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SELF-INTERSECTION OF FIXED MANIFOLDS AND RELATIONS FOR THE MULTISIGNATURE

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Abstract.

Let M^{2n} be a smooth, closed, orientable $2n$-manifold and K_x^{2n-2} be an orientable submanifold of M^{2n} dual to a cohomology class x. If $K_x^{(s)}$ is the s-fold self-intersection of K_x in M^{2n} and d is a nonnegative integer, then the signatures of $K_{dx}^{(s)}$ and $K_x^{(s)}$ are related by a numerical congruence. This congruence is used to study diffeomorphisms of odd prime order which fix a codimension 2 submanifold.

1. Introduction.

Let M^{2n} be a smooth, closed, orientable $2n$-manifold. If $K^{2n-2} \subset M^{2n}$ is a closed, orientable submanifold, and s is a nonnegative integer, then the s-fold self-intersection of K in M is defined inductively: $K^{(0)} = M, K^{(1)} = K$, and if $K^{(s)} \subset M$ and $j: K^{(s)} \to M$ is transverse to K, then $K^{(s+1)} = j^{-1}(K)$. The dimension of $K^{(s)}$ is $2n - 2s$. In particular, $K^{(n)}$ is a set of points. There is a chain of submanifolds $K^{(n)} \subset K^{(n-1)} \subset \ldots \subset K \subset M$. If $x \in H^2(M; \mathbf{Z})$, then K is dual to x if $i_*[K] = x \cap [M]$, where i: $K \subset M$ is the inclusion, and we will write K_x ro indicate this duality. If d is a nonnegative integer, our first theorem expresses the signature of $K_{dx}^{(s)}$ in terms of the signature of $K_x^{(s)}$. If n is a positive integer, let $f(n)$ be the quotient of $n!$ divided by a maximal power of 2.

THEOREM 1.1. *If $n - s$ is even and d is a nonnegative integer, then*

$$f(n) \operatorname{Sign} K_{dx}^{(s)} \equiv f(n)\, d^s \operatorname{Sign} K_x^{(s)} (\operatorname{mod} d^s(1 - d^2)).$$

The special case of Theorem 1.1 is which n is odd and $s = 1$ is a congruence for the signature of the submanifold K_{dx} itself. This special case was proved in [9]. The advantage of the more general formulation of Theorem 1.1 is that we need not consider only odd integers n in the applications. The principal application of Theorem 1.1 is to the study of finite group actions on M which fix a codimension 2 submanifold F. Let p be an odd prime and let G_p denote the cyclic group of order p. The Atiyah-Singer g-Signature Theorem [1, 2] expresses the value of the

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multisignature Sign (G_p, M) on a generator of G_p in terms of the action of G_p on the normal bundle of the fixed submanifold. Theorem 1.1 together with a formula of Berend and Katz [3] will enable us to find an expression for the contribution of F to the multisignature.

Let g be a generator of G_p and let Sign (g, M) be the value that Sign (G_p, M) takes on at g. Let ν be the normal bundle of F in M and let $\lambda = e^{i\theta}$, $\theta = 2\pi/p$, be the eigenvalue of the action of G_p on ν. The contribution of ν to Sign (g, M) in the signature formula is $L_\theta(\nu)L(F)[F]$, where $L(F)$ is the total Hirzebruch L-class of F and $L_\theta(\nu)$ is a nonstable characteristic class determined by θ and the Chern class of ν ([5], p. 492). The formula of Berend and Katz shows that $L_\theta(\nu)L(F)[F]$ is determined by the signatures of the self-intersections $F^{(s)}$, $s = 1, 2, \ldots, n$, and the algebraic number $\alpha = (\lambda + 1)(\lambda - 1)^{-1}$.

THEOREM 1.2 (Berend and Katz [3]). *If M^{2n} admits a smooth G_p action fixing a codimension 2 submanifold F, then the contribution of ν in the signature formula for* Sign (g, M) *is*

$$L_\theta(\nu)L(F)[F] = \alpha \,\mathrm{Sign}\, F + (\alpha^2 - 1) \sum_{s=1}^{n-1} (-1)^s \alpha^{s-1} \,\mathrm{Sign}\, F^{(s+1)}. \tag{1.3}$$

We remark that if the eigenvalue at ν is determined by $\theta = 2\pi ij/p$, $1 < j \leqq p - 1$, then formula (1.3) holds with α replaced by $\alpha_j = (\lambda^j + 1)(\lambda^j - 1)^{-1}$. It is clear that if Theorems 1.1 and 1.2 are used together, then $L_\theta(\nu)L(F)[F]$ can be expressed in terms of d and signatures of $K_x^{(s)}$ if it is known that F is dual to dx. The main theoretical result of this paper is a congruence in the ring $\mathrm{Z}[\alpha]$ for $f(n)L_\theta(\nu)L(F)[F]$. The congruence involves a certain polynomial function of a complex variable z, $P(z)$, (Definition 3.5). The coefficients of $P(z)$ are integers which depend on the cohomology class x.

THEOREM 1.4. *Suppose that M^{2n} admits a smooth G_p action fixing a codimension 2 submanifold F. If $x \in H^2(M; \mathrm{Z})$, d is a nonnegative integer, and F is dual to dx, then* $f(n)L_\theta(\nu)L(F)[F] \equiv$

$$\begin{cases} -f(n)\, d^2(\alpha^2 - 1)P(d\alpha) \,(\mathrm{mod}\, d^2(1 - d^2)(\alpha^2 - 1)), & n \text{ even}, \\ f(n)[\alpha \,\mathrm{Sign}\, F + d^3(\alpha^3 - \alpha)P(d\alpha)] \,(\mathrm{mod}\, d^3(1 - d^2)(\alpha^3 - \alpha)), & n \text{ odd}. \end{cases} \tag{1.5}$$

Formula (1.5) is produced using Theorem 1.1 together with formula (1.3). Note that in the case n odd, the congruence for $f(n)$ Sign F guaranteed by Theorem 1.1 does not appear in formula (1.5). The reason for this is that Sign F is often known exactly in the applications.

Let M be a cohomology $\mathrm{C}P^n$, that is, $H^*(M; \mathrm{Z}) = \mathrm{Z}[x]/(x^{n+1})$, where $x \in H^2(M; \mathrm{Z})$. Every closed, orientable, codimension 2 submanifold of M is dual to dx for some integer d, which we may take to be nonnegative, modula a change in orientation. We will refer to d as the degree of the submanifold. If G_p acts on M fixing a codimension 2 submanifold F, then the fixed point set consists of F and an isolated point ([5], Corollary 0.1). This means that Singn(g, M) is equal to $L_\theta(v)L(F)[F]$ plus a contribution from the isolated point and so formula (1.5) can be used to make inferences about the degree of F.

Let $D_{n,p}$ be the set of nonnegative integers d such that there exists a cohomology CP^n, M, together with a smooth action of G_p on M fixing a codimension 2 submanifold of degree d. Since CP^n itself admits a G_p action fixing CP^{n-1}, it is clear that $1 \in D_{n,p}$. If $d \in D_{n,p}$, then $d \not\equiv 0 \pmod p$ ([10], p. 587) and so, in particular, d is positive. Let $\tilde{D}_{n,p}$ be the subset of $D_{n,p}$ consisting of those positive integers d such that there exists a homotopy CP^n, M, together with an action of G_p on M fixing a codimension 2 submanifold of degree d. If $n \leqq 5$, then $\tilde{D}_{n,p} = \{1\}$ ([5], Theorem A, [8] Theorem 1.2, and [9], Theorem 1.4). A result in a different direction asserts that if M is a cohomology CP^n, then there is a constant which depends only on the Pontrjagin class of M such that if p is greater than this constant and F is a codimension 2 submanifold of M fixed by a G_p action, then the degree of F is 1 ([6], Theorem A). Other results seem to depend on the parity of n. If $f_p(n)$ is the quotient of $f(n)$ divided by a maximal power of p and $m \geqq 1$, *then* $D_{2m+1,p}$ is contained in the set of divisors of $f_p(2m+1)$ ([9], Theorem 1.3).

In this paper, we will apply formula (1.5) in the special case $p = 3$. The prime $p = 3$ is a good starting point since the contribution of the isolated fixed point to Sign(g, M) is simple in this case and the signature formula reduces to a numerical congruence. Our results support the conjecture that $D_{n,p} = \{1\}$ and the vague feeling that $D_{n,p}$ is more tractable if n is odd. They improve the result that $\tilde{D}_{n,3} = \{1\}$ if $n \leqq 6$ which was obtained by different methods ([8], Theorem 1.1). If n is a positive integer, let $a_3(n) = f_3(n)[3^{[n/2]} + (-1)^{[n/2]-1}]/4$.

THEOREM 1.6. *If* $n \geqq 3$ *and* $d \in D_{n,3}$, *then* d^2 *divides* $a_3(n)$ *if* n *is even and* d^3 *divides* $a_3(n)$ *if* n *is odd.*

THEOREM 1.7. *If* $n \leqq 7$, *then* $D_{n,3} = \{1\}$. *If* $m \leqq 6$, *then* $D_{2m+1,3} = \{1\}$. *If* $m \leqq 9$, *then* $\tilde{D}_{2m+1,3} = \{1\}$.

This paper is organized as follows. In Section 2, we prove Theorem 1.1. Section 3 contains a discussion of formula (1.3) and the proof of Theorem 1.4. In Section 4, we study smooth G_p actions on cohomology complex projective space which fix a codimension 2 submanifold. Section 5 is devoted to the special case $p = 3$ and contains the proofs of Theorems 1.6 and 1.7 as well as upper bounds for $D_{n,3}, n \leqq 22$. In Section 6, we discuss smooth G_p actions on CP^n itself which fix a codimension 2 submanifold.

2. The signatures of self-intersections.

If M^{2n} is a smooth, closed, orientable $2n$-manifold and $K^{2n-2} \subset M^{2n}$ is a closed, orientable submanifold, let $K^{(s)}$ denote the s-fold self-intersection of K in M as defined in the introduction. The notation K_x means that the submanifold K is dual to $x \in H^2(M;\mathbf{Z})$. In our first lemma, $L(M)$ denotes the total Hirzebruch L-class of M. The proof of the lemma can be found in the literature ([12], p. 84, Take $N = M$ and $d = 1$.).

LEMMA 2.1. *If $n-s$ is even and $x \in H^2(M;\mathbf{Z})$, then* $\operatorname{Sign} K_x^{(s)} = \{\tanh^s x\, L(M)\}[M]$.

DEFINITION 2.2. If d is a nonnegative integer and z is a complex number, then the function $T_d(z)$ is defined by

$$T_d(z) = [(1+z)^d - (1-z)^d]/[(1+z)^d + (1-z)^d]. \tag{2.3}$$

Note that $T_d(z)$ is an odd function of z and so its power series expansion has only odd powers. We will see that the coefficients of the series are rational numbers. Let $\mathbf{N}$ be the set of nonnegative integers.

DEFINITION 2.4. If $k \in \mathbf{N}$, then the function $r_k\colon \mathbf{N} \to \mathbf{Q}$ is defined by requiring that

$$T_d(z) = \sum_{k=0}^{\infty} r_k(d)\, z^{2k+1} \text{ for } d \in \mathbf{N}.$$

DEFINITION 2.5. If $k, s \in \mathbf{N}$, the function $R_{k,s}\colon \mathbf{N} \to \mathbf{Q}$ is defined by

$$R_{k,s}(d) = \sum_{i_1+i_2+\ldots+i_s=k} r_{i_1}(d) r_{i_2}(d) \ldots r_{i_s}(d). \tag{2.6}$$

The notation in formula (2.6) is meant to suggest that every possible choice of nonnegative integers $i_1, i_2, \ldots, i_s$ such that $i_1 + i_2 + \ldots + i_s = k$ occurs in the summation. For example, $R_{k,1}(d) = r_k(d)$ and

$$R_{k,2}(d) = 2r_0(d)r_k(d) + 2r_1(d)r_{k-1}(d) + \ldots + a_k r_{[k/2]}(d) r_{k-[k/2]}(d), \tag{2.7}$$

where $a_k = 1$, k even, and $a_k = 2$, k odd.

PROPOSITION 2.8. *If $n-s$ is even and $d \in \mathbf{N}$, then*

$$\operatorname{Sign} K_{dx}^{(s)} = d^s \operatorname{Sign} K_x^{(s)} + \sum_{k=1}^{(n-s)/2} R_{k,s}(d) \operatorname{Sign} K_x^{(2k+s)}, \tag{2.9}$$

$$f(n) \operatorname{Sign} K_{dx}^{(s)} \equiv f(n) d^s \operatorname{Sign} K_x^{(s)} \pmod{d^s(1-d^2)}. \tag{2.10}$$

Note that Proposition 2.8 contains Theorem 1.1 and that if n is odd and $s = 1$

in (2.10), then we retrieve formula (1.1) of [9], a congruence for the signature of K_{dx} itself. Before proceeding with the proof of Proposition 2.8, we single out an important special case. Let M be a homotopy CP^n with splitting invariants $(\sigma_2, \sigma_3, \ldots, \sigma_{n-1})$. Recall that the splitting invariants determine the PL homeomorphism type of M [11] and the splitting invariants with even subscript, σ_2, $\sigma_4, \ldots, \sigma_{2[(n-1)/2]}$, are integers which determine the Pontrjagin class of M ([8], Theorem 3.1). If $x \in H^2(M; \mathbf{Z})$ is a generator of the cohomology algebra and $n - s$ is even, then Sign $K_x^{(s)} = 1 + 8\sigma_{n-s}$ ([9], p. 593). We agree that $\sigma_0 = 0$ because $K_x^{(n)}$ is a single point in this case and hence Sign $K_x^{(n)} = 1$.

PROPOSITION 2.11. *Suppose that M^{2n} is a homotopy $\mathbf{C}P^n$ with integral splitting invariants $\sigma_2, \sigma_4, \ldots, \sigma_{2[(n-1)/2]}$. If $n - s$ is even, $x \in H^2(M; \mathbf{Z})$ is the generator of the cohomology algebra, and $d \in \mathbf{N}$, then*

$$\text{Sign}\, K_{dx}^{(s)} = d^s(1 + 8\sigma_{n-s}) + \sum_{k=1}^{(n-s)/2} R_{k,s}(d)(1 + 8\sigma_{n-2k-s}), \tag{2.12}$$

$$f(n)\, \text{Sign}\, K_{dx}^{(s)} \equiv f(n) d^s(1 + 8\sigma_{n-s}) (\text{mod}\, d^s(1 - d^2)). \tag{2.13}$$

The proof of Proposition 2.8 involves the next lemma which is exactly the same as Proposition 2.2 in [9].

LEMMA 2.14. *The functions $r_k(d)$, $k \in \mathbf{N}$, are polynomial functions in d such that $r_0(d) = d$, and, if $k \geqq 1$, then $r_k(d) = d(1 - d^2) q_k(d^2)$ where $q_k(d^2)$ is a rational polynomial in d^2 such that $f(2k + 1) q_k(d^2)$ is a polynomial in d^2 with integer coefficients.*

For the sake of completeness, we mention that for $k \geqq 1$,

$$r_k(d) = \binom{d}{2k+1} - r_{k-1}(d)\binom{d}{2} - r_{k-2}(d)\binom{d}{4} - \ldots - \\ - r_1(d)\binom{d}{2k-2} - d\binom{d}{2k}, \tag{2.15}$$

and we provide a table of the first five polynomials $q_k(d^2)$.

TABLE 2.16

k	$q_k(d^2)$
1	1/3
2	$(3 - 2d^2)/15$
3	$(45 - 53d^2 + 17d^4)/315$
4	$(315 - 503d^2 + 295d^4 - 62d^6)/2835$
5	$(14175 - 27702d^2 + 22568d^4 - 8848d^6 + 1382d^8)/155925$

These values follow from formula (2.15). The last two values were produced with the aid of a computer.

PROOF OF PROPOSITION 2.8. We begin with a proof of formula (2.9). It follows from Lemma 2.1 that $\operatorname{Sign} K_{dx}^{(s)} = \{\tanh^s dx\, L(M)\}[M]$. Formula (2.9) follows from this observation, the identity $T_d(\tanh x)^s = \tanh^s dx$ ([4], p. 208) and the power series expansion $(T_d(z))^s = d^s z^s + \sum_{k=1}^{\infty} R_{k,s}(d)\, z^{2k+s}$. Lemma 2.1 enters the argument again at the last step in the form of the equation $\{\tanh^{2k+s} x\, L(M)\}[M] = \operatorname{Sign} K_x^{(2k+s)}$.

The argument to establish (2.10) begins by noting that if $1 \leqq k \leqq (n - s)/2$ and $i_1 + i_2 + \ldots + i_s = k$, then in $\mathbf{Z}[d]$, the ring of polynomials in d with integer coefficients, we have the congruence

$$\prod_{j=1}^{s} f(2i_j + 1)\, r_{i_1}(d) r_{i_2}(d) \ldots r_{i_s}(d) \equiv 0 \pmod{d^s(1 - d^2)}. \tag{2.17}$$

Formula 2.17 follows from Lemma 2.14. It is clear that $f(2k + s)$ is divisible by $\prod_{j=1}^{s} f(2i_j + 1)$ because $i_1 + i_2 + \ldots + i_s = k$. It follows that $f(2k + s)\, R_{k,s}(d) \equiv 0 \pmod{d^s(1 - d^2)}$. Since $1 \leqq k \leqq (n - s)/2$, formula (2.10) follows by multiplying both sides of formula (2.9) by the integer $f(n)$.

3. The formula of Berend and Katz.

In this section, M^{2n} is an arbitrary smooth, closed orientable $2n$-manifold. Suppose that G_p acts smoothly on M fixing a codimension 2 submanifold F. If v is the normal bundle of the inclusion map i: $F \subset M$ and $s \in \mathbf{N}$, Berend and Katz define a quasi-signature $\mathscr{S}_s(v) = \{\tanh^s c_1(v)\, L(F)\}[F]$ ([3], p. 945). This quasi-signature is an integer and it measures the sth self-intersection of F in the total space of v. The relationship between these quasi-signatures and the contribution of v to $\operatorname{Sign}(g, M)$ is contained in

THEOREM 3.1 (Berend and Katz [3]). *If M^{2n} admits a smooth G_p action fixing a codimension 2 submanifold F, then*

$$L_\theta(v)\, L(F)[F] = \alpha \mathscr{S}_0(v) + (\alpha^2 - 1) \sum_{s=1}^{n-1} (-1)^s \alpha^{s-1} \mathscr{S}_s(v). \tag{3.2}$$

We remark that Theorem 3.1 is a special case of the analysis of Berend and Katz of the contribution of arbitrary slice types to the multisugnature ([3], Theorem 2.2). They specifically mention that in this special case, $\mathscr{S}_s(v) = \operatorname{Sign} F^{(s+1)}$ ([3], p. 967). This observation justifies the formulation of the theorem in the introduction.

Note that in formula (1.3), the signatures $\operatorname{Sign} F^{(s+1)}$, $0 \leqq s \leqq n-1$, are zero unless $n-s-1$ is even. Our next step is to make this dependence on the parity of n precise by rephrasing Theorem 1.2 as

PROPOSITION 3.3 ([3], Formula (8.1)). *If M^{2n} admits a smooth G_p action fixing a codimension 2 submanifold F, then*

$$L_\theta(v)L(F)[F] = \begin{cases} -(\alpha^2-1)\sum_{k=1}^{n/2} \alpha^{2k-2} \operatorname{Sign} F^{(2k)}, & n \text{ even}, \\ \alpha \operatorname{Sign} F + (\alpha^2-1)\sum_{k=1}^{[n/2]} \alpha^{2k-1} \operatorname{Sign} F^{(2k+1)}, & n \text{ odd}. \end{cases} \tag{3.4}$$

Now suppose that F is dual to $dx \in H^2(M;\mathbf{Z})$. We propose to show that Theorem 1.1 and formula (3.4) together yield Theorem 1.4.

DEFINITION 3.5. If $x \in H^2(M;\mathbf{Z})$ and $z \in \mathbf{C}$, then $P(z) = \sum_{k=1}^{[n/2]} c_k z^{2k-2}$, where

$$c_k = \begin{cases} \operatorname{Sign} K_x^{(2k)}, & n \text{ even}, \\ \operatorname{Sign} K_x^{(2k+1)}, & n \text{ odd}. \end{cases} \tag{3.6}$$

Note that the coefficients of the polynomial P are integers which depend only on the class $x \in H^2(M;\mathbf{Z})$. For example, if M is a homotopy $\mathbf{C}P^n$ and x is the generator of the cohomology algebra, then the coefficients of P are determined by the integral splitting invariants of M ([9], p. 593). The polynomial P plays a role in Theorem 1.4 which we restate as

THEOREM 3.7. *Suppose that M^{2n} admits a smooth G_p action fixing a codimension 2 submanifold F. If $x \in H^2(M;\mathbf{Z})$, $d \in \mathbf{N}$, and F is dual to dx, then* $f(n)L_\theta(v)L(F)[F] \equiv$

$$\begin{cases} -f(n)d^2(\alpha^2-1)P(d\alpha)(\operatorname{mod} d^2(1-d^2)(\alpha^2-1)), & n \text{ even}, \\ f(n)[\alpha \operatorname{Sign} F + d^3(\alpha^3-\alpha)P(d\alpha)](\operatorname{mod} d^3(1-d^2)(\alpha^3-\alpha)), & n \text{ odd}. \end{cases} \tag{3.8}$$

PROOF. The proof follows by multiplying formula (3.4) on both sides by the integer $f(n)$, applying Theorem 1.1 to the terms $f(n)\operatorname{Sign} F^{(s)}$, $s > 1$, and making minor adjustments to expose $P(d\alpha)$. The moduli of the congruences are obtained by multiplying the greatest common divisor of the moduli of the congruences for $f(n)\operatorname{Sign} F^{(s)}$ by the appropriate factor involving α.

4. Cohomology complex projective space.

In this section, M^{2n} is a cohomology $\mathbf{C}P^n$, that is, $H^*(M;\mathbf{Z}) = \mathbf{Z}[x]/(x^{n+1})$, where $x \in H^2(M;\mathbf{Z})$. If G_p acts on M fixing a codimension 2 submanifold F, then the fixed point set of the action consists of F and an isolated point ([5], Corollary 0.1). We propose to analyze the complete signature formula in this particular case in light of Theorem 3.7. This means that we must enhance the notation in order to describe the action near the isolated fixed point as well as near F.

Let $\mu = (p-1)/2$. The possible eigenvalue of the action of G_p on the eigenbundle summands in the decomposition of the tangent space at the isolated fixed point are $\lambda^j = \exp(2\pi ij/p)$, $1 \leqq j \leqq \mu$. Each eigenvalue is associated with an algebraic number $\alpha_j = (\lambda^j + 1)(\lambda^j - 1)^{-1}$. We will assume, as in the introduction, that the eigenvalue at the normal bundle of F is $\lambda^1 = \lambda$ and that it is associated with the algebraic number α_1. Note that α_1 was written as α in the previous sections of this paper where the other numbers $\alpha_j, j \neq 1$, did not appear in any formulas. The Atiyah-Singer g-signature Theorem ([5], formula (1.4)) asserts that there are integers $m_1, m_2, \ldots m_\mu$, such that $\sum_{j=1}^{\mu} m_j = n$ and

$$\text{Sign}(g, M) = \pm L_\theta(v) L(F)[F] \pm \alpha_1^{m_1} \alpha_2^{m_2} \ldots \alpha_\mu^{m_\mu}. \tag{4.1}$$

THEOREM 4.2. *Suppose that M^{2n} is a cohomology $\mathbf{C}P^n$ and that M^{2n} admits a smooth G_p action fixing a codimension 2 submanifold of degree d. If the multiplicities of the eigenvalues at the isolated fixed point are $m_1, m_2, \ldots, m_\mu$, then* $f(n)\alpha_1^{m_1}\alpha_2^{m_2}\ldots\alpha_\mu^{m_\mu} \equiv$

$$\begin{cases} \pm f(n) \pm f(n) d^2(\alpha_1^2 - 1) P(d\alpha_1) (\text{mod } d^2(1-d^2)(\alpha_1^2 - 1)), n \text{ even}, \\ \pm f(n)\alpha_1 \pm f(n) d^3(\alpha_1^3 - \alpha_1) P(d\alpha_1) (\text{mod } d^3(1-d^2)(\alpha_1^3 - \alpha_1)), n \text{ odd}. \end{cases} \tag{4.3}$$

PROOF. Formula (4.3) follows by multiplying both sides of (4.1) by $f(n)$, using (3.8) and the facts that $\text{Sign}(g, M) = \pm 1$ if n is even, and $\text{Sign}(g, M) = 0$ if n is odd ([5], p. 504) together with the fact that if n is odd, then $\text{Sign}\, F = \pm 1$ ([9], Lemma 3.1).

The congruence symbol in formula (4.3) means that the left hand side of the congruence minus the right hand side is equal to an element of $\mathbf{Z}[\alpha_1]$ multiplied by the modulus. Note that congruence (4.3) is similar to a general congruence of Katz ([7], Proposition 3.11) restricted to this special case. In our case, there are only two representations, the one at F and the one at the isolated fixed point, and we have multiplied by $f(n)$. Our congruence contains the added ingredient of the congruences involving the degree of the codimension 2 fixed submanifold but it does not contain the sign regulator in Katz's congruence. Note that F is represented on the right side of formula (4.3) solely by the rational integer d, the degree of F. The other ingredients on the right side of (4.3) are $f(n)$, algebraic numbers, and the integer coefficients of P which depend only on M. This is an improvement over previous efforts to deal with this problem which involved the Pontrjagin class of F and showed no clear pattern for arbitrary n ([8], formulas (11), (12), and (13)). Our next step is to show that formula (4.3) is a congruence of rational integers in the case $p = 3$.

5. Actions of the group G_3.

In this section M^{2n}, is a cohomology CP^n with $x \in H^2(M; Z)$ the generator of the cohomology algebra. In order to simplify the statement of the main result of this section, we introduce a simplification in notation. If K_x is dual to x, let $S^{(s)} = \text{Sign}\, K_x^{(s)}$, $0 \leqq s \leqq n$. We define a numerical function $a(n) = f(n)[3^{[n/2]} + (-1)^{[n/2]-1}]/4$.

THEOREM 5.1. *Suppose that M^{2n} is a cohomology CP^n. If M^{2n} admits a smooth G_3 actions fixing a codimension 2 submanifold of degree d, then* $\pm\, a(n) \equiv$

$$(5.2) \quad \begin{cases} f(n)d^2 \sum_{k=1}^{m} (-1)^{k-1}\, 3^{m-k}\, d^{2k-2}\, S^{(2k)} (\text{mod}\, d^2(1-d^2)),\ n = 2m, \\ f(n)d^3 \sum_{k=1}^{m} (-1)^{k-1}\, 3^{m-k}\, d^{2k-2}\, S^{(2k+1)} (\text{mod}\, d^3(1-d^2)),\ n = 2m+1. \end{cases}$$

We remark that formula (5.2) is an ordinary congruence on the ring of integers. Before we prove Theorem 5.1, we establish two consequences. First, note that if $a_3(n)$ is $a(n)$ with a maximal power of 3 divided out, then we have,

COROLLARY 5.3. *If $n \geqq 3$ and $d \in D_{n,3}$, then d^2 divides $a_3(n)$ if n is even and d^3 divides $a_3(n)$ if n is odd.*

PROOF. This follows immediately from formula (5.2) and the fact that $d \in D_{n,3}$ implies that $d \not\equiv 0 \pmod 3$.

Note that Corollary 5.3 is the same as Theorem 1.6 in the introduction. The second consequence of Theorem 5.1 we will establish is Theorem 1.7. We will do this by presenting a table of upper bounds for $D_{n,3}$, $n \leqq 22$, guaranteed by Corollary 5.3. The table lists only the maximal prime powers that can occur in the prime factorization of an element in $D_{n,3}$.

TABLE 5.4

n	$D_{n,3}$	n	$D_{n,3}$
3	1	13	1
4	1	14	1, 5, 7
5	1	15	1, 5
6	1	16	1, 2, 5^2, 7
7	1	17	1, 5
8	1, 2, 5	18	1, 5, 7
9	1	19	1, 5, 7
10	1, 5	20	1, 5^2, 7, 11
11	1	21	1, 5, 7, 11
12	1, 5, 7	22	1, 5^2, 7, 11

We give one example to illustrate the use of the table. If $d \in D_{22,3}$, then 5, 7, and 11 are the only prime divisors of d, the exponent of 5 is less than or equal to 2 and the exponents of 7 and 11 are less than or equal to 1.

The proof of Theorem 1.7 is contained in Table 5.4. It is clear from the table that $D_{n,3} = \{1\}$ if $n \leqq 7$ and $D_{2m+1,3} = \{1\}$ if $m \leqq 6$. This is because $n = 8$ is the smallest integer such that $a_3(n)$ is divisible by a perfect square, namely 100, and $n = 15$ is the smallest odd integer such that $a_3(n)$ is divisible by a perfect cube, namely 125. The statement in Theorem 1.7 about homotopy complex projective space also follows from the table because $d \in \tilde{D}_{2m+1,p}$ implies that $d \equiv 1 \pmod 8$ ([9], Theorem 1.3). This observation and the table imply that $\tilde{D}_{2m+1,3} = \{1\}$, $m \leqq 9$. Things go wrong at level $n = 21$, because the table indicates that $d = 385$ might be a member of $D_{21,3}$ and $385 \equiv 1 \pmod 8$. Table 5.4 was produced using Corollary 5.3, the formula $a_3(n) = f_3(n)[3^{[n/2]} + (-1)^{[n/2]-1}]/4$ and a calculator.

Proof of Theorem 5.1. Formula (5.2) is just congruence (4.3) in the special case $p = 3$ plus some additional information. Formula (4.3) states that the left hand side minus the right hand side is equal to an element of $\mathbf{Z}[\alpha_1]$ times the modulus and we need to know something about this element to produce formula (5.2).

If $p = 3$, then $\mu = 1$, $\alpha_1 = -i/\sqrt{3}$, and $\alpha_1^2 - 1 = -4/3$. If $n = 2m$ and F is the fixed submanifold, then it follows from formulas (3.4) and (4.1) that

$$(5.5) \qquad 1 = \pm(4/3) \sum_{k=1}^{m} (-1)^{k-1} 3^{-(k-1)} \operatorname{Sign} F^{(2k)} \pm (-1)^m 3^{-m}.$$

Formula (5.2) in the case $n = 2m$ follows by multiplying both sides of (5.5) by $3^m f(n)$ and using Theorem 1.1. If $n = 2m + 1$ and F is the fixed submanifold, then it follows from formulas (3.4) and (4.1) and the fact that $\operatorname{Sign} F = \pm 1$ that we have

$$(5.6) \qquad 1 = \pm(4/3) \sum_{k=1}^{m} (-1)^{k-1} 3^{-(k-1)} \operatorname{Sign} F^{(2k+1)} \pm (-1)^m 3^{-m}.$$

Formula (5.2) in the case $n = 2m + 1$ follows by multiplying both sides of (5.6) by $3^m f(n)$ and using Theorem 1.1.

6. Complex projective space.

We end this paper with a discussion of G_p actions on $\mathbf{C}P^n$ itself which fix a codimension 2 submanifold. If n is odd, then the degree of the fixed manifold is 1 ([9], Theorem 1.2). Our next result contains this fact and some new information about the fixed submanifold.

Theorem 6.1. *If $\mathbf{C}P^{2m+1}$ admits a smooth G_p action fixing a codimension 2 submanifold F, then the degree of F is 1 and* $\operatorname{Sign} F^{(s)} = 1$, $s = 1, 3, \ldots, 2m + 1$.

PROOF. Let d be the degree of F. There exists an orientation of F such that $\operatorname{Sign} F = \operatorname{Sign} F^{(1)} = 1$ and $d > 0$ ([9], Lemma 3.1). The integral splitting invariants are zero ([8], Corollary 3.3) and so $\operatorname{Sign} K_x^{(s)} = 1 + 8\sigma_{2m+1-s} = 1$, $s = 1, 3, \ldots, 2m+1$. This means that congruence (2.13) at level $s = 1$ reduces to $f(2m+1) \equiv f(2m+1)d \pmod{d(1-d^2)}$, and so $d = 1$ since $f(2m+1)$ is odd. If $d = 1$ is used with (2.13) at levels $s = 3, 5, \ldots, 2m+1$, we obtain $\operatorname{Sign} F^{(s)} = 1$, $s = 3, 5, \ldots, 2m+1$.

If $\mathbf{C}P^n$ admits a smooth G_3 action fixing a codimension 2 submanifold of degree d, then a result of Masuda states that $d^2 \equiv 1 \pmod 9$ ([10], p. 589). There is another congruence for the case $p = 3$.

THEOREM 6.2. *If $\mathbf{C}P^{2m}$ admits a smooth G_3 action fixing a codimension 2 submanifold of degree d, then*

$$(6.4) \qquad \pm a(2m) \equiv f(2m)\, d^2 \frac{3^m + (-1)^{m-1} d^{2m}}{3 + d^2} \pmod{d^2(1-d^2)}.$$

PROOF. Formula (6.4) follows from (5.2) in the case $n = 2m$, the fact that $S^{(2k)} = 1 + 8\sigma_{2m-2k} = 1$ in this case, and the formula for the sum of a geometric series.

It is now possible to return to Table 5.4 and, armed with Masuda's result and (6.4), investigate G_3 actions on $\mathbf{C}P^{2m}$ fixing a codimension 2 submanifold. For example, if $\mathbf{C}P^8$ admits a smooth G_3 action fixing a codimension 2 submanifold of degree d, then it follows from Table 5.4 and either (6.4) or Masuda's congruence that d is either 1 or 10. If $\mathbf{C}P^{10}$ admits a smooth G_3 action fixing a codimension 2 submanifold of degree d, then Table 5.4 and either (6.4) or Masuda's congruence implies that $d = 1$.

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