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A COUNTEREXAMPLE TO THE SEPARATION OF SEMIALGEBRAIC SETS

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1. Introduction.

A well known problem in the study of the geometry of semialgebraic sets is the separation problem, that is the problem of investigating when two given disjoint semialgebraic sets A and B in a real affine algebraic variety M are polynomially separated. Recall that the sets A and B are called (polynomially) separated if there exists a polynomial function f on M which is positive on A and negative on B (we will write $f(A) > 0$ and $f(B) < 0$).

It is an already classic result that in general such a function does not exist (see, for instance, the well known example due to Mostowski [7]), but many partial results are known ([8], [2], [4]).

A natural question in this context is: if A and B are open disjoint semialgebraic sets in a real affine algebraic variety M, say of pure dimension n, and they are quasi-separated (i.e. there exists a polynomial function f on M such that $f(A) \geqq 0$, $f(B) \leqq 0$ and $\dim V(f) \cap (A \cup B) < n$), then is it true that they can be polynomially separated?

Obviously, some hypothesis have to be added, otherwise the answer is trivially negative. For instance, consider the following elementary example in R^2: let $A' = \{h > 0\}$ and $B = \{h < 0\}$, where $h(x,y) = x^2 + y^2 - x^3$, and let A be the interior of $\overline{A'}$. It is clear that A and B are quasi-separated, but they cannot be separated by any polynomial function.

In [4] the previous question was studied under some reasonable hypothesis on the behaviour of the boundaries of the two semialgebraic sets; we recall here some of the positive results obtained there, still considering, for the sake of simplicity, only the pure dimensional case. Note that the Zariski closure of a set C will be denoted by $\overline{C}^Z$.

THEOREM 1.1 ([4], [5]). *Let A and B be open disjoint semialgebraic sets in a real affine algebraic pure dimensional variety M, such that $\overline{\partial A}^Z \cap A = \emptyset$ and*

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$\overline{\partial B^Z} \cap B = \emptyset$. Suppose that A and B are quasi-separated by a polynomial function f on M. If one of the following conditions is satisfied:

a) $\dim V(f) \cap (A \cup B) \leqq 1$,

b) *M is non singular,* $\dim \overline{V(f) \cap \partial B^Z} \cap A \leqq 1$ *and* $\dim \overline{V(f) \cap \partial A^Z} \cap B \leqq 1$,

then A and B are separated.

A question arises in a natural way: is the hypothesis on the relation between $V(f)$ and the boundaries of A and B in 1.1b) really needed, at least for basic open semialgebraic sets? (An open semialgebraic set is called basic open if it can be defined using only intersections of sets of solutions of strict polynomial inequalities). Such a question has been open for a long time.

We have to recall that one of the reasons of interest in that is the fact that 1.1a) has as an easy consequence ([5]) a simple proof, in the case $\dim M \leqq 3$, of Bröcker-Scheiderer's theorem ([3], [9], see also [6]) asserting that any basic open semialgebraic set in a variety M, $\dim M = n$, can be defined using no more than n inequalities. In [5] we remarked also that a positive answer to the separation problem in the general case, at least for basic open sets, would have implied in an easy and geometric way the result of Bröcker-Scheiderer without any restriction on the dimension of M.

Now we are able to show that, even in the non singular case and for basic open semialgebraic sets, the answer to the question "quasi-separation implies separation?" is negative. In fact in this note we will exhibit two basic open semialgebraic sets in R^4 which are quasi-separated, but not separated.

Finally, we would like to thank T. Mostowski for the helpful discussion we had on the subject.

2. The counterexample.

It can be useful to sketch here the strategy we used to find a counterexample to the separation problem.

First of all, consider a real algebraic set V of codimension 1 in R^n. Let V_{n-1} be the set of all points $x \in V$ such that the local dimension of V at x is $n-1$ and let $C = V - V_{n-1}$. Suppose that $\overline{V_{n-1}}^Z \supseteq C$ and that there exists no polynomial function P on R^n such that $P(C) > 0$ and $P(V_{n-1}) \leqq 0$.

Let h be a polynomial function on R^n which is a generator of the ideal of V. Clearly, $\forall x \in C$, the sign of h is "constant" (i.e. $\geqq 0$ or $\leqq 0$) in a small neighborhood U of x in R^n, because $U - V(h)$ is connected. Let's suppose this sign does not depend on x, that is suppose, for instance, that h is $\geqq 0$ locally at each point of C.

Consider now $B = \{x \in \mathsf{R}^n \mid h(x) < 0\}$; it is easy to check that $\partial B = V_{n-1}$. Moreover, let A be an open semialgebraic neighbourhood of C in R^n such that $h(A - C) > 0$ and that $\overline{\partial A \cap \partial B^Z}$ doesn't contain C, which is, informally speaking, the "minimal" necessary condition for the separability of A and B. We remark that such an A exists; take, for instance, $A = \{x \in \mathbb{R}^n \mid d(x, C) < d(x, B)\}$, where $d(x, C)$ (resp. $d(x, B)$) denotes the distance function from the semialgebraic set C (resp. B), which is a semialgebraic function.

As $h(A - C) > 0$, the semialgebraic sets A and B are quasi-separated by h, but they cannot be separated. In fact, suppose, on the contrary, there exists a polynomial function P on $\mathbb{R}^n$ such that $P(A) > 0$ and $P(B) < 0$. Such a P cannot identically vanish on ∂B, because otherwise $V(P) \supseteq \overline{\partial B}^Z = \overline{V_{n-1}}^Z \supseteq C$, which cannot happen. So on ∂B the function P should be generically negative, contradicting the assumption of the inseparability of V_{n-1} from C.

While B is evidently basic open, in general A is not; but if C happens to be basic open in $\overline{C}^Z$, we can modify A so that it becomes basic open. In fact in this situation there exists ([1], lemma 6.3) a basic open semialgebraic set A' in $\mathbb{R}^n$ such that $A' \subseteq A$ and $C \subseteq A'$. So if A fails to the basic, we have only to replace it by A'.

In conclusion, in order to obtain a counterexample to the separation problem it is enough to find an algebraic set V satisfying the properties described above. So we begin to construct our counterexample by presenting an algebraic set V having the properties we need.

Consider the following function on $\mathbb{R}^4$ (already used by Bröcker in [3] for a different aim)

$$h(x, y, z, t) = (z^2 \cdot f_1(x, y) + t^2) \cdot (t^2 \cdot f_2(x, y) + z^2),$$

where $\quad f_1(x, y) = x^2 + y^2 - 1 \quad$ and $\quad f_2(x, y) = (x - 1)^2 + y^2 - 1.$

Let $H = \{z = t = 0\}$.

It is easy to remark that the 3-dimensional variety $V = V(h)$ contains the plane H, which is not an irreducible component of V. Moreover $\operatorname{Sing} V = H$, thus the singular locus of V has codimension 1 in V. So in this example, with the notations introduced above, we have $V_3 = \overline{V - H}$ and $C = H - V_3$.

The variety V satisfies all the properties we need, in particular the set V_3 of the points of dimension 3 in V cannot be polynomially separated from the set C of the points of dimension 2 in V, in the sense previously precised. In fact suppose there exists a polynomial function P such that $P(C) > 0$ and $P(V_3) \leqq 0$; then $P(V_3 \cap H) \leqq 0$. But $V_3 \cap H$ is the union of two intersecting discs in H; so we get a contradiction, because no polynomial function on H can be $\leqq 0$ on these disks and positive outside.

Following the recipe described above, we use this algebraic set to construct our counterexample. In particular, as C is basic open in H, we know it is possible to choose A basic open in $\mathbb{R}^4$; that is immediately obtained by choosing $A = \{f_1 > 0, f_2 > 0\}$, which verifies also the other properties requested on A.

Therefore, for the reasons explained before, the sets

$$A = \{f_1 > 0, f_2 > 0\} \qquad B = \{h < 0\},$$

are the example of quasi-separated, but not separated, basic open sets we looked for. We remark only that $\overline{V(h) \cap \partial B^Z} \cap A = C$; that's the reason why in this situation 1.1b) does not apply.

3. Final remarks.

REMARK 3.1. The variety V of the previous section turns out to be a fruitful source of counterexamples; it was in such a role that Bröcker used it in [3]. Let us make only the following remarks.

a) The set $\overline{V - H} \cap H (= V_3 \cap H)$, being the union of two intersecting discs in H, is not basic closed in H (i.e. defined by simultaneous polynomial large inequalities). Then, although $V - H$ is basic open in V, the set $\overline{V - H}$ is not basic closed in V. So this example shows that, even in a variety of dimension 3, if the singular locus has codimension 1, the closure of a basic open semialgebraic set is not in general a basic closed set.

b) The set $\overline{V - H}$ is an example of a generically basic semialgebraic set which is not basic (recall that a semialgebraic set A is called generically basic if there exists a basic semialgebraic set B such that the codimension of $(A \cup B) - (A \cap B)$ is positive).

REMARK 3.2. The counterexample of the previous section gives also an answer to a problem of a different nature, arising from [4].

Given an open semialgebraic set B of a non singular affine variety M and a non singular subvariety H, say irreducible, $\operatorname{codim} H > 1$, it is known ([4], proof of 3.3) that it is possible to find an algebraic subvariety H' of M containing H, such that $\dim H' = \dim H + 1$ and $\overline{B} \cap H \subseteq \overline{B \cap H'}$. The question is if one can find H' such that Sing H' does not contain H.

Our counterexample shows that in general this is not possible. In fact consider B and H as above in the variety $M = \mathbb{R}^4$. We claim that every subvariety H', $\dim H' = 3$, satisfying the requested properties with respect to H and B must be singular in codimension 1. To see that, we will use Bröcker's result ([3], theorem 8.3) asserting that in a variety M such that $\dim M \leqq 3$ and $\dim \operatorname{Sing} M \leqq 1$, the closure of any basic open set is a basic closed set. So, if in our example we had $\dim \operatorname{Sing} H' \leqq 1$, the set $\overline{B \cap H'}$ would be basic closed in H' and, as a consequence, $\overline{B \cap H'} \cap H$ would be basic in H. But the condition $\overline{B} \cap H \subseteq \overline{B \cap H'}$ easily implies that $\overline{B} \cap H = \overline{B \cap H'} \cap H$. So we get a contradiction, because we know that $\overline{B} \cap H$ is not basic. So $\dim \operatorname{Sing} H' = 2$. Finally remark that $\operatorname{Sing} H'$ must contain H, because otherwise one could repeat the argument above in the affine variety $M - \operatorname{Sing} H'$ getting again a contradiction.

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