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CONSTANT SOLUTIONS OF YANG-BAXTER EQUATION FOR $\mathfrak{sl}(2)$ and $\mathfrak{sl}(3)$

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Abstract.

All constant solutions of the classical Yang-Baxter equation (CYBE) are listed for the function with values in $\mathfrak{sl}(2)$ and $\mathfrak{sl}(3)$ and an algorithm which allows one to obtain all constant solutions for a simple complex Lie algebra $\mathfrak{g}$ is given.

Introduction.

In what follows let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra over the field $\mathbf{C}$ of complex numbers, $X: \mathbf{C}^2 \to \mathfrak{g} \otimes \mathfrak{g}$ a function. Solutions of the classical Yang-Baxter equation

$$\text{CYBE} \quad [X^{12}(u_1,u_2), X^{13}(u_1,u_3)] + [X^{12}(u_1,u_2), X^{23}(u_2,u_3)] + [X^{13}(u_1,u_3), X^{23}(u_2,u_3)] = 0$$

where for $X = \sum a_i \otimes b_i \in \mathfrak{g} \otimes \mathfrak{g}$ we set $X^{12} = X \otimes 1$, $X^{13} = \sum a_i \otimes 1 \otimes b_i$, etc. are considered modulo equivalence relations

1) $X \sim cX$, for $c \in \mathbf{C}\backslash\{0\}$;
2) $X(u,v) \sim (\phi(u) \otimes \phi(v))X(u,v)$, where $\phi(u) \in \mathrm{Aut}(\mathfrak{g}[u])$.

In 1984 Drinfeld found all solutions for $\mathfrak{sl}(2)$ and made the following.

CONJECTURE (Drinfeld, 1984). *If $X(u,v)$ is a rational solution of* CYBE, i.e. $X(u,v) = C_2/(u-v) + r(u,v)$, *where r is a polynomial in u, v then* $\deg_u r = \deg_v r \leq 1$.

It seemed that there was some hope after all.

In [S1]–[S5] I proved this conjecture and reduced the problem of listing solutions of CYBE to classification of the so-called Lagrangian orders in $\mathfrak{g}$. They, in turn, are related with quasi-Frobenius subalgebras in $\mathfrak{g}$.

In [S5] it was shown that listing of all constant solutions of CYBE for a Lie algebra $\mathfrak{g}$ reduces to listing of quasi-Frobenius subalgebras L of $\mathfrak{g}$, their normalizers and $H^2(L)$. I will illustrate this algorithm with $\mathfrak{g} = \mathfrak{sl}(2)$ and $\mathfrak{sl}(3)$.

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Looking at nonconstant solutions we see how steeply their number increases for $\mathfrak{sl}(n)$ as n grows. Similar is the case of constant solutions.

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1. Statements.

1.1. PROPOSITION. *In* $\mathfrak{sl}(2)$ *there is only one up to conjugation 2-dimensional subalgebra: the Borel one,* $\mathfrak{B}$. *It is Frobenius and* $H^2(\mathfrak{B}) = 0$, *hence there is only one constant solution in* $\mathfrak{sl}(2)$.

1.2 To describe constant solutions in $\mathfrak{sl}(3)$ we have to consider its quasi-Frobenius subalgebras; their dimension must be even.

1.2.1. PROPOSITION. *Up to an automorphism, there is only one 6-dimensional subalgebra in* $\mathfrak{sl}(3)$, *the parabolic one,* P_1, *in notations of* [S5]. *Since* $H^2(P_1) = 0$, *there is only a constant solution of* CYBE *corresponding to* $L = P_1$.

1.2.2. In the description on 4-dimensional subalgebras the following statement is useful.

LEMMA. *Let* $\mathfrak{n}^+$ *be the algebra of uppertriangular matrices in* $\mathfrak{sl}(3)$, *Then* $\mathfrak{n}^+$ *has no subalgebras isomorphic to the Borel subalgebra of* $\mathfrak{sl}(2)$. $\mathfrak{n}^+$ *contains only the following 2-dimensional commutative subalgebras.*

$$Q^{\lambda,\nu} = \left\{ \begin{bmatrix} 0 & \nu a & b \\ 0 & 0 & \lambda a \\ 0 & 0 & 0 \end{bmatrix} \right\}$$

1.2.3. PROPOSITION. *Up to an automorphism a solvable 4-dimensional subalgebra of* $\mathfrak{sl}(3)$ *is of one of the following types:*

$$1)\ R = \left\{ \begin{bmatrix} q_1 & b & c \\ 0 & q_2 & 0 \\ & & q_3 \end{bmatrix} : \sum q_i = 0, b, c, q_i \in \mathbf{C} \right\}$$

Clearly, $R \cong \mathfrak{B} \oplus \mathfrak{B}$, where $\mathfrak{B}$ is the Borel subalgebra of $\mathfrak{sl}(2)$ and R is a Frobenius one.

$$2)\ R_{a_1,a_2,a_3} = \left\{ \begin{bmatrix} sa_1 & b & c \\ 0 & sa_2 & d \\ & & sa_3 \end{bmatrix} : \sum a_i = 0; s, b, c, d \in \mathbf{C} \right\}.$$

Clearly, R_{a_1,a_2,a_3} is $\mathrm{Aut}(\mathfrak{sl}(3))$-isomorphic to $R_{a'_1,a'_2,a'_3}$ if and only if either

$$(a_1, a_2, a_3) = t(a'_1, a'_2, a'_3) \quad \text{or}$$

$$(a_1, a_2, a_3) = (a'_3, a'_2, a'_1).$$

R_{a_1,a_2,a_3} is a Frobenius algebra for all values of parameters except $a_1 = a_3$,

1.2.4. Denote $\mathfrak{N}(\mathfrak{J})$ the normalizer of a Lie subalgebra $\mathfrak{J}$ in PGL(3) and by $\mathfrak{G}(\mathfrak{J})$ the group generated by exp(ad x) for $x \in \mathfrak{J}$.

PROPOSITION.
1) $H^2(R) \cong \mathbf{C}$ *and* $\mathfrak{N}(R)/_{\mathfrak{G}(R)} \cong \mathbf{Z}/2\mathbf{Z}$
2) $H^2(R_{a_1,a_2,a_3}) = 0$ *except for the following cases:*
$\dim H^2(R_{1,-1,0}) = \dim H^2(R_{1,1,-2}) = 1$
$\mathfrak{N}(R_{a,b,c})/\mathfrak{G}(R_{a,b,c}) \cong \mathbf{C}^*$ *for any* $a, b, c \in \mathbf{C}$.

REMARK. Since $H^2(R_{1,-2,1}) = 0$, there is no solution corresponding to this algebra.

1.2.5. PROPOSITION. *The commutative 2-dimensional algebra L can be embedded into* $\mathfrak{sl}(3)$ *in the following ways (up to an automorphism of* $\mathfrak{sl}(3)$).

1) $L \cong \mathfrak{H}$ *the diagonal Cartan subalgebras;* $\mathfrak{N}(\mathfrak{H})/_{\mathfrak{G}(\mathfrak{H})} \cong \{1\}$

2) $L \cong C_1 = \left\{\begin{bmatrix} a & b & 0 \\ 0 & a & 0 \\ 0 & 0 & -2a \end{bmatrix}\right\}$; $\mathfrak{N}(C_1)/_{\mathfrak{G}(C_1)} \cong \mathbf{C}^*$.

3) $L \cong Q^{0,1}$ or $Q^{1,1}$; $\mathfrak{N}(Q^{\lambda,\nu})/_{\mathfrak{G}(Q^{\lambda,\nu})} \cong (\mathbf{C}^*)^2$.

In either case $H^2(L) \cong \mathbf{C}$.

1.2.6. PROPOSITION. *The Borel subalgebra* $\mathfrak{B} \subset \mathfrak{sl}(2)$ *can be embedded into* $\mathfrak{sl}(3)$ *in the following ways up to an automorphism:*

1) $\mathfrak{B} \cong C_\lambda = \left\{\begin{bmatrix} \lambda a & b & 0 \\ 0 & (\lambda-1)a & 0 \\ 0 & 0 & (1-2\lambda)a \end{bmatrix}\right\}$;

2) $\mathfrak{B} \cong C^1_{1/3} = \begin{bmatrix} 2a/3 & 0 & b \\ 0 & -a/3 & a \\ 0 & 0 & -a/3 \end{bmatrix}\Bigg\}$;

3) $\mathfrak{B} \cong C'_0 = \left\{\begin{bmatrix} a & b & 0 \\ 0 & 0 & b \\ 0 & 0 & -a \end{bmatrix}\right\}$;

4) $\mathfrak{B} \cong C^{1,1}_{1/3} = \left\{\begin{bmatrix} 2a/3 & b & b \\ 0 & -a/3 & 0 \\ 0 & 0 & -a/3 \end{bmatrix}\right\}$.

THEOREM. *All the quasi-Frobenius subalgebras of* $\mathfrak{sl}(2)$ (*resp.* $\mathfrak{sl}(3)$) *up to automorphisms of* $\mathfrak{sl}(2)$ (*resp.* $\mathfrak{sl}(3)$) *are listed in Tables* 1 *and* 2.

TABLE 1

L	$H^2(L)$	The formula for r
$\mathfrak{B}$	0	$r = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

TABLE 2
List of quasi Frobenius subalgebras in $\mathfrak{sl}(3)$

N	L	dim L	$H^2(L)$	$\mathfrak{N}(L)/_{\mathfrak{G}(L)}$	Number of solutions with given L and remarks
1	$\mathfrak{H}$	2	$\mathbf{C}$	$\{1\}$	∞, $\mathfrak{H}$ is a Cartan subalgebra
2	C_1	2	$\mathbf{C}$	$\mathbf{C}^* = \mathbf{C}\backslash\{0\}$	1. $C_1 = \left\{\begin{bmatrix} a & b & 0 \\ 0 & a & 0 \\ 0 & 0 & -2a \end{bmatrix}\right\}$
3	$Q^{\lambda,1}$ $\lambda = 0, 1$	2	$\mathbf{C}$	$(\mathbf{C}^*)^2$	1. $Q^{\lambda,1} = \left\{\begin{bmatrix} 0 & a & b \\ 0 & 0 & \lambda a \\ 0 & 0 & 0 \end{bmatrix}\right\}$
4	C_λ	2	0		1. $C_\lambda = \left\{\begin{bmatrix} \lambda a & b & 0 \\ 0 & (\lambda-1)b & 0 \\ 0 & 0 & (1-2\lambda)a \end{bmatrix}\right\}$ is Frobenius
5	$C^1_{1/3}$	2	0		1. $C^1_{1/3} = \left\{\begin{bmatrix} 2a/3 & 0 & b \\ 0 & -a/3 & a \\ 0 & 0 & -a/3 \end{bmatrix}\right\}$ is a Frobenius one
6	C'_0	2	0		1. $C'_0 = \left\{\begin{bmatrix} a & b & 0 \\ 0 & 0 & b \\ 0 & 0 & -a \end{bmatrix}\right\}$ is a Frobenius one
7	$C^{1,1}_{1/3}$	2	0		1. $C^{1,1}_{1/3} = \left\{\begin{bmatrix} 2a/3 & b & b \\ 0 & -a/3 & 0 \\ 0 & 0 & -a/3 \end{bmatrix}\right\}$ is a Frobenius one
8	R_{a_1,a_2,a_3} $a_1 > a_3$ $a_1 \neq a_2$ $a_2 \neq a_3$ $a_1a_3 \neq 0$	4	0		1. $R_{a,b,c} = \left\{\begin{bmatrix} at & * & * \\ 0 & bt & * \\ 0 & 0 & ct \end{bmatrix}\right\}$ is a Frobenius one if $a \neq c$

Table 2 (cont.)

N	L	dim L	$H^2(L)$	$\mathfrak{N}(L)/_{\mathfrak{G}(L)}$	Number of solutions with given L and remarks
9	$R_{0,1,-1}$	4	C	C*	2
10	$R_{1,1,-2}$	4	C	C*	2
11	R	4	C	$\{\pm 1\}$	$\infty, R = \left\{\begin{bmatrix} * & * & * \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix}\right\}$
12	P_1	6	0		1. P_1 is a Frobenius one

2. Proofs.

2.1. Proof of Proposition 1.1. immediately follows from discussion in [S5].

2.2 Proof of Proposition 1.2.1. A 6-dimensional subalgebra must be reducible in standard representation of $\mathfrak{sl}(3)$. The statement follows by dimension considerations.

2.3. Proof of Lemma 1.2.2. is done by straightforward verification.

2.4. Proof of Proposition 1.2.3. Let L be an arbitrary 4-dimensional solvable subalgebra of $\mathfrak{sl}(3)$. Using an automorphism we may assume that $L \subset \mathfrak{B}^+$ where $\mathfrak{B}^+$ is the Lie algebra of uppertriangular matrices in $\mathfrak{sl}(3)$. Clearly, $\mathfrak{B}^+ = \mathfrak{H} + \mathfrak{n}^+$ where $\mathfrak{H}$ is the Cartan subalgebra and $\mathfrak{n}^+$ is the Lie algebra of strictly uppertriangular matrices, Consider the projection $L_{\mathfrak{H}}$ of L to $\mathfrak{H}$.

1) $\dim L_{\mathfrak{H}} = 2$.

Let $z_1, z_2 \in L$ be elements whose projections to $\mathfrak{H}$ generates $\mathfrak{H}$. We may assume that eigenvalues of z_1, z_2 are different and nonzero. Then without loss of generality we may assume that z_1 is diagonal matrices. By dimension considerations $\dim(L \cap \mathfrak{n}^+) = 2$ being ad z_1-invariant. By Lemma 1.2.2. $L \cap \mathfrak{n}^+ = Q^{\lambda,\nu}$.

Now, it is clear that $L = R \cong \mathfrak{B} \oplus \mathfrak{B}$ since $\lambda\nu = 0$ and therefore R is a Frobenius Lie algebra since too is $\mathfrak{B}$.

2) $\dim L_{\mathfrak{H}} = 1$.

Let the projection be generated by $(a_1, a_2, a_3) \in \mathfrak{H}$. Then, clearly, $L = R_{a_1,a_2,a_3}$. If $a_1 \neq a_3$, then the map f $R_{a_1,a_2,a_3} \to$ C given by the formula

$$f\colon \begin{bmatrix} sa_1 & b & c \\ 0 & sa_2 & d \\ 0 & 0 & sa_3 \end{bmatrix} \to c$$

determines a nondegenerate form $f([.,.])$.

If $a_1 = a_3$ then the algebra R_{a_1,a_2,a_3} has a center generated by the matrix

$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and therefore R_{a_1,a_2,a_3} can not be a Frobenius algebra.

There remained to prove statements on equivalence up to an automorphism. As is well-known $\mathrm{Aut}(\mathfrak{sl}(3)) \cong PGL(3) \times \mathbb{Z}/2\mathbb{Z}$ where $\mathbb{Z}/2\mathbb{Z}$ is the group of automorphisms of Dynkin diagram of $\mathfrak{sl}(3)$.

Let $X \in \mathfrak{SL}(3)$ and $X^{-1}R_{a_1,a_2,a_3}\,X = R_{a_1',a_2',a_3'}$. Clearly, that in this case $X^{-1}\mathfrak{n}^+X = \mathfrak{n}^+$ and therefore X is an uppertriangular matrix which does not move R_{a_1,a_2,a_3}.

The other equivalence mentioned in Proposition is obviously determined by an outer automorphism.

2.5. Proof of Proposition 1.2.4. Thanks to the low dimension of the algebras involved the calculations of cohomology can be performed directly.

Statement 2) on the normalizer is also obvious.

To prove statement of 1) on normalizer notice that R^t is an algebra with precisely 2 invariant 1-dimensional subspaces generated, respectively, by

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

2.6. Proof of Proposition 1.2.5. In $\mathfrak{sl}(3)$, consider the subalgebra L generated by commuting matrices T_1 and T_2.

1) Suppose that the eigenvalues of T_1 are distinct. This immediately implies that $L \cong \mathfrak{H}$.

2) Suppose that the eigenvalues of T_1 are $a, a, -2a$ for $a \neq 0$. Clearly, that in this case $L \cong \mathfrak{H}$ or $L \cong C_1$.

3) Both T_1 and T_2 have only 0 eigenvalues. We may assume that $L \subset \mathfrak{n}^+$. By Lemma 1.2.2. $L \cong C_2$.

Proposition is proved.

2.7. Proof of Proposition 1.2.6. Let us denote T subalgebra in $\mathfrak{sl}(3)$ generated by two matrices T_1 and T_2 such that $[T_1, T_2] = T_2$.

We may assume that $T \subset \mathfrak{B}^+$ and $T_2 \in \mathfrak{n}^+$. Let t_1, t_2, t_3 be the eigenvalues of T_1. Then the eigenvalues of $\mathrm{ad}(T_1)$ are of the form $t_i - t_j$. Since $[T_1, T_2] = T_2$ we have $t_i - t_j = 1$ for some i, j. There are four cases:

1) $t_i - t_j = 1$ only for $i = 1; j = 2$; in this case

$$\mathfrak{B} \cong C_\lambda = \begin{bmatrix} \lambda a & b & 0 \\ 0 & (\lambda - 1)a & 0 \\ 0 & 0 & (1 - 2\lambda)a \end{bmatrix}$$

2) $t_1 - t_2 = t_2 - t_3 = 1$; in this case

$$\mathfrak{B} \cong C_0' = \left\{ \begin{bmatrix} a & b & 0 \\ 0 & 0 & b \\ 0 & 0 & -a \end{bmatrix} \right\}$$

3) $t_1 - t_2 = 1$, $t_2 = t_3$; this case has two subcases:

a) T_1 is a diagonal matrix. Then

$$\mathfrak{B} \cong C_{1/3}^{1,1} = \left\{ \begin{bmatrix} 2a/3 & 0 & b \\ 0 & -a/3 & 0 \\ 0 & 0 & -a/3 \end{bmatrix} \right\}$$

b) T_1 is not diagonalizable. In this case

$$\mathfrak{B} \cong C_{1/3}^{1} = \left\{ \begin{bmatrix} 2a/3 & 0 & b \\ 0 & -a/3 & a \\ 0 & 0 & -a/3 \end{bmatrix} \right\}$$

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