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SIMPLE C^*-ALGEBRAS WITH THE PROPERTY WEAK (FU) *)

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Abstract.

We prove that the Bunce-Deddens algebras, "most" of the irrational rotation algebras, and the algebras of real rank 0 recently classified by Elliott, all have the property weak (FU): every unitary in the identity component of the unitary group is a norm limit of unitaries with finite spectrum. The proofs use approximation of unitaries by exponentials to derive this fact from the fact that the algebras involved have real rank 0. We also discuss the related properties (FU) (every unitary is a limit of unitaries with finite spectrum) and (FI) (every invertible is a limit of invertibles with finite spectrum).

Introduction.

In the last few years, the property (FS) (every selfadjoint element is a limit of selfadjoint elements with finite spectrum; now also called real rank 0) has been shown to be equivalent to a number of other properties, and many simple C^*-algebras have been shown to have this property. (See [7] and the references given there.) In this paper, we consider the very similar property (FU) (every unitary element is a limit of unitaries with finite spectrum) and several variants.

It is easily seen that (FU) implies (FS), and also that (FU) implies that the unitary group is connected. Since many interesting algebras with (FS) have disconnected unitary groups, it seems appropriate to consider the property weak (FU), which is (FU) applied only to the identity component of the unitary group. In this paper, we show that the irrational rotation algebras A_θ have weak (FU) for θ in a dense G_δ-subset of $[0, 1] - \mathbb{Q}$, that the Bunce-Deddens algebras have weak (FU), and that Elliott's C^*-algebras of real rank 0, obtained as direct limits of "basic buildings blocks" ([16]), all have weak (FU). We also give several examples of separable simple unital C^*-algebras which have (FU) but are not AF, using a modification of a construction in [26]. One of our examples is even stably finite.

The proofs that the algebras above have weak (FU) depend on the known results for (FS). (These results are due to Choi-Elliott [10] for "most" of the irrational rotation algebras, due to Blackadar-Kumjian [6] for the Bunce-

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Deddens algebras, and are part of the definition for Elliott's algebras.) Our contribution is to show that, in these algebras, every unitary in the identity component is a limit of exponentials. This result is obtained by approximation from similar results for algebras of sections of locally trivial M_n-bundles over compact spaces of dimension at most two and for Elliott's basic building blocks. The results on approximation by exponentials are part of a joint project with John Ringrose on the exponential rank for C^*-algebras; roughly speaking, the exponential rank of A is the smallest n such that $\exp(iA_{sa})^n$ is the whole identity component of the unitary group. The case we use here is exponential rank at most $1 + \varepsilon$.

Section 1 contains the definitions of (FU), weak (FU), and exponential rank, some connections between them, and results on exponential rank in direct limits and continuous fields. Section 2 proves that algebras of sections of locally trivial M_n-bundles over 2-dimensional spaces, and Elliott's basic building blocks, have exponential rank at most $1 + \varepsilon$. In Section 3 we then give the results and examples discussed in the second paragraph. We are still missing a number of examples; in particular, we do not even have an example of a simple C^*-algebra whose exponential rank is greater then $1 + \varepsilon$. In Section 4, we discuss the Banach algebra analogs of our concepts, obtained by substituting arbitrary invertible elements for unitaries. Our results are very incomplete, but we do show that the behavior is essentially different.

We will use the following notation throughout this paper. K is the algebra of compact operators on a separable infinite-dimensional Hilbert space. $C(X, A) = C(X) \otimes A$ is the algebra of continuous functions from X to A; if A is omitted, it is taken to be $\mathbf{C}$. For any C^*-algebra A, we let A_{sa} denote the set of selfadjoint elements of A. If A is unital then $U(A)$ is the unitary group of A and $U_0(A)$ is the connected component of $U(A)$ containing 1; similarly, $\mathrm{inv}(A)$ and $\mathrm{inv}_0(A)$ are the invertible group of A and its identity component. A^+ is the unitization of A.

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1. Definitions and elementary remarks.

In this section, we define the properties of interest, and prove a few elementary facts about them. The property (FU) was suggested to us by Shuang Zhang; it appears (without a name) in [8]. It implies that the unitary group is connected, which fails in many naturally occurring algebras with real rank 0. This fact led us to the property weak (FU), in which only the identity component of the unitary group is considered.

1.1 DEFINITION. (1) A unital C^*-algebra A is said to have the *property* (FU) if the elements of $U(A)$ with finite spectrum are dense in $U(A)$.

(2) A has *weak* (FU) if the elements of $U_0(A)$ with finite spectrum are dense in $U_0(A)$.

Von Neumann and AF algebras clearly have (FU). We will construct in Section 3 a separable C^*-algebra which has (FU) but is not AF. It is easy to see that the Calkin algebra has weak (FU). Most of this paper is devoted to proving that the Bunce-Deddens algebras, "most" of the irrational rotation algebras, and Elliott's algebras of real rank 0, all have weak (FU).

We will study (FU) and weak (FU) with the aid of the exponential rank, defined next. This notion is taken from a joint research project with the John Ringrose. A strong version of the condition exponential rank at most $1 + \varepsilon$ (the condition of most use here) has been used in [8].

1.2 DEFINITION. Let A be a unital C^*-algebra. We will define the *exponential rank* of A, written $\operatorname{cer}(A)$, to be the largest element of the set of symbols $\{1, 1+\varepsilon, 2, 2+\varepsilon, \ldots, \infty\}$ (with the obvious order) consistent with the following restrictions:

(1) $\operatorname{cer}(A) \leqq n$ if every $u \in U_0(A)$ is the product $\exp(ih_1) \ldots \exp(ih_n)$ for some $h_1, \ldots, h_n \in A_{sa}$.

(2) $\operatorname{cer}(A) \leqq n + \varepsilon$ if every $u \in U_0(A)$ is a norm limit of products as in (1).

For nonunital A, set $\operatorname{cer}(A) = \operatorname{cer}(A^+)$.

1.3 REMARK. If $u \in U(A)$ and $\|u - 1\| < 2$, then u has a logarithm by functional calculus. Thus, in the definition, $\operatorname{cer}(A) \leqq n + \varepsilon$ does indeed imply $\operatorname{cer}(A) \leqq n + 1$. This shows that the possible values of $\operatorname{cer}(A)$ are correctly ordered.

1.4 REMARK. It is obvious that $\operatorname{cer}(A \oplus \mathbf{C}) = \operatorname{cer}(A)$ for unital A. Thus, $\operatorname{cer}(A^+) = \operatorname{cer}(A)$ for arbitrary (not just nonunital) C^*-algebras A.

In this paper, no C^*-algebra A will be shown to have exponential rank greater than $1 + \varepsilon$. However, to show the limitations of the methods used here, we remark that $\operatorname{cer}(C(X) \otimes M_n)$ can be shown to be at least 2 whenever $n \geqq 2$ and X contains a homeomorphic image of an open subset of $\mathbf{R}^3$. As of this writing, we have found a C^*-algebra A with $\operatorname{cer}(A) \geqq 2 + \varepsilon$, but we have been unable to prove that any C^*-algebra has exponential rank 3 or greater.

(NOTE added February 1991: I have now proved that the exponential rank of a C^*-algebra can be arbitrarily large or even infinite. See [33] and [34].)

The connection between (weak) (FU) and exponential rank is given in the following easy proposition. Note the real rank 0 ($\mathrm{RR}(A) = 0$) is the same as (FS) ([7], Theorem 2.6).

1.5 PROPOSITION. *Let A be a unital C^*-algebra. Then A has weak* (FU) *if and only if A has real rank* 0 *and* $\mathrm{cer}(A) \leqq 1 + \varepsilon$. *Also A has* (FU) *if and only if* $\mathrm{RR}(A) = 0$, $\mathrm{cer}(A) \leqq 1 + \varepsilon$, *and $U(A)$ is connected.*

PROOF. First let A have weak (FU). Then $\mathrm{cer}(A) \leqq 1 + \varepsilon$ because every unitary with finite spectrum is an exponential. To see that $\mathrm{RR}(A) = 0$, let $a \in A_{sa}$; we want to show that a is a limit of selfadjoint elements with finite spectrum. Without loss of generality we may take $\|a\| < \pi$. Write $e^{ia} = \lim u_n$, where $u_n \in U(A)$ has finite spectrum. Let log be the branch of the logarithm function with range $i(-\pi, \pi]$. Then $-i \log(u_n)$ is selfadjoint, has finite spectrum, and converges to a.

For the converse, let $\mathrm{cer}(A) \leqq 1 + \varepsilon$. Let $u \in U_0(A)$ and write $u = \lim \exp(ia_n)$ with $a_n \in A_{sa}$. If also $\mathrm{RR}(A) = 0$, choose $b_n \in A_{sa}$ with finite spectrum such that $\|a_n - b_n\| < 1/n$. Then $u_n = \exp(ib_n)$ is a unitary with finite spectrum and $u_n \to u$. So A has weak (FU).

For the second part of the proposition, note that it is obvious that A has (FU) if and only if it has weak (FU) and $U(A)$ is connected.

1.6 PROPOSITION. *If φ: $A \to B$ is a surjective map of C^*-algebras, then* $\mathrm{cer}(B) \leqq \mathrm{cer}(A)$.

PROOF. Use the fact that $U_0(A) \to U_0(B)$ is surjective ([4], Proposition 3.4.5).

1.7 PROPOSITION. *Let $A = \varinjlim A_\alpha$ be a direct limit of C^*-algebras, and suppose that* $\mathrm{cer}(A_\alpha) \leqq n + \varepsilon$ *for all α. Then* $\mathrm{cer}(A) \leqq n + \varepsilon$.

PROOF. Let $\varphi_\alpha : A_\alpha \to A$ be the canonical maps. Unitizing everything (see Remark 1.4), we may assume that all the algebras and maps are unital. Let $u \in U_0(A)$, and write $u = \exp(ih_1) \ldots \exp(ih_N)$ for some N (presumably larger than n) and selfadjoint elements $h_1, \ldots, h_N \in A$. Since $\cup_\alpha \varphi_\alpha(A_\alpha)$ is dense in A, and since our index set is directed, we can find $\alpha(k)$ and $h_r^{(k)} \in (A_{\alpha(k)})_{sa}$ such that $\varphi_{\alpha(k)}(h_r^{(k)}) \to h_r$ for $r = 1, \ldots, N$. Then $v_k = \exp(ih_1^{(k)}) \ldots \exp(ih_N^{(k)})$ is an element of $U_0(A_{\alpha(k)})$ such that $\varphi_{\alpha(k)}(v_k) \to u$. By assumption, for each k there is $u_k \in U_0(A_{\alpha(k)})$ which is a product of n exponentials and satisfies $\|u_k - v_k\| < 1/k$. Then $\varphi_{\alpha(k)}(u_k)$ is a product of n exponentials and converges to u.

The corresponding result with n in place of $n + \varepsilon$ is false, as can be seen from Example 1.11.

Next we will consider fibers of continuous fields. We need a lemma.

1.8 LEMMA. *Let A be a unital continuous field of C^*-algebras over a compact* *metrizable space X, with fibers A_x. Assume that A is separable in the sense of* [13], 10.2.1. *That is, there is a countable subset S_0 of the C^*-algebra $\Gamma(A)$ of all continuous sections of A such that $\{s(x): s\in S_0\}$ is dense in A_x for every $x\in X$. Then there exists a countable subset $S\subset U(\Gamma(A))$ such that:*

(1) *$u(x)\in U_0(A_x)$ for all $u\in S$ and for all $x\in X$.*
(2) *$\{u(x): u\in S\}$ is dense in $U_0(A_x)$ for all x.*

PROOF. First observe that for any $s\in\Gamma(A)$, the set

$$\{x\in X: s(x)\in \operatorname{inv}(A_x)\}$$

is open. Indeed, if $a(x_0)$ is invertible, then there is $b\in\Gamma(A)$ such that $b(x_0)=a(x_0)^{-1}$ ([13], 10.1.10), and we have $\|a(x)b(x)-1\|$, $\|b(x)a(x)-1\|<1$ for x close enough to x_0. This proves openness.

Now we want to prove that

$$\{x\in X: s(x)\in \operatorname{inv}_0(A_x)\}$$

is open. Consider the continuous field $C([0,1])\otimes A$ with fibers $C([0,1])\otimes A_x$ and continuous sections $C([0,1])\otimes\Gamma(A)$. If $s(x_0)\in\operatorname{inv}_0(A_{x_0})$, choose a continuous path $\alpha\mapsto r_\alpha(x_0)$ in $\operatorname{inv}(A_{x_0})$ for $\alpha\in[0,1]$, with $r_0(x_0)=s(x_0)$ and $r_1(x_0)=1$. Regarding $r(x_0)$ as an element of $C([0,1])\otimes A_{x_0}$, let $r\in\Gamma(C([0,1])\otimes A)$ be a section with the given value at x_0 ([13], 10.1.10). Then for all x sufficiently close to x_0, we have $r(x)$ invertible in $C([0,1])\otimes A_x$, and $\|r_0(x)-s(x)\|$, $\|r_1(x)-1\|<1$. It follows that $s(x)\in\operatorname{inv}_0(A_x)$ for each such x, and openness is proved. In fact, more can be said. Extend $\alpha\mapsto r_\alpha(x)$ to a function on $[-1,2]$ by using straight line paths from $s(x)$ to $r_0(x)$ over $[-1,0]$ and from $r_1(x)$ to 1 over $[1,2]$, and reparameterize. This shows that if $s(x_0)\in\operatorname{inv}_0(A_{x_0})$, then there is a neighborhood U of x_0 and a section r of $C([0,1])\otimes A$ over U such that, for each $x\in U$, the path $\alpha\mapsto r_\alpha(x)$ is a continuous path of invertible elements connecting $s(x)$ to 1.

We will now construct a set S of sections in $\operatorname{inv}(\Gamma(A))$ satisfying the conclusions (1) and (2) for $\operatorname{inv}_0(A_x)$ in place of $U_0(A_x)$. The set whose existence is asserted by the lemma will then be $\{s(s^*s)^{-1/2}: s\in S\}$.

Fix temporarily $s\in S_0$. For each $x\in X$ such that $s(x)\in\operatorname{inv}_0(A_x)$, choose open sets $x\in U_x^{(s)}\subset\bar U_x^{(s)}\subset V_x^{(s)}$, a continuous section $(\alpha,y)\mapsto r_\alpha(y)$ of $C([0,1])\otimes A$ over $V_x^{(s)}$ with $r_0(y)=s(y)$ and $r_1(y)=1$, and a continuous function $f\colon X\to[0,1]$ such that $f(y)=1$ for $y\notin V_x^{(s)}$ and $f(y)=0$ for $y\in U_x^{(s)}$. Define $t_x^{(s)}\in\Gamma(A)$ by

$$t_x^{(s)}(y)=\begin{cases}1 & y\notin V_x^{(s)}\\ r_{f(y)}(y) & y\in V_x^{(s)}\end{cases}$$

Note that $t_x^{(s)}=s$ on $U_x^{(s)}$, that $t_x^{(s)}$ is invertible, and that $t_x^{(s)}(y)\in\operatorname{inv}_0(A_y)$ for $y\in X$.

Since X, being compact metrizable, is second countable, we can write

$$\{x \in X : s(x) \in \mathrm{inv}_0(A)\} = \bigcup_{n=1}^{\infty} U^{(s)}_{x_n(s)}$$

for appropriate $x_n(s) \in X$.

Set

$$S = \{t^{(s)}_{x_n(s)} : s \in S_0,\ n = 1, 2, \ldots\}.$$

Clearly $S \subset \mathrm{inv}(\Gamma(A))$ and $s(x) \in \mathrm{inv}_0(A_x)$ for $x \in X$ and $x \in S$. It remains to prove that $\{s(x) : s \in S\}$ is dense in $\mathrm{inv}_0(A_x)$ for $x \in X$. But this follows because $\mathrm{inv}_0(A_x)$ is open, $\{s(x) : s \in S_0\}$ is dense in A_x, and

$$\{s(x) : s \in S_0\} \cap \mathrm{inv}_0(A_x) \subset \{s(x) : s \in S\} \subset \mathrm{inv}_0(A_x).$$

1.9 PROPOSITION. *Let A be a continuous field as in the previous lemma, and fix $n \in \mathbf{N}$. Then the set*

$$G = \{x \in X : \mathrm{cer}(A_x) \leqq n + \varepsilon\}$$

is a G_δ-set in X.

PROOF. Let S be a set of sections as in the previous lemma. For each $u \in S$ and $\varepsilon > 0$, set

$$V_{u,\varepsilon} = \{x \in X : \text{there are } h_1, \ldots, h_n \in (A_x)_{sa} \text{ such that } \|u(x) - \exp(ih_1) \ldots \exp(ih_n)\| < \varepsilon\}$$

Note that $V_{u,\varepsilon}$ is open. Indeed, given x and $h_1, \ldots, h_n$, choose sections $k_1, \ldots, k_n$ through $h_1, \ldots, h_n$, replace k_i by $(k_i + k_i^*)/2$, and note that

$$\|u(y) - \exp(ik_1((y)) \ldots \exp(ik_n(y))\| < \varepsilon$$

for all y close enough to x.

Set

$$V = \bigcap_{u \in S} \bigcap_{k=1}^{\infty} V_{u,1/k}.$$

Then V is a G_δ-set. If $x \notin V$, then some $u(x)$ is not a limit of products of n exponentials, whence $\mathrm{cer}(A_x) > n + \varepsilon$. If $x \in V$, then every $u(x)$ for $u \in S$ is a limit of products of n exponentials. Since the $u(x)$ are dense in $U_0(A_x)$, it follows that $\mathrm{cer}(A_x) \leqq n + \varepsilon$. Thus $V = G$.

We will finish this section by computing the exponential ranks of a few important C^*-algebras.

1.10 PROPOSITION. (1) *Every finite dimensional C^*-algebra has exponential rank* 1.

(2) *Every von Neumann algebra has exponential rank* 1.

(3) *The Calkin algebra has exponential rank* 1.

(4) *Every commutative C^*-algebra has exponential rank* 1.

(5) *An* AF *algebra has exponential rank* 1 *or* $1 + \varepsilon$.

PROOF. Part (2) is obtained by using the Borel functional calculus on unitaries with any Borel branch of the logarithm function. Part (1) is a special case of part (2). Part (3) follows from part (2) by Proposition 1.6. Part (4) follows from the relation $\exp(a + b) = \exp(a)\exp(b)$ if a and b commute. Part (5) follows from (1) and Proposition 1.7.

1.11 EXAMPLE. A simple AF algebra can have exponential rank $1 + \varepsilon$. (This example was obtained from discussions with Bruce Blackadar.) Let G be the semidirect product $D \rtimes \mathbf{Z}/2\mathbf{Z}$, where D is the group of dyadic rationals mod 1, and the generator of $\mathbf{Z}/2\mathbf{Z}$ acts on $d \in D$ by inversion. Let D act on S^1 by rotation, and let $\mathbf{Z}/2\mathbf{Z}$ act by $z \mapsto -z$. This defines an action of G on S^1. Then $C^*(G, S^1)$ is simple ([4], 10.11.5 b), AF ([24]), and $C(S^1)$ is a maximal commutative subalgebra ([29], Proposition 4.14). (Also see [5], Remark 7.1.4b.) Let $u(z) = z$. If $u = \exp(ih)$ for some selfadjoint $h \in C^*(G, S^1)$, then h commutes with u and u^*, whence $h \in C(S^1)$. Since $u \notin U_0(C(S^1))$, this is a contradiction. So $\mathrm{cer}(C^*(G, S^1)) \neq 1$.

1.12 EXAMPLE. The algebra K is an AF algebra with exponential rank 1. To see this, note that the spectrum of any unitary $u \in K^+$ has at most one cluster point (since $u - \lambda \cdot 1 \in K$ for some $\lambda \in S^1$), and so can't be all of S^1. So u has a logarithm.

2. The exponential rank of some n-homogeneous C^*-algebras.

The purpose of this section is to prove the following theorem, and its analog for Elliott's "basic building blocks."

2.1 THEOREM. *Let X be a compact metric space with* $\dim(X) \leqq 2$ *in the sense of* [22]. *Let E be a locally trivial M_n-bundle over X. Then the C^*-algebra $\Gamma(E)$ of continuous sections of E has exponential rank at most* $1 + \varepsilon$.

This theorem is a generalization of results obtained jointly with John Ringrose. The proof consists of four steps: reduction to the case of a finite simplicial complex, reduction from unitary sections to $SU(n)$-valued sections, approximation of $SU(n)$-valued sections by ones which have n distinct eigenvalues, and proving that such sections have logarithms. The first and third steps are closely related to Theorem 4 of [10]. The first step is essentially the next lemma.

2.2 LEMMA. *Let X be a compact metric space of dimension at most d, in the sense of* [22], *and let E be a locally trivial M_n-bundle over X. Let $u \in U_0(\Gamma(E))$ and let $\varepsilon > 0$. Then there exist a finite simplicial complex L of dimensional at most d, a continuous surjective map $f: X \to L$, a locally trivial M_n-bundle E_0 over L with an* *isomorphism* $\varphi : E \to f^*(E_0)$, *and* $v \in U_0(\Gamma(E_0))$, *such that* $\|\varphi(u) - f^*(v)\| < \varepsilon$.

Here $f^*(E_0)$ is the pullback of E_0, and f^* also denotes the unital homomorphism $a \mapsto a \circ f$ from $\Gamma(E_0)$ to $\Gamma(f^*(E_0))$.

PROOF OF LEMMA 2.2. Write $X = \varprojlim X_n$, where the X_n are compact polyhedrons of dimension at most d, and with surjective maps $f_n : X \to X_n$. (See [18], Satz 1, page 229; also see [25].) Let $F = K \otimes E$ be the locally trivial bundle whose fiber over $x \in X$ is $K \otimes E_x$. Let δ be the Dixmier-Douady class of this bundle ([13], 10.7.14), which is an element of the Čech cohomology group $H^3(X, \mathbf{Z})$. By Theorem X.10.1 of [15], we have $H^3(X, \mathbf{Z}) \cong \varinjlim H^3(X_n, \mathbf{Z})$. Without loss of generality, we may therefore assume that $\delta = f_0^*(\delta_0)$ for some $\delta_0 \in H^3(X_0, \mathbf{Z})$.

Choose ([13], Theorem 10.8.4) a locally trivial bundle F_0 over X_0 with fiber K and Dixmier-Douady class δ_0. Let $f_{mn} : X_m \to X_n$ be the maps of the inverse system, and set $F_n = f_{n0}^*(F_0)$. Then $\varprojlim F_n$ is a locally trivial bundle over X with fiber K, and it is isomorphic to $f_0^*(F_0)$. Therefore it has Dixmier-Douady class $f_0^*(\delta_0) = \delta$. Hence $\varprojlim F_n \cong F$ by [13], Theorem 10.8.4. Furthermore, $\Gamma(F) \cong \varinjlim \Gamma(F_n)$, where the direct limit is formed using the maps $f_{mn}^* : \Gamma(F_n) \to \Gamma(F_m)$.

Let $p \in \Gamma(F)$ be the tensor product of a rank one projection in K and the identity of $\Gamma(E)$. Choose projections $p_n \in \Gamma(F_n)$ such that $f_n^*(p_n) \to p$. For large enough n, there is a unitary $w_n \in \Gamma(F)^+$ such that $w_n f_n^*(p_n) w_n^* = p$, and we can require $w_n \to 1$. Then $\cup_n w_n f_n^*(\Gamma(F_n)) w_n^*$ is dense in $\Gamma(F)$, so $\cup_n w_n f_n^*(p_n)\Gamma(F_n)p_n) w_n^*$ is dense in $p\Gamma(F)p \cong \Gamma(E)$. The theorem is then proved by taking, for n large enough, $L = X_n$, $E_0 = p_n F_n p_n$, $\varphi(a) = w_n^* a w_n$, and $v = b(b^*b)^{-1/2}$ for some $b \in p_n \Gamma(F_n) p_n$ with $\|w_n f_n^*(b) w_n^* - u\|$ sufficiently small.

For the second step, we introduce the following notation. If E is a locally trivial M_n-bundle over a space X, then U_E, SU_E, E_{sa} and L_E denote the sets of elements of E which are unitary, unitary with determinant 1, selfadjoint, and selfadjoint with trace 0 respectively. Since the determinant and trace are preserved by all automorphisms of M_n, these four sets are well defined locally trivial fiber bundles over X, with the same group (namely $\mathrm{Aut}(M_n)$) and with fiber $U(n)$, $\mathrm{SU}(n)$, $(M_n)_{sa}$, and $L = \{h \in (M_n)_{sa} : \mathrm{tr}(h) = 0\}$ respectively. Furthermore, we will denote by $2\pi\mathbf{Z}/n + L_E$ the bundle over X whose fiber over x is $\dfrac{2\pi}{n}\mathbf{Z} \cdot 1 + (L_E)_x$. Note that it is again a locally trivial fiber bundle with the same group, because 1 is a continuous section of E which is invariant under the group of the bundle.

As before, we use the letter Γ for spaces of continuous sections. We let $\Gamma_0(U_E)$ and $\Gamma_0(SU_E)$ denote the sets of sections of U_E and SU_E which are homotopic, via sections of the appropriate bundle, to the constant section 1. Thus $\Gamma_0(U_E) = U_0(\Gamma(E))$. For $a \in \Gamma(E)$ we let $\det(a)$ denote the continuous function $x \mapsto \det(a(x))$; note that it is well defined and continuous. Thus, $\Gamma(SU_E) = \{u \in \Gamma(U_E) : \det(u) = 1\}$. Similarly, the usual trace is a well defined map from $\Gamma(E)$ to $C(X)$.

2.3 LEMMA. *Let X be a compact space, and let E be a locally trivial M_n-bundle over X. Then* $\operatorname{cer}(\Gamma(E)) \leqq r$ *(respectively,* $\operatorname{cer}(\Gamma(E)) \leqq r + \varepsilon$*) if and only if for every* $u \in \Gamma_0(SU_E)$ *there are* $h_1, \ldots, h_r \in \Gamma(2\pi\mathbb{Z}/n + L_E)$ *such that* $u = \exp(ih_1)\ldots\exp(ih_r)$ *(respectively, u is a uniform limit of products of this sort).*

PROOF. We first establish the following claim:

(*) If $u \in \Gamma_0(SU_E)$ is a product of r elements of $\exp(i\Gamma(E_{sa}))$, then u is a product of r elements of $\exp(i\Gamma(2\pi\mathbb{Z}/n + L_E))$.

To prove this, let $u \in \Gamma_0(SU_E)$ and let $u = \exp(ih_1) \ldots \exp(ih_r)$ with $h_1, \ldots, h_r \in \Gamma(E_{sa})$. Set

$$\alpha = \frac{1}{n}\sum_{j=1}^{r} \operatorname{tr}(h_j) \quad \text{and} \quad z = \exp(i\alpha).$$

Then

$$z^n = \prod_{j=1}^{r} \det(\exp(ih_j)) = \det(u) = 1.$$

Therefore the range of z is contained in $\exp(2\pi i\mathbb{Z}/n)$, and the range of α is contained in $2\pi\mathbb{Z}/n$. Define

$$k_1 = h_1 + \left[\alpha - \frac{1}{n}\operatorname{tr}(h_1)\right]\cdot 1 \quad \text{and} \quad k_j = h_j - \frac{1}{n}\operatorname{tr}(h_j)\cdot 1$$

for $j \geqq 2$. Then the perturbations $h_j - k_j$ are all in the center of $\Gamma(E)$ and sum to 0, so

$$\exp(ik_1)\ldots\exp(ik_r) = \exp(ih_1)\ldots\exp(ih_r) = u.$$

Furthermore, $k_1 \in \Gamma(2\pi\mathbb{Z}/n + L_E)$ and $k_j \in \Gamma(L_E)$ for $j \geqq 2$. This proves the claim (*).

Next we prove the claim:

(**) If $u \in \Gamma_0(U_E)$ then there is a continuous function $\alpha: X \to \mathbb{R}$ such that $z = \exp(i\alpha)$ is an nth root of $\det(u)$ and $z^{-1}u \in \Gamma_0(SU_E)$.

To prove the claim, let $u \in \Gamma_0(U_E)$, and let $t \mapsto u_t$ be the homotopy from u to 1. Then $t \mapsto \det(u_t)$ is a homotopy from $\det(u)$ to 1, so there is a homotopy $t \mapsto \beta_t$ of continuous functions from X to $\mathbb{R}$ such that $\exp(i\beta_t) = \det(u_t)$. Then $\alpha = \beta_0/n$ (yielding $z = \exp(i\beta_0/n)$) is the required function. (Note that the homotopy $t \mapsto \exp(-i\beta_t/n)u_t$ shows that $z^{-1}u \in \Gamma_0(SU_E)$.) This proves (**).

To prove the lemma for the case $\operatorname{cer}(\Gamma(E)) \leqq r$, note that the definition of $\operatorname{cer}(\Gamma(E)) \leqq r$ is obtained by replacing SU_E by U_E and $2\pi\mathbb{Z}/n + L_E$ by E_{sa} in the condition in the lemma. That $\operatorname{cer}(\Gamma(E)) \leqq r$ implies the condition in the lemma is now just (*). For the converse, assume the condition in the lemma, and let $u \in \Gamma_0(U_E)$. Let z and α be as in (**), write $z^{-1}u = \exp(ih_1)\ldots\exp(ih_r)$ with $h_1,\ldots,h_r \in \Gamma(2\pi\mathbb{Z}/n + L_E)$, and observe that $u = \exp(ih_1 - \alpha\cdot 1)\exp(ih_2)\ldots\exp(ih_r)$.

We now turn to the case involving $\operatorname{cer}(\Gamma(E)) \leqq r + \varepsilon$. Since it is not clear that $\Gamma_0(SU_E) = \Gamma_0(U_E) \cap \Gamma(SU_E)$ for nontrivial E, we must be a little careful.

Assume $\operatorname{cer}(\Gamma(E)) \leqq r + \varepsilon$, let $u \in \Gamma_0(SU_E)$, and write $u = \lim u_m$ where each u_m is a product of r elements of $\exp(i\Gamma(E_{sa}))$. Since $\det(u) = 1$, we may assume $\|\det(u_m) - 1\| < 2$ for all m. Using the usual continuous branch of log, set

$$\beta_m = -\frac{i}{n}\log(\det(u_m)) \quad \text{and} \quad \zeta_m = \exp(i\beta_m).$$

Further choose α_m and z_m for u_m as in (**).

Since z_m is the exponential of an element of the center of $\Gamma(E)$, the elements $z_m^{-1}u_m$ are, along with u_m, products of r elements of $\exp(i\Gamma(E_{sa}))$. It follows from (*) that $z_m^{-1}u_m$ is a product of r elements of $\exp(i\Gamma(2\pi\mathbf{Z}/n + L_E))$. Furthermore, $\beta_m - \alpha_m$ takes values in $2\pi\mathbf{Z}/n$, since $\exp(in\beta_m) = \exp(in\alpha_m) = \det(u_m)$. Therefore also $\zeta_m^{-1}u_m = \exp(i(\beta_m - a_m))z_m u_m$ is a product of r elements of $\exp(i\Gamma(2\pi\mathbf{Z}/n + L_E))$. Since $\beta_m \to 0$, we have $\zeta_m^{-1}u_m \to u$, so that u is a limit of products of r elements of $\exp(i\Gamma(2\pi\mathbf{Z}/n + L_E))$, as desired.

For the converse, assume that every element $v \in \Gamma_0(SU_E)$ is a limit of products v_m of r elements of $\exp(i\Gamma(2\pi\mathbf{Z}/n + L_E))$. Let $u \in \Gamma_0(U_E)$. Choose $z = \exp(i\alpha)$ as in (**), let $v = z^{-1}u$, and let $v = \lim v_m$ as above. Since α is in the center, we get that $u = \lim zv_m$ is the limit of products of r elements of $\exp(i\Gamma(E_{sa}))$. So $\operatorname{cer}(\Gamma(E)) \leqq r + \varepsilon$.

The third step of the proof requires the following preliminary lemma.

2.4 Lemma. *The set of elements in $SU(n)$ with at least one repeated eigenvalue is the union of finitely many submanifolds of $SU(n)$, all of codimension at least* 3.

Proof. Let P be a partition of n, that is, a sequence $(n_1, \dots, n_k)$ of positive integers such that $n_1 + \dots + n_k = n$ and $n_1 \geqq n_2 \geqq \dots \geqq n_k$. Let M_P be the set of all $u \in SU(n)$ having exactly k distinct eigenvalues, with multiplicities $n_1, \dots, n_k$. Let G_P be the set of sequences $(V_1, \dots, V_k)$ of orthogonal subspaces of $\mathbf{C}^n$ such that $\dim(V_j) = n_j$ for each j. Let W_P be the set of k-tuples of *distinct* elements $(\lambda_1, \dots, \lambda_k) \in (S^1)^k$ such that $\lambda_1^{n_1} \cdot \dots \cdot \lambda_k^{n_k} = 1$. Then W_P and G_P are smooth manifolds. Define $f_P: G_P \times W_P \to M_P$ by sending $(V_1, \dots, V_k, \lambda_1, \dots, \lambda_k)$ to the unitary $u \in SU(n)$ such that $u\xi = \lambda_j \xi$ for $\xi \in V_j$. Then f_P is a smooth surjective local homeomorphism from $G_P \times W_P$ to M_P.

To show that M_P is a smooth manifold, we must show that f_P is a local diffeomorphism, that is for each $x \in G_P \times W_P$ there is a smooth map g from a neighborhood of $f_P(x)$ in $SU(n)$ to $G_P \times W_P$ such that $g \circ f_P$ is the identity near x and $f_P \circ g$ is the identity on a neighborhood of $f_P(x)$ in M_P. To construct g, let $x = (V_1, \dots, V_k, \lambda_1, \dots, \lambda_k)$, and let $u = f_P(x)$. Choose $\varepsilon > 0$ such that the ε-disks about $\lambda_1, \dots, \lambda_k$ in $\mathbf{C}$ are disjoint. For v close enough to u, let p_j be the spectral projection corresponding to $\{\lambda \in \mathbf{C} : |\lambda - \lambda_j| < \varepsilon\}$ and let W_j be the corresponding subspace. Let $\mu_n = \det(p_j v p_j)^{1/n_j}$, where $p_j v p_j$ is regarded as an operator on W_j and the n_jth root is the branch going through λ_j. Then $g(v) = (W_1, \ldots, W_k, \mu_1, \ldots, \mu_j)$ will do. (Note that it is smooth because the projections p_j can be obtained via holomorphic functional calculus.)

$SU(n)$ is the disjoint union of the manifolds M_P as P runs through all partitions. So the lemma is proved if we can show that codim $(M_P) \geqq 3$ for $P \neq (1, \ldots, 1)$. It is easily seen that the map g above extends to a local diffeomorphism $v \mapsto (W_1, \ldots, W_k, \mu_1, \ldots, \mu_k, \mu_1^{-1} p_1 v p_1, \ldots, \mu_k^{-1} p_k v p_k)$ to a manifold locally diffeomorphic to $G_P \times W_P \times SU(n_1) \times \ldots \times SU(n_k)$, and the dimension of the last part is at least 3 if some $n_j \neq 1$.

2.5 LEMMA. *Let X be a finite simplicial complex of dimension at most 2. Let E be a locally trivial M_n-bundle over X, let $u \in \Gamma(SU_E)$, and let $\varepsilon > 0$. Then there exists $v \in \Gamma(SU_E)$ such that $\|u - v\| < \varepsilon$ and $v(x)$ has no repeated eigenvalues for all $x \in X$.*

PROOF. We make use of the smooth retraction $S: U \to \mathrm{SU}(n)$, where U is a neighborhood of $\mathrm{SU}(n)$ in M_n, defined as follows:

$$S(a) = \det(a(a^*a)^{-1/2})^{-1/n}\, a(a^*a)^{-1/2}.$$

Also let

$$D_x = \{v \in (SU_E)_x : v \text{ has no repeated eigenvalues}\},$$

which is a dense open subset of $(SU_E)_x$. We define the perturbation v first on the 0-skeleton X_0 of X, then on the 1-skeleton X_1, and finally on the 2-skeleton X_2.

For $x \in X_0$, choose $v_0(x) \in D_x$ close to $u(x)$. Define v_0 elsewhere by choosing local sections w_x through $v_0(x)$ for $x \in X_0$, and setting

$$v_0 = S\left(\sum_{x \in X_0} f_x w_x + \left(1 - \sum_{x \in X_0} f_x\right) u\right)$$

for appropriate continuous $f_x: X \to [0, 1]$ supported in the domain of w_x. This can be done so that $\|v_0 - u\| \leqq \varepsilon/3$. The result satisfies $v_0(x) \in D_x$ for all x in some neighborhood W_0 of X_0.

Now we want to do the same thing for the 1-simplexes in X. On each individual 1-simplex L, we will construct an arbitrarily small perturbation v_1 of v_0, equal to v_0 on a neighborhood of the boundary of L, such that $v_1(x) \in D_x$ for all $x \in L$. The resulting function can be extended over X to satisfy $v_1(x) \in D_x$ for $x \in X_1$ and $\|v_1 - v_0\| < \varepsilon/3$ by the method used in the previous paragraph.

The 1-simplex L is homeomorphic to $[0, 1]$, which is contractible. Thus, it suffices to consider a function $v_0: [0, 1] \to \mathrm{SU}(n)$, where $v_0(x)$ has no repeated eigenvalues for $x \in [0, 3\delta]$ and $[1 - 3\delta, 1]$, for some $\delta > 0$. Choose a function a:

$[\delta, 1-\delta] \to M_n$ which is close to v_0 on this interval, agrees with v_0 at δ and $1-\delta$ and is smooth on $[2\delta, 1-2\delta]$. Then $S(a)$ is close to v_0, agrees with v_0 at δ and $1-\delta$, is smooth on $[2\delta, 1-2\delta]$, and has values in SU(n). If a is close enough to v_0, then $S(a)$ will have no repeated eigenvalues on $[\delta, 2\delta] \cup [1-2\delta, 1-\delta]$. Using the proof of the Homotopy Transversality Theorem ([19], page 70), choose w: $[2\delta, 1-2\delta] \to SU(n)$ close to $S(a)$ such that w is transverse to the finitely many submanifolds of Lemma 2.4, and agrees with $S(a)$ at 2δ and $1-2\delta$. Since these manifolds have codimension greater than 1, it follows that they do not intersect the range of w. The desired v_1 is now equal to v_0 on $[0, \delta] \cup [1-\delta, 1]$, equal to $S(a)$ on $[\delta, 2\delta] \cup [1-2\delta, 1-\delta]$, and equal to w on $[2\delta, 1-2\delta]$.

We now have $v_1 \in \Gamma(SU_E)$ such that $v_1(x) \in D_x$ for all x in a neighborhood of X_1. For each 2-simplex L of X, note that L is homeomorphic to the contractible smooth manifold with boundary $\{x \in \mathbb{R}^2 : \|x\| \leqq 1\}$. Repeat the argument used to produce v_1 on the 1-simplexes, where now $v_1(x) \in D_x$ for $1-3\delta \leqq \|x\| \leqq 1$, and with $\{x : \|x\| \in [1-2\delta, 1-\delta]\}$ in place of $[\delta, 2\delta] \cup [1-2\delta, 1-\delta]$ and $\{x : \|x\| \in [1-\delta, 1]\}$ in place of $[0, \delta] \cup [1-\delta, 1]$. The transversality argument still yields the same result, because the manifolds in Lemma 2.4 have codimension greater than 2. The result is $v \in \Gamma(SU_E)$ with $v(x) \in D_x$ for all x and $\|v - v_1\| < \varepsilon/3$. Using the triangle inequality gives $\|v - u\| < \varepsilon$.

2.6 LEMMA. *Let E be a locally trivial M_n-bundle over a path-connected space X. Using the notation Lemma 2.3, let $u \in \Gamma(SU_E)$ be a section such that $u(x)$ has no repeated eigenvalues for all $x \in X$. Then there is $h \in \Gamma(2\pi\mathbb{Z}/n + L_E)$ such that* $\exp(ih) = u$.

Note that this lemma implies $u \in U_0(\Gamma(E))$. Roughly speaking, $K_1(\Gamma(E))$ should be thought of as the odd twisted cohomology of X. The restriction $\det(u) = 1$ prevents u from representing a nonzero 1-dimensional class, and the eigenvalue restriction (obtained in Lemma 2.5 from $\dim(X) \leqq 2$) prevents u from representing a class of dimension 3 or larger.

PROOF OF LEMMA 2.6. Let γ: $[0, 1] \to X$ be any continuous path, and fix $k_0 \in (2\pi\mathbb{Z}/n + L_E)_{\gamma(0)}$ such that $\exp(ik_0) = u(\gamma(0))$. Since the eigenvalues of $u(\gamma(t))$ are all distinct for all t, there is a continuous decomposition of the identity $1 = p_1(t) + \ldots + p_n(t)$ into rank one projections onto the eigenspaces of $u(\gamma(t))$. (Note that E is locally trivial, and construct this decomposition in a neighborhood of any given $x \in X$ by using functional calculus as in the construction of the map g in the proof of Lemma 2.4.) We then have continuous functions $z_1, \ldots, z_n$ (the eigenvalues of u) such that $p_j(t)u(\gamma(t))p_j(t) = z_j(t)p_j(t)$. Now choose continuous logarithms $ir_1, \ldots, ir_n$ of $z_1, \ldots, z_n$ such that $p_j(0)k_0p_j(0) = r_j(0)p_j(0)$. (This can be done because $[0, 1]$ is contractible.) Set $k(t) = r_1(t)p_1(t) + \ldots + r_n(t)p_n(t)$.

Then $k(t) \in (2\pi \mathbb{Z}/n + L_E)_{\gamma(t)}$ and $\exp(ik(t)) = u(\gamma(t))$. Note that the function k is uniquely determined by u, γ, and k_0.

We want to define h as follows. Fix $x_0 \in X$, fix $h_0 \in (2\pi \mathbb{Z}/n + L_E)_{x_0}$ such that $\exp(ih_0) = u(x_0)$, and define $h(x)$ by choosing any path γ from x_0 to x and letting $h(x)$ be the corresponding $k(1)$ as in the previous paragraph. For this to yield a well defined $h \in \Gamma(2\pi \mathbb{Z}/n + L_E)$ it suffices to prove that if $x = x_0$ then $k(1) = h_0$. For convenience, we will take h_0 to have all eigenvalues in $[0, 2\pi)$.

Thus, let the notation be as in the first paragraph, with $\gamma(0) = \gamma(1) = x_0$ and $k_0 = h_0$. By renumbering the p_j etc., we can assume $0 \leqq r_1(0) < r_2(0) < \ldots < r_n(0) < 2\pi$. Since the r_j are continuous, and the numbers $\exp(ir_j(t))$ are distinct for all t, we have $r_1(t) < r_2(t) < \ldots < r_n(t) < r_1(t) + 2\pi$ for all t. Furthermore, we have $p_1(1) = p_{j+1}(0)$ for some j, since $\exp(ik(1)) = \exp(ik(0)) = u(x_0)$. Therefore there is an integer m such that

$$r_j(1) = r_{i+j}(0) + 2\pi m \quad \text{for } i + j \leqq n$$

and

$$r_i(1) = r_{i+j-n}(0) + 2\pi(m + 1) \quad \text{for } i + j > n.$$

Now $\mathrm{tr}(k(t))$ is continuous and has values in $2\pi\mathbb{Z}$, so is constant. Thus

$$\sum_{i=1}^{n} r_i(0) = \mathrm{tr}(k(0)) = \mathrm{tr}(k(1)) = \sum_{i=1}^{n} r_i(1) = 2\pi(nm + j) + \sum_{i=1}^{n} r_i(0),$$

whence $m = j = 0$. (Note that $0 \leqq j < n$). It follows that $k(1) = k(0)$, as desired.

PROOF OF THEOREM 2.1. If L is a connected finite simplicial complex of dimension at most 2, and F is a locally trivial M_n-bundle over L, then Lemmas 2.3, 2.5, and 2.6 combine to prove $\mathrm{cer}(\Gamma(F)) \leqq 1 + \varepsilon$. By an obvious direct sum decomposition, the assumption that L is connected can be dropped. Now let E be an arbitrary locally trivial M_n-bundle over a space X of dimension at most 2, and let $u \in U_0(\Gamma(E))$. Given $\varepsilon > 0$, choose $f: X \to L, F$, and $v \in U_0(\Gamma(F))$ as in Lemma 2.2. Choose $h \in \Gamma(E)_{sa}$ such that $\|\exp(ih) - v\| < \varepsilon$. Then $\|\exp(if^*(h)) - u\| < 2\varepsilon$, proving that $\mathrm{cer}(\Gamma(E)) \leqq 1 + \varepsilon$.

2.7 REMARK. Let E be a locally trivial M_n-bundle over X, with $n \geqq 2$. It can be shown that $\mathrm{cer}(\Gamma(E)) \geqq 1 + \varepsilon$ whenever X contains a subset homeomorphic to $[0, 1]$. (Problem 4.6.9 of [23] shows that $\mathrm{cer}(C(S^1) \otimes M_2) \geqq 1 + \varepsilon$.) As mentioned after Remark 1.4, it can also be shown that $\mathrm{cer}(\Gamma(E)) \geqq 2$ whenever X contains a subset homeomorphic to an open subset of $\mathbb{R}^3$.

We also prove an analog of Theorem 2.1 for Elliott's "basic building blocks with spectrum the interval" ([16], 4.1). (The other sort of basic building block is covered by Theorem 2.1.) Recall that these are algebras of the form

$$A_{k,n} = \{a \in C([0, 1], M_k \otimes M_n): a(0), a(1) \in \mathbb{C} \cdot 1 \otimes M_n\}$$

for fixed positive integers k, n.

2.8 PROPOSITION. *The algebras* $A_{k,n}$ *satisfy* $\operatorname{cer}(A_{k,n}) \leqq 1 + \varepsilon$.

PROOF. Let $u \in U(A_{k,n})$. Write $u(0) = 1 \otimes u_0$ and $u(1) = 1 \otimes u_1$. Perturbing u slightly, and using the function $a \mapsto a(a^*a)^{-1/2}$ appropriately, we may assume that u_0 and u_1 each have n distinct eigenvalues. Now multiply u by the central element $f(t) = \det(u(t))^{-1/(kn)} \cdot 1$ (for a continuous choice of the root). Since f is the exponential of a central skewadjoint element, this changes neither the connected component of $U(A_{k,n})$ containing u nor whether u is a limit of exponentials. Thus, we may assume $u(t) \in \mathrm{SU}(kn)$ for all t. Next, choose unitaries $v_0, v_1 \in M_n$ such that $v_0 u_0 v_0^*$ and $v_1 u_1 v_1^*$ are diagonal with eigenvalues $\exp(i\beta_1), \ldots, \exp(i\beta_n)$ and $\exp(i\gamma_1), \ldots, \exp(i\gamma_n)$ respectively, and

$$0 \leqq \beta_1 < \beta_2 < \ldots < \beta_n < 2\pi \quad \text{and} \quad 0 \leqq \gamma_1 < \gamma_2 < \ldots < \gamma_n < 2\pi.$$

Since $U(M_{kn})$ is connected, we can find $v \in C([0, 1], M_{kn})$ such that $v(0) = 1 \otimes v_0$ and $v(1) = 1 \otimes v_1$. This v is actually in $A_{k,n}$. Replacing u by vuv^* does not change the connected component containing u or whether u is a limit of exponentials, and we can now assume u_0 and u_1 are diagonal with eigenvalues as given.

Let f_j denote the projection in M_k or M_n onto the jth standard basis vector, and let $e_{j+(l-1)k} = f_j \otimes f_l \in M_k \otimes M_n$. Then $e_1, \ldots, e_{kn}$ are orthogonal rank one projections which sum to 1. We have

$$u(0) = \exp(i\beta_1) \sum_{j=1}^{k} e_j + \ldots + \exp(i\beta_n) \sum_{j=(n-1)k+1}^{nk} e_j$$

and a similar formula for $u(1)$ with γ_l in place of β_l. By perturbing u near 0 and 1, and using the operator S from the proof of Lemma 2.5, we may assume that for t close enough to 0 we have

$$u(t) = \sum_{j=1}^{nk} \exp(i\alpha_j(t)) e_j, \tag{1}$$

where $\alpha_j(0) = \beta_l$ for $(l-1)k + 1 \leqq j \leqq lk$, and where for $0 < t < \varepsilon$ we have

$$\alpha_1(t) < \alpha_2(t) < \ldots < \alpha_{nk}(t) < \alpha_1(t) + 2\pi. \tag{2}$$

Similarly, for t near 1 we can arrange to have

$$u(t) = \sum_{j=1}^{nk} \exp(i\alpha'_j(t)) e_j \tag{3}$$

where $\alpha_j(1) = \gamma_l$ for $(l-1)k + 1 \leqq j \leqq lk$, and where for $1 - \varepsilon < t < 1$ we have

$$\alpha'_1(t) < \alpha'_2(t) < \ldots < \alpha'_{nk}(t) < \alpha'_1(t) + 2\pi. \tag{4}$$

The transversality argument in the 1-simplex part of the proof of Lemma 2.5 can now be applied to obtain one final perturbation, resulting in a u such that all of the eigenvalues of $u(t)$ are distinct for $t \in (0, 1)$, and the relations (1)–(4) still hold. The proof will be completed by showing that the perturbed u is either an exponential or not in $U_0(A_{k,n})$.

Imitate the proof of Lemma 2.6 to write

$$u(t) = \sum_{j=1}^{nk} \exp(i\alpha_j(t))p_j(t) \tag{5}$$

for continuous $\alpha_j: [0, 1] \to \mathrm{R}$ (given by (1) for t near 0) and continuously varying orthogonal projections $p_j(t)$ which sum to 1 (with $p_j(t) = e_j$ for j near 0). Note that, in view of (1) and (3), the argument from Lemma 2.6 need only be applied over an appropriate interval $[\varepsilon, 1-\varepsilon]$, on which all eigenvalues of $u(t)$ are in fact distinct. It follows that (2) holds for all $t \in (0, 1)$.

Comparing (3) and (5) gives $p_j(1) = e_{\sigma(j)}$ for some permutation σ of $\{1, \dots, nk\}$. Now comparing (2) and (4) for $t \in (1-\varepsilon, 1)$, we see that σ must be a cyclic permutation, $\sigma(j) = j + m \pmod{nk}$ for some fixed m in $\{0, 1, \dots, nk-1\}$. Furthermore, there is an integer r such that:

$$\alpha'_j(1) = \alpha_{j-m}(1) + 2\pi r \text{ for } j > m \text{ and } \alpha'_j(1) = \alpha_{j-m+nk}(1) + 2\pi(r-1) \text{ for } j \leqq m.$$

Suppose k divides m. We show u is an exponential. Set

$$h(t) = \sum_{j=1}^{nk} \alpha_j(t)p_j(t).$$

Then the coefficient of e_j in $h(1)$ is constant on the ranges $(l-1)k + 1 \leqq j \leqq lk$, being equal to either $\gamma_l - 2\pi r$ or $\gamma_l - 2\pi(r-1)$ depending on whether $l > m/k$ or $l \leqq m/k$. Therefore $h \in A_{k,n}$, and $\exp(ih) = u$.

Now suppose $m = l_0 k + j_0$ with $0 \leqq l_0 \leqq n-1$ and $1 \leqq j_0 \leqq k-1$. We show $u \notin U_0(A_{k,n})$. Define $\varepsilon_j(t) = 2\pi t$ for $l_0 k + 1 \leqq j \leqq m$, and $\varepsilon_j(t) = 0$ otherwise. Set

$$h(t) = \sum_{j=1}^{nk} (\alpha_j(t) + \varepsilon_j(t))p_j(t).$$

The coefficient of e_j in $h(1)$ is $\alpha_{j-m}(1) + \varepsilon_{j-m}(1)$ for $j > m$ and $\alpha_{j-m+nk}(1) + \varepsilon_{j-m+nk}(1)$ for $j \leqq m$. These expressions are equal to $\alpha'_j(1) - 2\pi r$ for $j > l_0 k$ and to $\alpha'_j(1) - 2\pi(r-1)$ for $j \leqq l_0 k$, and are thus constant for j in the ranges $(l-1)k + 1 \leqq j \leqq lk$. Therefore $h \in A_{k,n}$, and $\exp(ih) \in U_0(A_{k,n})$. We further have $u = \exp(ih)\exp(ia)$, where

$$a(t) = -\sum_{j=0}^{nk} \varepsilon_j(t)p_j(t).$$

(Note that a is *not* in $A_{k,n}$.) Clearly $\exp(ia)$ is homotopic to the unitary

$$v(t) = \exp(-2\pi i t)(e_1 + \dots + e_{j_0}).$$

The proof of Lemma 2.1 of [17], applied to our case, shows that an element of this form is trivial in $K_1(A_{k,n})$ if and only if k divides j_0. Since that is not the case here, we conclude that v, and hence also u, is not in $U_0(A_{k,n})$.

(NOTE added February 1991: George Elliott has introduced in [30] another sort of basic building block, namely $C(S^1) \otimes A_{k,n}$. The proof just given can be generalized to show that these algebras also have exponential rank at most $1 + \varepsilon$. Also see [31].)

3. Simple C^*-algebras with (FU) and weak (FU).

In this section, we combine the results of the previous two sections to conclude that "most" of the irrational rotation algebras, the Bunce-Deddens algebras, and Elliott's algebras of real rank 0, have weak (FU). We also exhibit a separable unital C^*-algebra which has exponential rank at most $1 + \varepsilon$ but does not have real rank 0, and one which has (FU) but is not AF. Many more examples remain to be found (or, conceivably, ruled out). We know of no separable simple C^*-algebras of the following sorts: $\mathrm{cer}(A) > 1 + \varepsilon$; A has real rank 0 but not weak (FU) (such A also satisfies $\mathrm{cer}(A) > 1 + \varepsilon$); or A is nuclear, stably finite, has (FU), but is not AF.

(NOTE (added February 1991). George Elliott has constructed in [30] a nuclear stably finite simple C^*-algebra A with (FU) such that $K_0(A)$ has torsion. Therefore A is not AF. The proof of (FU) uses the note added at the end of the last section. Also see [31].)

3.1 THEOREM. *The rotation algebras A_θ satisfy* $\mathrm{cer}(A_\theta) \leqq 1 + \varepsilon$ *for θ in a dense G_δ-subset of* $[0, 1]$.

PROOF. Rieffel has shown [27] that there is a continuous field A over $[0, 1]$ whose fiber over $\theta \in [0, 1]$ is the (rational or irrational) rotation algebra A_θ. Furthermore, every element of $l^1(\mathbf{Z}, C(S^1))$ defines a continuous section of this field, via the obvious inclusion of $l^1(\mathbf{Z}, C(S^1))$ as a dense subalgebra of each crossed product $C^*(\mathbf{Z}, C(S^1), \theta) = A_\theta$. Therefore A is separable. Consequently $\{\theta \in [0, 1] : \mathrm{cer}(A_\theta) \leqq 1 + \varepsilon\}$ is a G_δ-set by Proposition 1.9.

If $\theta = p/q$ is rational (in lowest terms), then it is shown in [21] (see especially Section 2) that A_θ is the algebra of sections of a locally trivial M_q-bundle over $S^1 \times S^1$. So $\mathrm{cer}(A_\theta) \leqq 1 + \varepsilon$ by Theorem 2.1. Thus $\{\theta \in [0, 1] : \mathrm{cer}(A_\theta) \leqq 1 + \varepsilon\}$ is dense.

3.2 COROLLARY. *The irrational rotation algebras A_θ have weak* (FU) *for θ in a dense G_δ-set of* $[0, 1]$.

PROOF. By Proposition 1.5, A_θ has weak (FU) for θ in the intersection of the dense G_δ-set of the previous theorem, and the dense G_δ-set of numbers θ such that

A_θ has real rank 0 ([10]). (The authors of [10] do not say that their dense set is a G_δ-set, but it is clear from their proof.)

3.3 THEOREM. *Any algebra which can be written as a direct limit of finite direct sums of the "basic building blocks" of* [16], 4.1, *has exponential rank at most* $1 + \varepsilon$.

PROOF. A basic building block with spectrum the circle has exponential rank at most $1 + \varepsilon$ by Theorem 2.1, and one with spectrum the interval has exponential rank at most $1 + \varepsilon$ by Proposition 2.8. So finite direct sums have exponential rank at most $1 + \varepsilon$, and the result follows from Proposition 1.7.

3.4 COROLLARY. *The algebras classified in* [16], *that is, direct limits A of finite direct sums of the basic building blocks such that A has real rank* 0, *all have weak* (FU).

3.5 COROLLARY. *The Bunce-Dence algebras have weak* (FU) *and exponential rank at most* $1 + \varepsilon$.

PROOF. It is shown in [6] how to write the Bunce-Deddens algebras as direct limits of algebras of the form $C(S^1) \otimes M_n$; see (2) at the beginning of Section 3 of that paper. (This description is obtained from (1) there by writing the torsion subgroup H as a direct limit $\varinjlim H_n$ of finite groups. Then $C^*(H, S^1) = \varinjlim C^*(H_n, S^1)$.) It is also shown in [6] that the Bunce-Deddens algebras have (HP), which is the same as real rank 0 ([17], Theorem 2.6). So these algebras are covered by the previous theorem and corollary.

3.6 REMARK. Neither the irrational rotation algebras nor the Bunce-Deddens algebras have (FU), because both contain unitaries representing nonzero K_1-classes. Similarly, the algebras A of Corollary 3.4 cannot have (FU) except in the trivial case that they are AF. Indeed, if $K_1(A) \neq 0$ then it is easily seen that A contains a unitary representing a nonzero K_1-class (by examining an appropriate basic building block); otherwise, A is AF by Theorem 7.1 of [16].

Since we know of no simple C^*-algebras with exponential rank greater than $1 + \varepsilon$, we are not in a position to prove that real rank 0 does not imply exponential rank at most $1 + \varepsilon$. However, we can show that exponential rank at most $1 + \varepsilon$ does not imply real rank 0.

3.7 EXAMPLE. Let A be the algebra A_2 of Example 1.6 in [6] with $X = S^1 \times S^1$ as there. This algebra is a direct limit of algebras of the form $C(X) \otimes M_{2^n}$, and therefore has exponential rank at most $1 + \varepsilon$ by Proposition 1.7 and Theorem 2.1. It is shown in [6] that A does not have the property (HP), which is equivalent to real rank 0 ([7], Theorem 2.6).

We will now construct an example of a separable simple C^*-algebra A which has (FU) but is not AF. This example is a small modification of Proposition 16 of [26].

3.8 EXAMPLE. Let M be a type III factor on a separable Hilbert space. Construct separable unital C^*-algebras $A_0 \subset A_1 \subset \ldots \subset M$ by induction as follows. A_0 is a separable C^*-subalgebra of M containing a proper isometry u. Given A_k for k even, A_{k+1} is generated by A_k and, for each u in a countable dense subset of $U(A_k)$, the countable collection of spectral projections of u corresponding to the sets $\exp(2\pi i[r, s))$ for $r, s \in \mathbb{Q}$. Given A_k for k odd, A_{k+1} is a separable simple C^*-subalgebra of M containing A_k ([3], Proposition 2.2). Set $A = \overline{\bigcup_k A_k}$. Considering the even $k \neq 0$, Lemma 4.5 of [2] implies that A is simple. Considering the odd k, and noting that $U(A) = \overline{\bigcup_{k \text{ even}} U(A_k)}$ by functional calculus, we easily see that A has (FU). But A is not AF because it contains a proper isometry.

3.9 REMARK. By adding more elements to A_{k+1} for k even, we can get some stronger properties. Adding the matrix entries of an appropriate countable set of spectral projections for unitaries in $M_n(A_k)$, for $n \geqq 1$, we can arrange that $M_n(A)$ has (FU) for all n. Using an analogous procedure for the normal elements, not just the unitaries, we can have all matrix algebras $M_n(A)$ satisfy the stronger property (FN) ([1], 2.6): every normal element is the limit of normal elements with finite spectrum. By adding a partial isometry from p to a proper subprojection of p for one p in each of the countably many Murray-von Neumann equivalence classes of projections in A_k, we can ensure that A is purely infinite. (Every projection in A is equivalent to one in $\bigcup_{k \text{ even}} A_k$.) Doing the same with partial isometries from one projection to another, we can guarantee that $K_0(A) = 0$.

The resulting algebra A is thus unital, separable, simple, purely infinite, has stable (FN), and satisfies $K_*(A) = 0$.

The preceding example leaves open the question of whether a *finite* separable simple non-AF algebra can have (FU). We now show how the construction can be modified to produce such an example.

3.10 EXAMPLE. Let R be the hyperfinite type II_1 factor. We first prove that there exists a separable C^*-subalgebra A_0 of R which is not isomorphic to a subalgebra of any nuclear C^*-algebra. The proof is similar to that of Theorem 4.1 of [3] and Theorem 10.2 of [14].

Begin by observing that, as a C^*-algebra, R is not nuclear ([28], Corollary 1.9). As in the proof of Theorem 4.1 of [3], we get from Corollary 6.5(4) of [9] a finite dimensional operator system N and a non-nuclear completely positive map $\eta: N \to R$. Let A_0 be the unital C^*-algebra generated by $\eta(N)$, which is certainly separable. If $\varphi: A_0 \to B$ were an injective homomorphism to a nuclear C^*-algebra, then the injectivity of R as a von Neumann algebra (*not* as a C^*-algebra) would yield a completely positive map $\psi : B \to R$ such $\psi \circ \varphi$ is the inclusion of A_0 in R. Nuclearity of id_B would then imply nuclearity of the composite $\eta = \psi \circ \mathrm{id}_B \circ \varphi \circ \eta$, a contradiction.

Now repeat the construction of Example 3.8 inside R instead of M. The resulting algebra A is separable, simple, has (FU), is stably finite (since it is contained in R), and is not AF (because it can't be nuclear).

3.11 REMARK. As in Remark 3.9, we can get some stronger properties. By the same argument as there, we may arrange that $M_n(A)$ has (FN) for all n. By including the matrix entries of countably many partial isometries, we can arrange that two projections in $M_n(A)$ are equivalent if and only if they are equivalent in $M_n(R)$.

The resulting C^*-algebra A is unital, separable, simple, stably finite, has stable (FN), has $K_0(A)$ order isomorphic to a countable subgroup of $\mathbb{R}$, and is not nuclear (hence not AF).

It should be pointed out that it is not known whether all AF algebras have (FN).

3.12 QUESTION. Do any "naturally occurring" separable simple non-AF algebras have (FU)? The Cuntz algebras (which have real rank 0 by Corollary 3.10 of [7]) seem to be obvious possibilities.

(NOTE (added February 1991): It is shown in [32] that the Cuntz algebras have (FU).)

4. Banach exponential rank and property (FI).

In this section we discuss the Banach algebra versions of exponential rank and properties (FU) and weak (FU), as applied to the algebras considered in the previous section. While our results are far from complete, we prove enough to show that the behavior of these properties is quite different.

4.1 DEFINITION. A unital Banach algebra is said to have the property (FI) if the elements of $\mathrm{inv}(A)$ with finite spectrum are dense in $\mathrm{inv}(A)$.

The analog of weak (FU) is obtained by replacing $\mathrm{inv}(A)$ by $\mathrm{inv}_0(A)$. The analog of (FS) (real rank 0) is that the elements of A with finite spectrum are dense in A. In contrast to Proposition 1.5, it turns out that all three notions are equivalent. In particular, weak (FI) implies that $\mathrm{inv}(A)$ is connected.

4.2 PROPOSITION. (Contrast with Proposition 1.5.) *Let A be a unital Banach algebra. Then the following three conditions are equivalent:*

(1) *Every element of A is a limit of elements with finite spectrum.*

(2) *A has* (FI).

(3) *Every element of* $\mathrm{inv}_0(A)$ *is a limit of elements of* $\mathrm{inv}_0(A)$ *with finite spectrum.*

PROOF. (1) implies (2): Let $a \in \mathrm{inv}(A)$, and let $a_n \in A$ with $\mathrm{sp}(a_n)$ finite and $a_n \to a$. Since $\mathrm{inv}(A)$ is open, we have $a_n \in \mathrm{inv}(A)$ for all sufficiently large n.

(2) implies (3): Similarly, the subgroup $\mathrm{inv}_0(A)$ is open in $\mathrm{inv}(A)$.

(3) implies (1): Let $a \in A$. By scaling, we may assume $\|a\| < \pi/2$. Set $U = \{\zeta \in \mathbf{C} : |\zeta| < \pi\}$, and note that the exponential map is injective on this set. Let $\log = (\exp|_U)^{-1}$. Let $b_n \in \mathrm{inv}_0(A)$ be a sequence with $\mathrm{sp}(b_n)$ finite and $b_n \to \exp(a)$. Then for n large enough, we have $\mathrm{sp}(b_n) \subset \exp(U)$, so that holomorphic functional calculus yields elements $a_n = \log(b_n)$ with finite spectrum which converge to a.

Finite dimensional algebras, and hence also AF algebras, have (FI). We know of no other C^*-algebras with (FI). Note that the irrational rotation and Bunce-Deddens algebras do not have (FI) since their invertible groups are not connected. Similarly, Elliott's algebras do not have (FI) unless they are AF. (See Remark 3.6.) $L(H)$, the algebra of bounded operators on an infinite dimensional separable Hilbert space, does not have (FI), since index considerations show that the unilateral shift cannot be approximated by elements with finite spectrum. For similar reasons, infinite simple C^*-algebras cannot have (FI). (See Corollary 4.8 below.)

4.3 QUESTION. Does there exist a simple unital non-AF algebra with (FI)?

Remark 3.9 and Corollary 4.8 show that even stable (FN) for simple C^*-algebras does not imply (FI). We know nothing about possible results in the other direction.

4.4 QUESTION. Does (FI) imply any of (FS) (= real rank 0), (FU), or (FN)?

Note that a proof that (FI) implies (FN) would show that AF algebras have (FN), which is presently unknown.

4.5 DEFINITION. The *Banach exponential rank* $\mathrm{ber}(A)$ of a Banach algebra A is defined by modifying Definition 1.2 as follows: replace $U_0(A)$ by $\mathrm{inv}_0(A)$, replace A_{sa} by A, and replace $\exp(ih_j)$ by $\exp(h_j)$.

Clearly (FI) implies $\mathrm{ber}(A) \leqq 1 + \varepsilon$. The converse is false in general ($C([0,1]) \otimes M_2$ is a counterexample), but we do not know whether it holds for simple C^*-algebras.

There is one obvious relation between the Banach and C^* exponential ranks. The expression $\mathrm{cer}(A) + 1$ is interpreted to mean $(n+1) + \varepsilon$ if $\mathrm{cer}(A) = n + \varepsilon$.

4.6 PROPOSITION. *Let A be a C^*-algebra. Then* $\mathrm{ber}(A) \leqq \mathrm{cer}(A) + 1$.

PROOF. We may assume A is unital. Let $\mathrm{cer}(A) \leqq n$, and let $a \in \mathrm{inv}_0(A)$. Then $a(a^*a)^{-1/2} \in U_0(A)$ and is therefore a product of n exponentials. Also $(a^*a)^{1/2}$ has spectrum contained in $(0, \infty)$ and therefore has a logarithm. So $a = [a(a^*a)^{-1/2}][(a^*a)^{1/2}]$ is a product of $n+1$ exponentials, and $\operatorname{ber}(A) \leqq n+1$. The case $\operatorname{cer}(A) \leqq n + \varepsilon$ is similar.

We have no estimate in the other direction. The possibility $\operatorname{ber}(A) = \operatorname{cer}(A) + 1$ occurs for $L(H)$. Indeed, Theorem 4.1 of [11] implies that there are invertible elements which are not limits of exponentials, so that $\operatorname{ber}(L(H)) \geqq 2$; but $\operatorname{cer}(L(H)) = 1$.

Actually, more can be proved.

4.7 THEOREM. *If A is an infinite simple unital C^*-algebra, then* $\operatorname{ber}(A) \geqq 2$.

PROOF. According to 2.2 and the following remark in [12], there are orthogonal projections $p, q, p_0, q_0 \in A$ with $p \sim q \sim p_0 \sim q_0 \sim 1$ (Murray-von Neumann equivalence). Let x be a partial isometry implementing $p \sim q$, that is, $x^*x = p$, $xx^* = q$. Let v be a partial isometry with $v^*v = p$ and $vv^* < p$. (For example, v could be y^2 where $y^*y = 1$ and $yy^* = p$.) Now define the following elements in A:

$$w = xvx^*$$

$$e = q - ww^*$$

$$z = x(p - vv^*)$$

$$u = (w^* + z^* + v) + (1 - p - q)$$

$$a = (2w^* + z^* + v) + (1 - p - q).$$

Then one can check that $w^*w = q$ and $ww^* < q$, so that e is a nonzero projection. One also checks that $z^*z = p - vv^*$ and $zz^* = e$, that u is unitary, and that a is invertible with inverse

$$a^{-1} = \left(\frac{1}{2}w + z + v^*\right) + (1 - p - q).$$

With respect to the decomposition $1 = p + q + (1 - p - q)$, we can write u and a in matrix form as

$$u = \begin{matrix} v & z^* & 0 \\ 0 & w^* & 0 \\ 0 & 0 & 1 \end{matrix} \quad \text{and} \quad a = \begin{matrix} v & z^* & 0 \\ 0 & 2w^* & 0 \\ 0 & 0 & 1 \end{matrix} .$$

The only essential properties are that $v^*v = p$, $w^*w = q$, and $z \neq 0$ is a partial isometry with $z^*z = p - vv^*$ and $zz^* = q - ww^*$.

We now modify the construction slightly so as to ensure $a \in \operatorname{inv}_0(A)$. Replacing 2 with $1 + \alpha$, for $\alpha \in [0, 1]$, in the definition of a yields a path of invertibles connecting a to u. Now let s be a partial isometry with $s^*s = p + q$ and

$ss^* = p_0 + q_0$. Then the element

$$(1 - ss^* + su^*s^*)u = (w^* + z^* + v) + s(w^* + z^* + v)^*s^* + (1 - p - q - p_0 - q_0)$$

is in $U_0(A)$, since in an obvious matrix decomposition it has the form $c \oplus c^* \oplus 1$. The equivalence $q \sim 1$ yields a partial isometry t with $t^*t = p_0 + q_0$ and $tt^* \leqq q$. Standard 2×2 matrix arguments show that $1 - ss^* + su^*s^*$ is homotopic to $1 - tt^* + tsu^*s^*t^*$. Therefore $(1 - tt^* + tsu^*s^*t^*)u \in U_0(A)$. But this element differs from u only in that w has been replaced by wc, where $c = q - tt^* + tsu^*s^*t^* \in U(qAq)$. Making this change has no effect on the essential properties given above, so we may replace w by wc and assume $u \in U_0(A)$ and $a \in \mathrm{inv}_0(A)$.

Let π be a unital representation of A on a Hilbert space H. We will show that $\pi(a)$ is not in the closure of the range of the exponential map from $L(H)$ to $L(H)$. Let $f_0 = \pi(e)$, and let $f_n = \pi(u)^n f_0 \pi(u^*)^n$ for $n \in \mathbf{Z}$. The relations $upu^* \leqq p$ and $u^*(q - e)u \leqq q - e$ are easily seen to imply that $u^n e(u^*)^n$ is orthogonal to e for $n \in \mathbf{Z} - \{0\}$. Applying π we can now show that the f_n are mutually orthogonal projections in $L(H)$. Let $f = \sum_{n=-\infty}^{\infty} f_n$. Since $\pi(u)^n$ determines an isomorphism $f_n H \cong f_0 H$ for any n, we have a decomposition

$$H \cong (1 - f)H \oplus l^2(\mathbf{Z}) \otimes f_0 H,$$

with respect to which $\pi(u)$ becomes $(1 - f) \oplus (s \otimes 1)$, with s being the bilateral shift. Furthermore, $\pi(a)$ becomes the operator $b \oplus (c \otimes 1)$, where

$$b = \pi(1 - p - q) + \left(\pi(p) - \sum_{n=1}^{\infty} f_n\right) + 2\left(\pi(q) - \sum_{n=0}^{-\infty} f_n\right)$$

and $c : l^2(\mathbf{Z}) \to l^2(\mathbf{Z})$ is

$$(c\xi)(n) = \begin{cases} \xi(n-1) & n \geqq 1 \\ 2\xi(n-1) & n \leqq 0 \end{cases}$$

We want to prove that $\pi(a) - \lambda$ is semi-Fredholm with nonzero index for $1 < |\lambda| < 2$. Since b has spectrum $\{1, 2\}$, it suffices to prove that $c - \lambda$ is surjective but not injective for $1 < |\lambda| < 2$. The vector

$$\xi(n) = \begin{cases} \lambda^{-n} & n \geqq 1 \\ (2\lambda^{-1})^n & n \leqq 0 \end{cases}$$

is a nonzero element of $\mathrm{Ker}(c - \lambda)$. Furthermore, if $\xi \in l^2(\mathbf{Z})$, we let

$$\xi_+(n) = \begin{cases} \xi(n) & n \geqq 1 \\ 0 & n \leqq 0 \end{cases}$$

and $\xi_- = \xi - \xi_+$. Let

$$\eta_+ = \sum_{n=0}^{\infty} (-\lambda)^{-n+1} s^n(\xi_+) \quad \text{and} \quad \eta_- = \lambda^{-1} \sum_{n=1}^{\infty} (\lambda/2)^n (s^*)^n(\xi_-).$$

Then $\eta = \eta_+ + \eta_-$ is in $l^2(\mathbf{Z})$, since

$$\|\eta\| \leqq |\lambda|^{-1}(1 - |\lambda|^{-1})^{-1} \|\xi_+\| + \frac{1}{2}(1 - |\lambda|/2)^{-1} \|\xi_-\|,$$

and a computation shows that $(c - \lambda)(\eta) -= \xi$. So $c - 1$ is surjective.

Theorem 4.1 of [11] now implies that $\pi(a)$ is not in the closure of the range of the exponential map. (I am grateful to Kevin Clancey for pointing out this reference.) Therefore a is not the closure of the range of the exponential map of A.

4.8 COROLLARY. *An infinite simple unital C^*-algebra cannot have* (FI).

The basic properties of the C^* exponential rank from Section 1 also hold for ber. Finite dimensional algebras have Banach exponential rank 1, by holomorphic functional calculus. Therefore AF algebras have Banach exponential rank 1 or $1 + \varepsilon$. Again, $\mathrm{ber}(K) = 1$ by the argument from Example 1.12. (Even a nonnormal compact operator cannot have a nonzero cluster point in its spectrum.) Using the Putnam-Fuglede Theorem ([20], Problem 52), one can make the argument of Example 1.11 show that the algebra used there has Banach exponential rank $1 + \varepsilon$. This argument now even applies to the 2^∞ UHF algebra, since the maximal commutative subalgebra $C(X)$ of [5], Corollary 7.1.3 is generated by a and a^* for a single normal element $a \in \mathrm{inv}(C(X)) - \mathrm{inv}_0(C(X))$.

The methods of Section 2, however, do not carry over. A version of Lemma 2.4 is still true, but the codimension can now be as small as 2. (The set of elements in $SL(2, \mathbf{C})$ whose Jordan canonical form is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has codimension 2.) Thus, $C([0, 1], M_2)$ can be shown to have Banach exponential rank $1 + \varepsilon$. However, the argument in Lemma 2.6 to deal with noncontractibility fails.

4.9 EXAMPLE. Let $A = C(S^1) \otimes M_2$, where $S^1 \subset \mathbf{C}$ is the unit circle. We show $\mathrm{ber}(A) \geqq 2$. Let $a(\zeta) = \begin{pmatrix} \frac{1}{2}\zeta & 0 \\ 0 & 2\zeta^{-1} \end{pmatrix}$ for $\zeta \in S^1$. Then a is easily seen to be in $\mathrm{inv}_0(A)$.

Choose $\rho > 0$ so that $b \in A$ and $\|b - a\| < \rho$ imply $\mathrm{sp}(b) \cap (S^1 \cup \{0\}) = \emptyset$. Fix any b satisfying this estimate; we show b is not an exponential. Set $c_t = (1 - t)a + tb$ for $t \in [0, 1]$. Then $\|a - c_t\| < \rho$ for all t. Regard c as an element of $C([0, 1] \times S^1) \otimes M_2$, and apply holomorphic functional calculus to c with the functions χ and $1 - \chi$, where

$$\chi(\zeta) = \begin{cases} 1 & 0 < |\zeta| < 1 \\ 0 & |\zeta| > 1 \end{cases}$$

The result is idempotents e and f with $e + f = 1$. The ranks of e and f are locally constant, hence both constant equal to 1. Thus, we can write $c_t(\zeta) = \lambda_t(\zeta)e_t(\zeta) + \mu_t(\zeta)f_t(\zeta)$, with λ and μ jointly continuous in t and ζ, and $\lambda_0(\zeta) = \frac{1}{2}\zeta$, $\mu_0(\zeta) = 2\zeta^{-1}$. By continuity, we must have $|\lambda_t(\zeta)| < 1$ and $|\mu_t(\zeta)| > 1$ for all t, ζ.

Suppose $b = \exp(x)$ for some $x \in C(S^1) \otimes M_2$. Since x commutes with b, and $b = c_1$ has distinct eigenvalues, we must have $x(\zeta) = \alpha(\zeta)e_1(\zeta) + \beta(\zeta)f_1(\zeta)$ for $\zeta \in S^1$, with α, β continuous. Therefore $\lambda_1(\zeta) = \exp(\alpha(\zeta))$, so the winding number of λ_1 about the origin is 0. But the homotopy $t \mapsto \lambda_t$, from $\lambda_1(\zeta)$ to $\frac{1}{2}\zeta$, shows that this winding number is 1, a contradiction. So b is not an exponential, and $\operatorname{cer}(A) \geqq 2$ has been proved.

Because of this example, our earlier methods fail to show that any Bunce-Deddens algebras or irrational rotation algebras have Banach exponential rank at most $1 + \varepsilon$. They also fail to show that direct limits of Elliott's "basic building blocks" [16] have Banach exponential rank at most $1 + \varepsilon$. Indeed, one can show that even the algebra $A_{1,2}$, as defined just before Proposition 2.8, has Banach exponential rank at least 2. (The element $a \in A_{1,2}$ given by

$$a(t) = \begin{pmatrix} (2 + \sin(\pi t))\exp(2\pi it) & 0 \\ 0 & (2 - \sin(\pi t))\exp(-2\pi it) \end{pmatrix}$$

is not a limit of exponentials; the proof is a bit more complicated than in the previous example.)

4.10 QUESTION. Is there a simple unital C^*-algebra (necessarily finite) which is not AF and whose Banach exponential rank is at most $1 + \varepsilon$?

We don't even know the Banach exponential rank of a type II_1 factor.

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