$$+ ([\rho(h), \rho(f)] - \rho([h, f]), g) = ([f + \rho(f), g + \rho(g)], h + \rho(h)) = 0$$

Thus, $W(F(r))$ is a subalgebra. Clearly, the proof is convertable. The remaining statements of Theorem are obvious.

6.2. Proof of Theorem 1.2. Let R be a rational solution of CYBE, $W(R)$ the corresponding order.

First, consider $R = R_0 = \dfrac{c_2}{u - v}$. Then let

$$W(R_0) = u^{-1}\mathfrak{g}[[u^{-1}]], r_1 = (\sigma(u) \otimes \sigma(v)) \frac{c_2}{u - v} = \sum_m \frac{\sigma(u) I_m \otimes \sigma(v) I_m}{u - v}$$

Let us show that $W(R_1) = \sigma(u) W(R_0)$. Let us make use of the following two obvious identities

(1)
$$\frac{c^2}{u - v} = \sum_{m, k \geq 0} I_m u^{-k-1} \otimes I_m v^k$$

(2)
$$f = \sum_{m, k \geq 0} (f, I_m v^k) \cdot I_m u^{-k-1} \text{ for any } f \in u^{-1}\mathfrak{g}[[u^{-1}]]$$

The identities imply that $W(R_1)$ is generated by

$$Q_f = \sum_{k \geq 0, m} (f, \sigma(v) I_m v^k) \cdot \sigma(u) I_m u^{-k-1}$$

$$= \sum_{k \geq 0, n, m, l} (f, \sigma(v) I_m v^k) \cdot (\sigma(u) I_m u^{-k-1}, I_l u^n) \cdot I_n u^{-n-1}$$

$$= \sum_{k \geq 0, n, m, l} ((\sigma^{-1} f, I_m v^k) \cdot I_m u^{-k-1}, \sigma^{-1} I_l u^n) I_n u^{-n-1};$$

Set $g = \sum_{k \geq 0, m} (\sigma^{-1} f, I_m v^k) \cdot I_m u^{-k-1}$. Then g is the projection of $\sigma^{-1} f$ onto

$u^{-1}\mathfrak{g}[[u^{-1}]]$ with respect to the decomposition $\mathfrak{g}((u^{-1})) = \mathfrak{g}[u] \oplus u^{-1}\mathfrak{g}[[u^{-1}]]$ and

$$Q_f = \sum_{l,n} (g, \sigma^{-1} I_l u^n) I_l u^{-n-1} = \sigma(g).$$

This proves Theorem in our particular case. The general case is quite similar.

6.3. Proof of Theorem 2.1. Let $W \subset \mathfrak{g}((u^{-1}))$ be an order. We can assume that W is an O-module. It suffices to prove that there exists an n-dimensional O-module M of finite type such that $WM \subset M$.

Indeed such a module M must be free and of rank n over O. Therefore $M = g^{-1}\mathrm{O}^n$ for some $g \in GL(n, K)$ which still prove Theorem.

It remains to construct M. Set

$$M = \mathrm{O}^n + W\mathrm{O}^n + \ldots + W \ldots W\mathrm{O}^n + \ldots$$

Clearly $WM \subset M$. Let us show that $M \subset u^l\mathrm{O}^n$ for some l. Let $x_1, \ldots, x_N$ be a basis of the O-module W. Then

$$M = \sum_{k_i \geqq 0, 1 \leqq i \leqq N} x_1^{k_1} \ldots x_N^{k_N} \mathrm{O}^n$$

For any $A \in W$ define its norm setting

$$\|A\| = 2^g, \text{ where } g = \inf_k (A\mathrm{O}^n \subset u^k \mathrm{O}^n)$$

For $\lambda = a_n u^n + \ldots + a_0 + \ldots + \ldots \in K$ set $|\lambda| = 2^n$. This absolute value on K can be continued to $\bar{K}$. The norm is well-defined since, clearly,

$$\|AB\| \leqq \|A\| \, \|B\|, \ \|\lambda A\| = \|\lambda| \, \|A\|, \ \|A + B\| \leqq \sup\{\|A\|, \|B\|\}.$$

Thus it suffices to show that

$$\sup_{(k_1, \ldots, k_N)} \|x_1^{k_1} \ldots x_N^{k}\| < \infty \Leftrightarrow \sup_k \|x_i\|^k \leqq 1 \text{ for any } i.$$

The latter condition is equivalent to the fact that the absolute values of the eigenvalues of $x_i \in W$ for the action of the x_i on K^n are $\leqq 1$.

Notice, that ad $x_i(W) \subset W$ since W is a subalgebra.

Let $\lambda_1^i, \ldots, \lambda_n^i$, be eigenvalues of x_i on K^n, then the set of eigenvalues of ad x_i then W acts of $\mathfrak{sl}(n; \mathrm{K})$ is $\{\lambda_p^i - \lambda_q^i\}$, where

$$(*) \qquad \sum_p \lambda_p^i = 0$$

Since an order $W \subset \mathfrak{sl}(n; K)$ is an O-submodule of finite type,

$$(**) \qquad |\lambda_p^i - \lambda_q^i| \leqq 1 \text{ for all } p, q.$$

Formulas (*) and (**) imply $|\lambda_p^i| \leqq 1$ for all p.

PROOF OF PROPOSITION 2.3. The statement of Proposition 2.3 is a corollary of 2.2 and the following almost obvious.

LEMMA. *Let* $g = \operatorname{diag}(u^{m_1}, \ldots, u^{m_k})$, $m_l \leqq \ldots \leqq m_k$, *and* $g^{-1}\mathfrak{sl}(n; \mathrm{O})g + \mathfrak{sl}(n; \mathrm{C}[u]) = \mathfrak{sl}(n; K)$.

Then $m_i - m_j \geqq -1$ *for all* i, j.

PROOF OF THEOREM 2.4. 1) Let $R(u, v)$ be a rational solution of CYBE, $W(R)$ the corresponding order. Proposition 2.3 implies that there exists a gauge transformation $\sigma(u)$ such that

$$\sigma(u)W(R) \subset d_k^{-1}\mathfrak{sl}(n; \mathrm{O})d_k$$

By Theorem 1.2

$$\sigma(u)W(R) = W(R_1), \text{ where } R_1 = (\sigma \otimes \sigma)R$$

The proof of Theorem 1.1 implies that

$$W(R_1) = \{f + \rho(R_1)f \colon f \in u^{-1}\mathfrak{sl}(n; \mathrm{O})\}$$

Since $W(R_1) \subset d_k^{-1}\mathfrak{sl}(n; \mathrm{O})d_k$, then $\deg_u \rho(R_1)f \leqq 1$. Formula (1) in 6.1. implies that

$$R_1(u, v) = \frac{c_2}{u - v} + \sum_{m,k} \rho(R_1)(I_m u^{-k-1}) \otimes I_m v^k$$

Thus,

$$\deg_u\left(R_1 - \frac{c_2}{u - v}\right) \leqq 1$$

the remaining statements of 1) follow from skew symmetricity.

2) Proof of this is absolutely similar to that of 1).

7. Proof of Theorem 3.

The details of this proof are of independent interest; therefore we have singled them out in a separate section.

The following statement is obvious.

7.1. LEMMA

1) $\mathfrak{sl}(n; \mathrm{O})^{\perp} = u^{-2}\mathfrak{sl}(n; \mathrm{O})$.

2) $\mathfrak{sl}(n, \mathrm{O})/u^{-2}\mathfrak{sl}(n, \mathrm{O}) \cong \mathfrak{sl}(n, \mathrm{C}[\varepsilon])$, *where* $\varepsilon^2 = 0$.

3) *The inner product in* $\mathfrak{sl}(n, \mathrm{O})$ *induces one in* $\mathfrak{sl}(n, \mathrm{C}[\varepsilon])$; *explicitly the induced inner product is given by the formula*

$$(x_1 + y_1\varepsilon, x_2 + y_2\varepsilon) = \operatorname{tr}(x_1 y_2 + y_1 x_2)$$

REMARK. This trace is Berezin integral over ε, see [B].

7.2. PROPOSITION. *There is a natural one-to-one correspondence between subalgebras* $W \subset d_k^{-1}\mathfrak{sl}(n, \mathbb{Q})d_k$ *satisfying conditions* 1)–3) *of Theorem* 1.1 *of Lagrangian (with respect to the inner product introduced in Lemma* 7.1 *subalgebras* X_W *in* $\mathfrak{sl}(n, \mathbb{C}[\varepsilon])$ *such that* $X_W \oplus (P_k + \varepsilon P_k^\perp) = \mathfrak{sl}(n, \mathbb{C}[\varepsilon])$. *The subalgebra* $X_W \subset \mathfrak{sl}(n, \mathbb{C}[\varepsilon])$ *corresponding to* W *is defined to be the image of* $d_k W d_k^{-1} \subset \mathfrak{sl}(n, \mathbb{Q})$ *in* $\mathfrak{sl}(n, \mathbb{C}[\varepsilon])$.

REMARK. Since $P_k + \varepsilon P_k^\perp$ is a Lagrangian subspace in $\mathfrak{sl}(n, \mathbb{C}[\varepsilon])$ (i.e., it equals its orthogonal complement) a subalgebra X satisfying the conditions of Proposition 7.2 is also Lagrangian.

The proof of Proposition 7.2 is based on the following observation: since $d_k W d_k^{-1} \subset \mathfrak{sl}(n, \mathbb{Q})$ and $W = W^\perp$ we have $d_k W d_k^{-1} \supset u^{-2}\mathfrak{sl}(n, \mathbb{Q})$.

7.3. Let us return to the proof of Theorem 3.1. Proposition 7.2 shows that to a rational solution of CYBE corresponds a subalgebra in $\mathfrak{sl}(n, \mathbb{C}[\varepsilon]) = \mathfrak{sl}(n, \mathbb{C}) \oplus \varepsilon\mathfrak{sl}(n, \mathbb{C})$.

Set $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$. Denote by $L \subset \mathfrak{g}$ the image of X_W under the projection ($\varepsilon \mapsto 0$) to $\mathfrak{g}$, then $X_W \subset L + \varepsilon\mathfrak{g}$. Since X_W is Lagrangian, $X_W \supset (L + \varepsilon\mathfrak{g})^\perp = \varepsilon L^\perp$. Clearly, $L + \varepsilon\mathfrak{g}/\varepsilon L^\perp \cong L + \varepsilon L^*$ since $\mathfrak{g}/L^\perp \cong L^*$ due no nondegeneracy of the inner product on $\mathfrak{g}$.

Thus, X_W is determined by a Lagrangian subalgebra $\tilde{X}_W \subset L + \varepsilon L^*$ such that the image of X_W under the projection is onto L.

Clearly, such subalgebras are in one-to-one correspondence with 2-cocycles on L, i.e. $\tilde{X}_W = \{x + \varepsilon f_B(x): x \in L\}$ where $f: L \to L^*$ is obtained by $f_B(x)(y) = B(x, y)\}$.

Hence, $X_W = \{x + \varepsilon f_B(x) + \varepsilon X^\perp: x \in L\}$.

7.4. LEMMA. *The following three condition are equivalent*

1) $X_W \cap (P_k + \varepsilon P_k^\perp) = \{0\}$
2) $X_W + P_k + \varepsilon P_k^\perp = \mathfrak{sl}(n; \mathbb{C}[\varepsilon])$
3) $L + P_k = \mathfrak{g}$ *and* B *is nondegenerate on* $L \cap P_k$.

PROOF. Obviously, 1) $\Leftrightarrow$ 2).

3) $\Rightarrow$ 1). Let $x + \varepsilon f_B(x)^\perp = p + \varepsilon p^\perp$. Then $x = p \in L \cap P_k$ and $f_B(x) + x^\perp = p^\perp$. Hence, $p^\perp - x^\perp \in (P_k \cap L)^\perp$ and therefore $f_B(x) \in (P_k \cap L)^\perp$. This, clearly, contradicts to nondegeneracy of B on $L \cap P_k$.

1), 2) $\Rightarrow$ 3). Clearly, $L + P_k = \mathfrak{g}$. Let B be degenerate on $L \cap P_k$. This means that there exists $x \in L \cap P_k$, $x \neq 0$, such that $f_B(x) \in (P_k \in (P_k \cap L)^\perp = L^\perp + P_k^\perp$. Then $f_B(x) = p^\perp - x^\perp$ for some $x^\perp \in L^\perp, p^\perp \in P_k^\perp$. So $x + \varepsilon f_B(x) + \varepsilon x^\perp = x + \varepsilon p^\perp \in X_W \cap (P_k + \varepsilon P_k^\perp)$. So we have proved Lemma 7.4 and Theorem 3.1.

7.5. Proof of Corollary 3.4.

Lemma. (Proof see in [E2}). *Let a Lie algebra L be a semidirect sum of a Lie algebra R and a commutative ideal N,* i.e. *is determined by a triple (R, N, φ), where $\varphi: R \to$* End(N) *is a Lie algebra homomorphism. Then L is Frobenius if and only if*

1) *(R, N^*, φ^*) is locally transitive,* i.e. *there exists $n \in N^*$ such that $\{Rn\} = N^*$.*

2) *R_n, the stationary subalgebra (Lie algebra of the stationary group) of n is Frobenius.*

Clearly, $L_{r_1,r_r,n}$ is not Frobenius if $r_1 + r_2 = n$. In the other cases we have:

$$L_{r_1,r_r,n} = \begin{array}{l} (\mathfrak{sl}(r_1) \oplus \mathfrak{gl}(r_2) \oplus \mathfrak{gl}(n - r_1 - r_2), (\mathbf{C}^{r_1} \oplus \mathbf{C}^{r_2}) \otimes \mathbf{C}^{n-r_1-r_2} \\ (\Lambda_1 \oplus \Lambda_1) \otimes \Lambda_1), \text{if } r_1 + r_2 < n \\ (\mathfrak{sl}(n - r_1) \oplus \mathfrak{gl}(n - r_2) \oplus \mathfrak{gl}(r_1 + r_2 - n), (\mathbf{C}^{n-r_1} \oplus \mathbf{C}^{n-r_2}) \otimes \mathbf{C}^{r_1+r_2-n}, \\ (\Lambda_1 \oplus \Lambda_1) \otimes \Lambda_1), \text{if } r_1 + r_2 < n \end{array}$$

By Lemma 7.5 we have to find out when the stationary subalgebra of a generic point is a Frobenius one since the local transitivity of the action is clear.

Thus, $\mathfrak{sl}(r_1) \oplus \mathfrak{gl}(r_2) \oplus \mathfrak{gl}(k)$ acts on $(\mathbf{C}^{r_1} \oplus \mathbf{C}^{r_2}) \otimes \mathbf{C}^k$. Denote by (r_1, r_2, k) the Lie algebra obtained with the help of this action semidirect sum of the algebra and its module considered with the trivial bracket. Let us calculate the stationary algebras.

1) $K < r_1, r_2$. It is easy to see that the stationary algebra of a generic points is isomorphic to $(r_1 - k, r_2 - k, k)$ and $(r_1 - k, r_2 - k, k)$ is Frobenius if and only if so is $(r_1 - k, r_2 - k, k)$.

2) $k > r_1 + r_2$. By castling pass to the case $(r_1, k - r_1 - r_2, r_2)$.

3) $k > r_1, r_2$, but $k < r_1 + r_2$. By castling pass to the case $(k - r_2, r_1 + r_2 - k, k - r_1)$.

4) $r_1 > k > r_2$. The stationary subalgebra of a generic point is isomorphic to the following subalgebra of $\mathfrak{sl}(r_1)$:

	$r_1 - k$	r_2	$k - r_2$
$r_1 - k$	*	0	0
r_2	*	*	0
$k - r_2$	*	*	*

Let for definiteness sake $r_1 - k \geqq k - r_2$. Take $P^t_{r_1,k} \subset \mathfrak{sl}(r_1 + r_2 - k)$; clearly, $P^t_{r_1,k} \oplus \mathfrak{gl}(k - r_2)$ acts locally transitivily on $\mathbf{C}^{k-r_2} \otimes \mathbf{C}^{r_1+r_2-k}$ and the stationary subalgebra of a generic point of this action is isomorphic to $(k - r_2, r_1 + r_2 - 2k, r_2)$.

5) $r_1 - k < k - r_2$. By castling to the case $(r_1 - k, 2k - r_1 - r_2, r_2)$ it follows from the results of [E2] that $(r_1, r_2, 1)$ is Frobenius algebra and $P_k \subset \mathfrak{sl}(n)$ is Frobenius if and only if $(k, n) = 1$.

Thus, 1) of our Proposition follows by induction. 2) is similarly proved.

8. Proof of statements of section 4.

8.1. Proof of Lemma 4.1. Let X_1 and X_2 be Lie subalgebras of $\mathfrak{sl}(n, \mathbb{C}[\varepsilon])$ recovered from (L_1, B_1) and (L_2, B_2), respectively. By Proposition 7.3 we have

$$X_1 = \{a + \varepsilon f_{B_1}(a) + \varepsilon a^{\perp}\} \text{ and } X_2 = \{\{b + \varepsilon f_{B_2}(b) + \varepsilon b^{\perp}\}.$$

Then

$$(\mathrm{Ad}\, X^{-1})(X_1) = \{X^{-1} aX + \varepsilon X^{-1} f_{B_1}(a)X + \varepsilon X^{-1} a^{\perp} X\}$$

Let $X^{-1} aX = b_0 \in L_2$. Then

$$(X^{-1} f_{B_1}(a)X)(y) = (X^{-1} f_{B_1}(a)X, y) =$$
$$= (f_{B_1}(a) XyX^{-1}) = B_2(b_0, y) = f_{B_2}(b_0)(y).$$

Hence

$$X^{-1} f_{B_1}(a)X = f_{B_2}(b_0) \text{ implies } (\mathrm{Ad}\, X^{-1})(X_1) = X_2.$$

Accordingly, the corresponding orders W_1 and W_2 from $\mathfrak{sl}(n, K)$ are conjugate by $T = d_k^{-1} X d_k \in \mathrm{SL}(n, \mathbb{C}([u])$.

8.2. Proof of Lemma 4.2. Let X be the set of subalgebras of the Lie algebra N of the form $(\mathrm{Ad}\, g)(L_2)$, where $g \in G(N)$. Then X is a connected complex-analytic variety. Set

$$X' = \{L \in X : L + P = N\}.$$

Then $X \backslash X'$ is an analytic space since it is distinguished by a system of equations corresponding to vanishing of certain minors. Since $\dim_{\mathbb{R}} X \backslash X' \leqq 2$, then $X \backslash X'$ is connected. Therefore X' is also connected.

The group $G(P)$ acts on X'. Let us show that this action is transitive. It suffices to show that the orbit $\mathrm{Ad}\, G(P) \cdot L_2$ is open. Indeed, $\mathrm{Ad}\, G(L_2) L_2 = L_2$, hence $\mathrm{Ad}\, G(P)\, L_2 = \mathrm{Ad}\,(G(P) \cdot G(L_2) \cdot L_2$. But $L_2 + P = N$, hence $G(P) \cdot G(L_2)$ is an open subset in $G(N)$. The statement of Lemma follows from the transitivity of the $\mathrm{Ad}\, G(N)$-action on X.

8.3. Proof of Proposition 4.2. 1) Let X_1 and X_2 be Lagrangian subalgebras in $\mathfrak{sl}(n; \mathbb{C}[\varepsilon])$ corresponding to (L, B_1) and (L, B_2). Let

$$b_1(x, y) = B_2(x, y) + l([x, y])$$

Let l be given by a matrix $T \in \mathfrak{sl}(n, \mathbb{C})$ i.e.

$$l(x) = \operatorname{tr} Tx$$

Clearly, $\operatorname{Ad}(E + \varepsilon T)X_1 = X_2$. By Lemma 4.2 and Proposition 7.2 there exists $Y \in G(P_k + \varepsilon P_k^{\perp})$ such that $(\operatorname{Ad} Y)(X_1) = X_2$. Clearly, Y can be lifted to an element from $\mathrm{SL}(n, \mathbb{C}[u])$.

2) Let X_1 and X_2 be Lagrangian subalgebras in $\mathfrak{sl}(n, \mathbb{C}[u])$, corresponding to (L, B_1) and (L, B_2), respectively. Similarly to proof of Lemma 4.1 we get $(\operatorname{Ad} X)(X_1) = X_2$. By Lemma 4.2 there exists $P + \varepsilon P^{\perp} \in G(P_k + \varepsilon P_k^{\perp})$ such that $\operatorname{Ad}(P + \varepsilon P^{\perp})(X_1) = X_2$.

8.4. Proof of Lemma 4.3. For each cohomology class we have to construct a 2-cocycle B on L nondegenerate on $L \cap P_k$. Let f be a functional on $L \cap P_k$ such that $B_f(\cdot,\cdot) = f([\cdot,\cdot])$ is nondegenerate.

Let $\tilde{f}$ be an extension of f to L. Then for $|\lambda| \gg 0, \lambda \in \mathbb{C}$, we see that $B(x, y) + \lambda \tilde{f}([x, y])$ is a nondegenerate 2-cocycle on $L \cap P_k$ from the same cohomology class as B, as required.

Proof of Proposition 4.3. This is an easy corollary of Lemma 4.3. Indeed, to this Lemma there exists a solution for a given L and all such solutions are gauge equivalent.

8.5. Proof of Lemma 4.4. Let V be on L-invariant r-dimensional subspace of $\mathbb{C}^n$. Set

$$Q = \{A \in \mathfrak{sl}(n) \colon AV \subset V\}$$

Clearly, $L \subset Q$ and $Q + P_k = \mathfrak{sl}(n)$. Obviously, P_{n-r}^t preserves an r-dimensional subspace in $\mathbb{C}^n$ and therefore there exists $X \in \mathrm{SL}(n, \mathbb{C})$ such that $X^{-1}QX = P_{n-r}^t$. Manifestly, $P_{n-r}^t + P_k = \mathfrak{sl}(n)$. Thanks to Lemma 4.2 there exists $Y \in G(P_k)$ such that $Y^{-1}QY = P_{n-r}^t$. Thus, we may assume that $L \subset P_{n-r}^t$. Obviously

$$L + u^{-1}\mathfrak{sl}(n, \mathbb{O}) \subset \mathfrak{sl}(n, \mathbb{O}) \cap d_{n-r}\mathfrak{sl}(n, \mathbb{O})\, d_{n-r}^{-1}$$

where d_{n-r} is defined in Proposition 2.3.

Now, Propositions 2.3 and 7.2 imply that the order W recovered from L must belong to

$$d_k^{-1}\mathfrak{sl}(n, \mathbb{O})\, d_k \cap d_k^{-1} d_{n-r}\mathfrak{sl}(n, \mathbb{O})\, d_{n-r}^{-1} d_k,$$

whereas it is easy to see that $d_k^{-1} d_{n-r}\mathfrak{sl}(n, \mathbb{O})\, d_{n-r}^{-1} d_k = \sigma(u)(d_m^{-1}\mathfrak{sl}(n, \mathbb{O})\, d_m)$ for some polynomial $\sigma(u) \colon \mathbb{C} \to \operatorname{Aut} \mathfrak{g}$ where $m = |k + r - n|$ if $|k + r - n| \leqq n/2$ and $m = n - |k + r - n|$ if $|k + rr - n| > n/2$.

8.7. Proof of Proposition 4.4 immediately follows from Lemma 4.4.

9. Proof of statements from section 5.

9.1. LEMMA.

1) $H^2(P_r) = 0$.

2) *The subalgebra* P_r^t *determines the only solution of class* k *in* $\mathfrak{sl}(n) \Leftrightarrow (k - r, n) = 1$.

PROOF. 1) is obvious from Hochschild-Serre spectral sequence, cf. [F]. 2) follows from 1) and Propositions 3.4 and 4.3.

9.2. PROOF OF PROPOSITION 5.2. 1) Solutions for $\mathfrak{sl}(2)$. There is only one class which can provide with nonconstant solutions According to Proposition 4.4 L cannot be solvable so the only possible case is $L = \mathfrak{sl}(2)$. Proposition 4.3 implies that there is precisely one solution corresponding to $L = \mathfrak{sl}(2)$.

2) Solutions for $\mathfrak{sl}(3)$. Proposition 2.3 and Theorem 2.4 imply that nonconstant solution is gauge equivalent to a solution of class 1. Theorem 5.2 and Lemma 4.3 imply that the only solution with an irreducible L is for class 1 the one with $L = \mathfrak{sl}(3)$.

All other L's are reducible. Proposition 4.4 implies that the semisimple part of L must be nontrivial; furthermore Lemma 4.4, implies that L should have an invariant 1-dimensional subspace in $\mathbf{C}^3$ and can not have a 2-dimensional invariant subspace.

It is clear (with Lemma 9.1 being taken in account) that such an L is isomorphic either to P_2^t or to the semidirect sum $\mathfrak{sl}(2) \hat{\oplus} \mathbf{C}^2$ where $\mathbf{C}^2$ is the space of the identity (standard) representation with the trivial bracket. The latter case, however, is excluded by the following Lemma.

LEMMA. *If* L *satisfies conditions* (*) *from Theorem* 3.1 *then* $\dim L \equiv k(n - k) \bmod 2$.

Thus, there are two nontrivial solutions for $\mathfrak{sl}(3)$.

3) Solutions for $\mathfrak{sl}(4)$. Theorem 5.2 gives the following solutions of class 1 corresponding to an irreducible subalgebra:

a) $L = \mathfrak{sl}(4)$; b) $L = \mathfrak{sl}(2)$ and the embedding of $\mathfrak{sl}(2)$ into $\mathfrak{sl}(4)$ is the principal one ($R(3A_1)$).

Looking at Tables we see that there are no solutions of class 2 corresponding to an irreducible subalgebra. Lemma 4.4 implies that all nonconstant solutions are gauge equivalent to those of class 1 and moreover the semisimple part of L is nontrivial. The same Lemma together with Theorem 2.4 implies that L can only preserve a 1- or 2-dimensional subspace. Let us consider, separately, the arising possibilities

A) L preserves a 1-dimensional subspace and does not preserve 2-dimensional subspaces. By the same arguments as in the proof of Lemma 4.4 we may assume that $L \subset P_3^t$. Lemma 9.2 shows that $\dim L$ must be odd. So there are two possibilities:

a) $L = \mathfrak{sl}(3) \hat{\oplus} \mathbf{C}^3$ (here $\hat{\oplus}$ stands for the semidirect sum; the ideal to the right) with $\mathbf{C}^3$ being the standard $\mathfrak{sl}(3)$-module, i.e. $\mathbf{C}^3 = R(\Lambda_1)$. The Hochschild-Serre spectral sequence shows that $H^2(L) = 0$ and $P_1 \cap L$ is a Frobenius algebra. Thus, there is exactly one solution corresponding to this case.

b) L is the algebra of matrices

$$\begin{bmatrix} t & a & b & 0 \\ -a & t & c & 0 \\ -b & -c & t & 0 \\ * & * & * & -3t \end{bmatrix} a, b, c, t, * \in \mathbf{C}$$

As follows immediately from Hochschied-Serre spectral sequence, $H^2(L) = 0$.

Let us show that $P_1 \cap L$ is a Frobenius Lie algebra. Indeed, direct calculations show that $P_1 \cap L$ is 4-dimensional Lie algebra isomorphic to $N \hat{\oplus} \mathbf{C}^2$, where N is an abelian 2-dimensional Lie algebra locally transitively acting on $(\mathbf{C}^2)^*$. From the dimension considerations the stationary subalgebra is 0 and therefore by Lemma 1 from [E2] $P_1 \cap L$ is a Frobenius algebra. Thus there, is a solution corresponding to this case.

B) L preserves a 2-dimensional subspace and does not preserve 1-dimensional subspaces.

As earlier we may assume that $L \subset P_2^t$. Let us list all the possibilities.

1) Example 1 from Table 1 for $n = m = 2$. Then $L = \mathfrak{sl}(2)$, $H^2(L) = 0$ and by Lemma 5.1.2 we get an only solution.

2) $L = \mathfrak{gl}(2) \oplus \mathfrak{sl}(2)$, $H^2(L) = 0$ and by Proposition 3.4 we get an only solution.

3) $L = P_2^t$ and as above this subalgebra determines an only solution.

4) $$L = \left[\begin{array}{c|c} A & 0 \\ \hline \begin{matrix} 0 & x \\ -x & 0 \end{matrix} & -A^t \end{array}\right] : A \in \mathfrak{gl}(2)$$ Then as follows from Hochschild-Serre spectral sequence, $H^2(L) = 0$ and $L \oplus \mathfrak{gl}(1)$ acts on $\mathbf{C}^4$ locally transitively with a generic point of the form, say

$$\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

The stationary subalgebra of this vector is a 2-dimensional noncommutative Lie algebra. As is known it is a Frobenius one. Thus, we have an only solution.

5) $L = \left[\begin{array}{cc|c} A & & 0 \\ \hline u & x & \\ x & v & -A^t \end{array}\right]$: $A \in \mathrm{gl}(2)$ Then $L + P_1 = \mathrm{sl}(4)$ and $H^2(L) = 0$ by usual appellation to Hochschild-Serre spectral sequence. Clearly $P_1 \cap L$ is Frobenius since it is isomorphic to the Lie algebra of matrices

$$\begin{bmatrix} a & b & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & -a & 0 \\ 0 & v & -b & -d \end{bmatrix}$$

Let $A \in L \cap P_1$ be a matrix as above. The form $f(A) = b + v$ determines a non-degenerate bilinear form on $L \cap P_1$. Therefore in this case we have an only solution.

$$6)\ L = \left[\begin{array}{c|c} A & 0 \\ \hline B & -A^t \end{array}\right] : A \in \mathrm{sl}(2),\ B \in \mathrm{gl}(2)$$

Then $L + P_1 = \mathrm{sl}(4)$ and $H^2(L) = 0$.

$$L \cap P_1 = \begin{bmatrix} a & b & 0 & 0 \\ 0 & -a & 0 & 0 \\ 0 & x & -a & 0 \\ 0 & y & -b & a \end{bmatrix}$$

Since $H^2(L) = 0$ this algebra should be a Frobenius one. However, $L \cap P_1 \cong N \oplus \mathbf{C}^2$, where

$$N = \begin{bmatrix} a & b & 0 & 0 \\ 0 & -a & 0 & 0 \\ 0 & 0 & -a & 0 \\ 0 & 0 & -b & a \end{bmatrix}, \quad \mathbf{C}^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & y & 0 & 0 \end{bmatrix}$$

N does not act locally transitively on $\mathbf{C}^2$ but N acts locally transitively on $(\mathbf{C}^2)^*$. Therefore by Lemma 1 from [E2] $L \cap P_1$ is Frobenius. Hence, we have a solution in this case.

C) L preserves a 1-dimensional subspace and a 2-dimensional subspace. These subspces are not transversal because L does not preserve 3-dimensional subspaces. Thus, L preserves $(1, 2)$-flag.

$$L \subset \begin{bmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \\ * & * & * & * \end{bmatrix} \subset \mathfrak{sl}(4)$$

As above, the semisimple part of L must be nontrivial and we can assume that $L \supset \mathfrak{a}$ where

$$\mathfrak{a} = \begin{bmatrix} a & b & 0 & 0 \\ c & -a & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since $(\mathfrak{a}, L] \subseteq L$ we have $L = \mathfrak{a} + B + C$ where

$$B \subseteq \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & 0 & 0 \end{bmatrix} = D$$

$$C \subseteq \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & * & * \end{bmatrix} = D$$

$[\mathfrak{a}, B] \subseteq B$. If $B = 0$ then there is a 3-dimensional L-invariant subspace. If $B \neq 0$ and $B \neq D$ then $V + BV$ is a 3-dimensional L-invariant subspace where V is generated by the first two vectors of our basis. Hence, $B = D$. Since $\dim L$ must be odd we have only the following possibilities:

1) $$L = \begin{bmatrix} a & b & 0 & 0 \\ c & -a & 0 & 0 \\ x & y & 0 & 0 \\ z & u & 0 & 0 \end{bmatrix}$$

2) $$L = \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ x & y & p & 0 \\ z & u & 0 & q \end{bmatrix} : a + d + p + q = 0$$

3) $$L = \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ x & y & p & 0 \\ z & u & 0 & q \end{bmatrix} : a + d + 2p = 0$$

4) $$L = C_\lambda^1 = \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ x & y & \lambda p & 0 \\ z & u & q & (\lambda - 1)p \end{bmatrix} : a + d + \lambda p + (\lambda - 1)p = 0$$

1) $L \cong \mathfrak{sl}(2) \hat{\oplus} (\mathbf{C}^2 \oplus \mathbf{C}2)$

Let N denote a noncommutative 2-dimensional algebra. Then $P_1 \cap L \cong N \hat{\oplus} \mathbf{C}^2$ and N does not act locally transitively on $(\mathbf{C}^2)^*$. Thus $P_1 \cap L$ is not Frobenius algebra but direct calculations show that $H^2(P_1 \cap L) = 0$. Thus, $P_1 \cap L$ is not quasi-Frobenius algebra and there are no solutions in this case.

2) As follows from Hochschild-Serre spectral sequence, $H^2(L) = \mathbf{C}$. As above $L \cap P_1$ is Frobenius algebra. Thus, we have solutions in this case. The quotient group of the normalizer of L modulo the inner automorphisms is $\mathbf{Z}/2\mathbf{Z}$. The action of $\mathbf{Z}/2\mathbf{Z}$ is not trivial and an element of $H^2(L)$ is important up to a sign.

3) In this case $H^2(L) = \mathbf{C}$. $\dim L/[L, L] = 2$ and a non-trivial skew-symmetric form on $L/[L, L]$ generates $H^2(L)$. $P_1 \cap L$ is Frobenius algebra as above. The quotient group of the normalizer of L modulo the inner automorphisms is isomorphic to $\mathbf{C}^*$. Thus, we have two solutions.

4) Let N denote a noncommutative 2-dimensional algebra $P_1 \cap L \cong (N \oplus N) \hat{\oplus} \mathbf{C}^2$. N acts on $(\mathbf{C}^2)^*$ locally transitively which is evident and the stationary algebra of a generic point is isomorphic to N iff $\lambda \neq 3/4$. Let us consider $H^2(L)$. As follows from Hochschild-Serre spectral sequence, $H^2(L)$ can be generated only by $H^0(K, \Lambda^2 \langle x^*, y^*, z^*, u^* \rangle)$ where K is subalgebra in $L = C_\lambda^1$ such that $x = y = z = u = 0$. Calculations show that $x^* \wedge y^*$ is K-invariant if $\lambda = 1/4$ and other elements are not K-invariant. Thus, $H^2(L) = 0$ if $\lambda \neq 1/4$ and $\mathbf{C}$ if $\lambda = 1/4$. Therefore there are no solutions for $\lambda = 3/4$, one solution for each $\lambda \neq 3/4, 1/4$ and two solutions if $\lambda = 1/4$ (in this case the action of the normalizer of L on $H^2(L)$ has two orbits).

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