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REMARKS ON LOCALLY COMPACT GROUP EXTENSIONS

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Introduction.

In this note we present some results concerning extensions of locally compact groups, which has applications in the representation theory of such groups and their lattice subgroups, see Proposition 1.15. Further applications will appear in a subsequent article.

The paper consists of two independent but related parts. In Section 1 our emphasis will be on central extensions of connected Lie groups G. To be more precise, we shall focus on the correspondence between elements of the second cohomology group $H^2(G, A)$ and bilinear cocycles on the Lie algebra $\mathfrak{g}$ of G. Here A denotes a connected abelian group.

Next, recall the existence of locally well behaved (usually Borel) cross sections plays an essential role in the representation theory of second countable locally compact groups ([Ma1], [GR]). Section 2, in which the extensions are not assumed central, deals with almost fibered extensions and we show that any extension of locally compact groups is almost fibered. This means that all such extensions can be described in terms of cross sections continuous at the identity or, equivalently, by a cocycle continuous at the identity element, [Ca].

1. Central extensions.

In the present section we shall focus on the correspondence between central extensions of connected Lie groups and their Lie algebras. In the simply connected case this correspondence is classic, even for noncentral extensions, [H]. However, keeping later applications in mind, we shall need to argue in terms of cocycles rather than group extensions. Moreover, the explicit relation given in eq. (1.7) of Theorem 1 (a) below, does not seem to be available in the litterature. It is equation (1.7) that we shall find particularly helpful. Throughout this section all cocycles considered will be central 2-cocycles, as defined below. It has become

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customary to require, particularly when G is second countable, that cocycles be Borel maps, cf. [Ma 1,2]. We prefer here, partly in light of Theorem 2, to follow the convention of Calabi, [Ca].

1.1 DEFINITION. (a) Let G be a locally compact group, A be a connected abelian Lie group, $A = \mathsf{R}^m \times \mathsf{T}^n$, to be written additively. By a *central 2-cocycle on G with coefficients in A*, we understand a function $\omega: G \times G \to A$ which satisfies the cocycle identities

$$\partial\omega(x, y, z) = \omega(x, y) + \omega(xy, z) - \omega(x, yz) - \omega(y, z) = 0 \tag{1.1}$$

$$\omega(x, e) = \omega(e, x) = 0 \quad (x, y, z \in G)$$

and, in addition, ω and the maps $y \mapsto \omega(x^{-1}, x)^{-1} + \omega(x^{-1}y) + \omega(x^{-1}y, x)$ are continuous at the identity in $G (\forall x \in G)$. ω is said to be normalized in case $\omega(x, x^{-1}) = 0$ $(x \in G)$. Here e and 0 denote the neutral elements of G and A, respectively. ω is trivial if it can be written

$$\omega(x, y) = f(xy) - f(x) - f(y) \quad (x, y \in G) \tag{1.2}$$

in which the map $f: G \to A$ is continuous at e. We denote by $C^2(G, A)$, $B^2(G, A)$, $H^2(G, A)$ the groups (with pointwise addition of cocycles as group composition) consisting of all central 2-cocycles, all trivial cocycles, respectively the quotient group $C^2(G, A)/B^2(G, A)$. These groups will also be regarded as real vector spaces, in the natural fashion, whenever A is simply connected. Two cocycles are said to be similar (cohomologous) if their difference falls in $B^2(G, A)$.

(b) Let $\mathfrak{g}$ and $\mathfrak{a}$ be Lie algebras over R, and assume $\mathfrak{a}$ is abelian. By *a central 2-cocycle of $\mathfrak{g}$ with values in $\mathfrak{a}$* we understand an antisymmetric bilinear map $B: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{a}$ satisfying the following cocycle identity,

$$B(x, [y, z]) + B(z, [x, y]) + B(y, [z, x]) = 0 \quad (x, y, z \in \mathfrak{g}) \tag{1.3}$$

in which $[\cdot,\cdot]$ denotes the Lie product on $\mathfrak{g}$. B is trivial if there is a linear map $f: \mathfrak{g} \to \mathfrak{a}$ such that

$$B(x, y) = f[x, y] \quad (x, y \in \mathfrak{g}) \tag{1.4}$$

By $H^2(\mathfrak{g}, \mathfrak{a})$ we shall understand the real vector space of all central 2-cocycles which is the quotient $C^2(\mathfrak{g}, \mathfrak{a})$ modulo the space $B^2(\mathfrak{g}, \mathfrak{a})$ of trivial cocycles.

Higher order cohomology groups/spaces are defined similarly, however, we shall not pursue this any further.

A cocycle will be identified with its cohomology class whenever convernient.

1.2. We shall always assume the cocycles are normalized. This is justified by the fact that any $\omega \in C^2(G, A)$ is similar to a normalized cocycle ω'. Take e.g.

$$\omega'(x, y) = \omega(x, y) - \tfrac{1}{2}\omega(x, x^{-1}) + \omega(y, y^{-1})) - \tfrac{1}{2}\omega(xy, y^{-1}x^{-1}) \quad (x, y \in G).$$

Assume that G is a connected and simply connected Lie group with Lie algebra $\mathfrak{g}$. Let C be a (central 2-)cocycle on $\mathfrak{g}$ taking values in the abelian Lie algebra $\mathfrak{a}$ and denote by $\mathfrak{g}(C)$ the central extension of $\mathfrak{g}$ by $\mathfrak{a}$ defined by C. Thus the underlying vector space of $\mathfrak{g}(C)$ is a direct sum of $\mathfrak{a}$ and $\mathfrak{g}$ and the Lie product is given as follows,

$$[(a,x),(b,y)] = (C(x,y),[x,y]) \quad (x,y\in\mathfrak{g}; a,b\in\mathfrak{a}) \tag{1.5}$$

Now, the connected and simply connected Lie group $\tilde{G}$ with Lie algebra $\mathfrak{g}(C)$ is a central extension of G by a vectorgroup A, accordingly is determined by a cocycle $\omega: G \times G \to A$, which may be assumed analytic [Pa], such that

$$(a,\exp x)(b,\exp y) = (a + b + \omega(\exp x,\exp y),\exp x\, exp\, y) \quad (a,b\in A; x,y\in\mathfrak{g}) \tag{1.6}$$

We remark that the topology on $\tilde{G}$ is in general not identical to the product topology on $A \times G$. In fact, it is derived from the product Borel structure together with the invariant Borel measure on $A \times G$, [Mal,2] [Pa]. Observe that Theorem 2 below permits a different construction of this topology.

Conversely, given an analytic cocycle ω on the group G we can find a corresponding cocycle C of the Lie algebra $\mathfrak{g}$, upon forming the extended group $G(\omega)$ and its Lie algebra.

1.3. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$, A be a connected abelian Lie group and $\mathfrak{a}$ its Lie algebra. We denote by $\tilde{G}$ the simply connected covering group of G. Let $p:\tilde{G} \to G$ be the covering homomorphism. If $\omega: G \to A$ is a cocycle, we form the corresponding cocycle $\tilde{\omega} = \omega \circ p$ on $\tilde{G}$.

THEOREM 1. *With notation as above the following assertions hold,*

(a) *If ω is a central 2-cocycle on G taking values in A, there exists an antisymmetric 2-cocycle C on the Lie algebra $\mathfrak{g}$, $C:\mathfrak{g} \times \mathfrak{g} \to \mathfrak{a}$, such that $\tilde{\omega}$ is cohomologous to the cocycle $\tilde{\omega}_C$ on $\tilde{G}$ associated to C in the followng way,*

$$\begin{aligned} &\tilde{\omega}_C(\exp x, \exp y) \\ &\quad = \exp(\tfrac{1}{2}\, C(x,y) + \tfrac{1}{12}(C([x,y],y) + C([y,x],x)) + \ldots + C_n(x:y) + \ldots) \end{aligned} \tag{1.7}$$

Here, each term $C_n(x:y)$ is obtained from the similar term occuring in the Campbell-Hausdorff formula of $\exp x \exp y$ *on replacing the outermost Lie bracket by the form $C(.,.)$.*

(b) *Assume ω and ω' are cocycles on G, and C and C′ are cocycles on $\mathfrak{g}$ corresponding to ω and ω' respectively, as in part* (a). *Then ω and ω' are cohomologous on the group G only if C and C′ are cohomologous on the Lie algebra $\mathfrak{g}$. The converse statement holds if G is simply connected.*

(c) *The correspondence $\omega \mapsto \tilde{\omega} \mapsto \tilde{\omega}_C \mapsto C$ given by* (a) *and* (b) *induces a homomorphism $\Phi: H^2(G,A) \to H^2(\mathfrak{g},\mathfrak{a})$ between cohomology groups. Its kernel* *consists of all classes* ω *such that* $\omega \circ p$ *is cohomologous to* 0, *its image of all classes in* $H^2(\mathfrak{g}, \mathfrak{a})$ *in which there exists a cocycle* C *satisfying* $\tilde{\omega}_C|\mathrm{Ker}\, p \equiv 0$.

We remark that the arguments given below are valid even for noncentral extensions, with some reasonably appearant modifications.

The proof of part (a) of Theorem 1 will extend over the next three sections. The main part of it consists of verifying (1.7). Clearly, it suffices to assume G is simply connected.

1.4. We proceed to study the connection between such cocycles ω and C. It follows from the uniqueness property of the one-parameter group $t \mapsto \exp t(0, x)$, $\mathsf{R} \to G$, that we can write

$$\exp(0, x) = (f(x), \exp x) \quad (x \in \mathfrak{g}) \tag{1.8}$$

where $f : \mathfrak{g} \to A$ is a C^∞-map satisfying $\frac{d}{dt} f(tx)|_{t=0} = 0$ and

$$f((s+t)x) - f(tx) - f(sx) = \omega(\exp sx, \exp tx) \quad (s, t \in \mathsf{R}) \tag{1.9}$$

In fact, using eq. (1.8) and eq. (1.6) we derive:

$$\begin{aligned} \exp(0, sx)\exp(0, tx) &= (f(sx), \exp sx)(f(tx), \exp tx) \\ &= (f(sx) + f(tx) + \omega(\exp sx, \exp tx), \exp(s+t)x) \\ &= (f(sx) + f(tx) + \omega(\exp sx, \exp tx), e)(0, \exp(s+t)x), \end{aligned}$$

which yields eq. (1.9) when combined with

$$\exp(0, (s+t)x) = (f(s+t)x, \exp(s+t)x) = (f((s+t)x), e)(0, \exp(s+t)x).$$

Moreover, $\exp x = \exp y$ implies $f(x) = f(y)$, so that $f = g \circ \exp$ for some C^∞-function g on the group G. If $(a, \exp x) \in \tilde{G}$ we have,

$$(a, \exp x) = (a, e)(0, \exp x) = (a - f(x), e)(f(x), \exp x) \tag{1.10}$$

Further, by eqs. (1.6), (1.8), (1.10) and the Campbell-Hausdorff formula, (CH), we calculate, writing $a' = \exp a$ $(a \in \mathfrak{a})$,

$$(a', \exp x)(b', \exp y) = (a' - f(x), e)(b' - f(y), e)\exp(0, x)\exp(0, y) \quad \text{(using eq. (1.10))}$$

$$= (a' + b' - f(x) - f(y), e)\exp((0, x+y) + \tfrac{1}{2}(C(x, y), [x, y]) + \ldots) \quad \text{(by eq. (1.6) and (CH))}$$

$$= (a' + b' - f(x) - f(y), e)\exp((\tfrac{1}{2}C(x, y) + \tfrac{1}{12}C([y, x], x) + \tfrac{1}{12}C([x, y], y)$$

$$+ \ldots, x + y + \tfrac{1}{2}[x, y] + \ldots) \quad \text{(using (CH))}$$

$$= (a' + b' - f(x) - f(y), e)\exp(\tfrac{1}{2}C(x, y) + \tfrac{1}{12}(C([x, y], y) + C([y, x], x)) + \ldots, 0)$$

$$\cdot \exp(0, x + y + \tfrac{1}{2}[x, y] + \ldots)$$

$$= (a' + b' - f(x) - f(y) + \exp\{\tfrac{1}{2}C(x,y) + \tfrac{1}{12}(C([x,y],y) + C([y,x],x)) + \ldots\}, e)$$

$$\cdot(f(x + y + \tfrac{1}{2}[x,y] + \ldots), \exp x \exp y) \quad \text{(using } \exp(a,0) = (\exp a, e) \text{ and eq. (1.8))}$$

In view of eq. (1.6) we have shown,

(1.11) $(a', \exp x)(b', \exp y) =$

$$(a' + b' - f(x) - f(y) + f(x + y + \tfrac{1}{2}[x,y] + \ldots) + \exp\{\tfrac{1}{2}C(x,y) + \tfrac{1}{12}(C([x,y],y) + C([y,x],x)) + \ldots\}, \exp x \exp y)$$

Let us introduce the

1.5. NOTATION.

(1.12) $\omega_C(\exp x, \exp y)$

$$= \exp(\tfrac{1}{2}C(x,y) + \tfrac{1}{12}(C([x,y],y) + C([y,x],x)) + \ldots + C_n(x:y) + \ldots)$$

where the terms $C_n(x:y)$ on the right hand side are analogous to those in the Campbell-Hausdorff formula for $\exp x \exp y$. We shall see below that ω_C is always a cocycle on G.

1.6. It follows from eq. (1.11) that the above series converges. Comparing eq. (1.11) with eq. (1.6) we see immediately that

$$\omega(\exp x, \exp y) = \omega_C(\exp x, \exp y) + f(x + y + \tfrac{1}{2}[x,y] + \ldots) - f(x) - f(y) \tag{1.13}$$

Next, we choose a neighborhood U of 0 in $\mathfrak{g}$ for which exp becomes a diffeomorphism onto $\exp U$ in G. Hence, for (x, y) in a suitable neighborhood $V \times V$ of $(0,0)$ in $\mathfrak{g} \times \mathfrak{g}$ such that $V \subset U$, we may assume $\exp x \exp y \in \exp U$ and therefore

$$f(x + y + \tfrac{1}{2}[x,y] + \ldots) = g(\exp(x + y + \tfrac{1}{2}[x,y] + \ldots)) = g(\exp x \exp y),$$

in which g denotes the C^∞-function on $\exp U$ defined by the relation $g \circ \exp = f$. It follows that

(1.14) $\omega(\exp x, \exp y) =$

$$\omega_C(\exp x, \exp y) + g(\exp x \exp y) - g(\exp x) - g(\exp y) \quad (\forall x, y \in V)$$

In particular, ω restricted to the neighborhood $\exp V \times \exp V$ is cohomologous to ω_C, so that ω_C is a cocycle on this neighborhood. Invoking [Pa; Theorem 3.2] this result holds globally. Thus we have proved statement (a) of Theorem 1.

For convenience we give some corollaries before completing the proof. We note first the following corollary to the above argument.

1.7. COROLLARY. *Let G and A be connected Lie groups, A abelian. Assume that C is a central 2-cocycle on $\mathfrak{g}$ taking values in the abelian Lie algebra $\mathfrak{a}$. Let ω_C denote the cocycle on G corresponding to C as in Theorem* 1. *Then ω_C is cohomologous to* 0 *if and only if there exists a group homomorphism $f:\mathfrak{g}\to A$ such that*

$$\omega_C(\exp x, \exp y) = f\circ\log(\exp x \exp y) - f(x) - f(y)$$

on every coordinate patch in G (regarding $\mathfrak{g}$ as a vectorgroup).

1.8. COROLLARY. *Assume G is a simply connected and connected Lie group, $\omega = \omega_C$ an analytic cocycle of G into a connected abelian Lie group A. Let $\hat{\omega}$ denote the map defined by*

$$\hat{\omega}(x, y) = \omega(x, y) - \omega(y, x) \quad (x, y \in G).$$

Then $\hat{\omega}$ is an analytic, antisymmetric map, and

$$\frac{d}{dt}\hat{\omega}(\exp X, \exp tY)|_{t=0} = C(X, Y) \quad (t \in \mathrm{R}; X, Y \in \mathfrak{g}).$$

Here C denotes the cocycle on $\mathfrak{g}$ associated to ω as in Theorem 1 (a).

PROOF. In light of Theorem 1 (a),

$$\frac{d}{dt}\omega(\exp X, \exp tY)|_{t=0} = \tfrac{1}{2}C(X, Y) + \tfrac{1}{12}C(X, [X, Y]),$$

and

$$\frac{d}{dt}\omega(\exp tY, \exp X)|_{t=0} = \tfrac{1}{2}C(Y, X) + \tfrac{1}{12}C(X, [X, Y]),$$

for all X, $Y \in \mathfrak{g}$, and the corollary follows, C being antisymmetric.

1.9. We note that usually $\hat{\omega}_C \circ \exp$ is not equal to $\exp C$. In fact,

$$\hat{\omega}_C(\exp X, \exp Y) = \exp \sum_{n=0}^{\infty} 2C_{2n+1}(X : Y)$$

since the terms C_{2n} are symmetric and C_{2n-1} are antisymmetric ($n = 1, 2, 3, \ldots$). Now, $\sum_{n=1}^{\infty} C_{2n+1} = 0$ if and only if each term C_{2n+1} ($n = 1, 2, 3, \ldots$) vanishes, as the $C_{2n+1} - s$ form a linear independent family of polynomials (their degrees are increasing with n). We have proved,

COROLLARY. *Let $\omega = \omega_C$ be a cocycle on a simply connected and connected Lie group G. Then $\hat{\omega}_C(\exp X, \exp Y) = \exp C(X, Y)$ ($\forall X, Y \in \mathfrak{g}$) if and only if $C_{2n+1} = 0$ ($n = 1, 2, 3, \ldots$). If G is 2-step nilpotent, this holds for all cocycles ω_C.*

1.10. We proceed with the proof of Theorem 1, cf. Section 1.6.

(b) First, let G be simply connected. Assume that $\omega \sim \omega_C$ is cohomologous to 0. We are to show that C is cohomologous to 0 on $\mathfrak{g}$. By assumption there exists a measurable function $f : \mathfrak{g} \to A$ such that on each coordinate neighborhood W in $G \times G$,

$$\begin{aligned}\omega_C(\exp tx, \exp y) &= f \circ \log(\exp tx \exp y) - f(tx) - f(y) \\ &= f(tx + y + \tfrac{t}{2}[x, y] + o(t)) - f(tx) - f(y), \quad ((x, y) \in W),\end{aligned}$$

where $t \in \mathsf{R}$ is sufficiently small and $\frac{1}{t}o(t) \to 0$ as $t \to 0$. Now, since ω_C is differentiable on G, we may assume that f is differentiable, [Pa; Theorem 4.1, Theorem 3.2]. Thus by Corollary 1.8 and Theorem 1 (a)

$$\begin{aligned}&\frac{d}{dt}(\omega_C(\exp tx, \exp y) - \omega_C(\exp y, \exp tx))|_{t=0} = C(x, y) \\ &= (Df)(0)\cdot(x + \tfrac{1}{2}[x, y]) - (Df)(0)\cdot x - (Df)(0)\cdot(x + \tfrac{1}{2}[x, y]) + (Df)(0)\cdot x \\ &= Df(0)\cdot[x, y],\end{aligned}$$

where $x \cdot y$ denotes Euclidean inner product on $\mathfrak{g}$. Accordingly $C = Df(0)\cdot[\cdot,\cdot]$ is a trivial cocycle on $\mathfrak{g}$.

Conversely, assume that ω is cohomologous to ω_C where C is cohomologous to 0 on the Lie algebra. Then we can find a linear $F : \mathfrak{g} \to \mathfrak{a}$ satisfying $C = F \circ [\cdot,\cdot]$. Put $f = \exp \circ F$, $\mathfrak{g} \to A$. Choosing local coordinates on G we derive via the Campbell-Hausdorff formula and Theorem 1 (a),

$$\begin{aligned}\omega_C(\exp x, \exp y) &= f(\tfrac{1}{2}[x, y] + \tfrac{1}{12}([[y, x], x] + [[x, y], y]) + \dots) \\ &= f \circ \log(\exp x \exp y) - f(x) - f(y),\end{aligned}$$

from which it is evident that ω_C, and hence ω, are cohomologous to 0.

Next, if G is not simply connected, we have

$$\omega \sim \omega' \Rightarrow \tilde{\omega} \sim \tilde{\omega}' \Rightarrow \tilde{\omega}_C \sim \tilde{\omega}_{C'} \Rightarrow C \sim C',$$

by the first part of (b).

(c) Let $\omega \in H^2(G, A)$ be arbitrary. Now, if C and C' satisfy $\omega \sim \omega_C \sim \omega_{C'}$, we have $C \sim C'$ by part (b), and Φ is well defined. That Φ is a homomorphism follows readily, $\omega \mapsto \tilde{\omega} = \omega \circ p$ and $\exp : \mathfrak{a} \to A$ being group homomophisms. Finally, the last two statements in (c) are clear.

1.11. Examples. We illustrate some extreme possibilities in Theorem 1. First, if G is semisimple, we always have $H^2(\mathfrak{g}, \mathfrak{a}) = (0)$, however, $H^2(G, A)$ need not be trivial (take e.g. $G = \mathrm{SL}_2(\mathsf{R})$). On the other hand, let $\tilde{G}$ denote the 3-dimensional Heisenberg group and fix a lattice $Z \approx \mathsf{Z}$ in its one-dimensional center. The factor group $G = \tilde{G}/Z$ has $H^2(G, \mathsf{R}) = \{0\}$, whereas its Lie algebra satisfies

$H^2(\mathfrak{g}, \mathsf{R}) = \mathsf{R}\, e_1^* \wedge e_3^* \oplus \mathsf{R}\, e_2^* \wedge e_3^*$. Here, $e_i\,(i = 1,2,3)$ denotes the "standard" basis for $\mathfrak{g}$ whose nonzero Lie relations are $[e_1, e_2] = e_3 = -[e_2, e_1]$. The cocycle on $\tilde{G}$ corresponding to $C = e_1^* \wedge e_3^*$ is obtained after some calculations as

$$\omega_C((a,b,c),(x,y,z)) = \tfrac{1}{2}(az - cx) + \tfrac{1}{6}ax(b - 2y) + \tfrac{1}{12}(a^2y + x^2b).$$

ω_C is seen to be cohomologous with the (somewhat simpler) cocycle

$$((a,b,c),(x,y,z)) \mapsto \tfrac{1}{2}(az - cx) - \tfrac{1}{2}axy, \quad (a,b\,c),(x,y,z) \in \tilde{G}.$$

Here we have used the following group composition on $\tilde{G}$,

$$(a,b,c)(x,y,z) = (a + x, b + y, c + z + ay).$$

1.12. REMARK. Let ω be a cocycle on $G, \omega_C + \alpha$, where C is as in Theorem 1 and α is a trivial cocycle. On the centralizer Z_x of x in G we have

$$\hat{\omega}(x,y) = \omega_C(x,y) - \omega_C(y,x) = \exp C(x,y) \quad (y \in Z_x)$$

since $\alpha(x,y) = \alpha(y,x)\,(\forall y \in Z_x)$. Consequently $\hat{\omega}(x,\cdot)$ is an analytic character on Z_x.

1.13. For any 2-cocycle ω on G, we let $S_\omega = \{x \in G \mid \omega(x,y) = \omega(y,x)\,\forall y \in G\}$. We say that ω is symmetric in case $S_\omega = G$, totally skew provided that $S_\omega = \{e\}$. If G is Lie and $\omega = \omega_C$, ω is totally skew if and only if the alternating bilinear map C is nondegenerate. In fact, the Lie algebra of S_ω equals the radical of C. It is well known that if G is abelian, ω is trivial if and only if it is symmetric. This is a consequence of the fact that any abelian extension of G by $A = \mathsf{R}^n \times \mathsf{T}^k$ is a direct product, [Ca, Proposition 18.5]. In the connected Lie case the only if part is obviously wrong, simply because cocycles ω_C are trivial and nonsymmetric whenever C is nonzero and can be written $C = f \circ [\cdot,\cdot]$ (f a real linear functional on $\mathfrak{g}$). Furthermore, trivial cocycles ω_C can even be totally skew (i.e., C is nondegenerate). For an example, let $\omega = \omega_C$, $C = e_1^* \wedge e_2^* = e_2^* \circ [.,.]$ (e_2^* = the linear functional dual to e_2) in which $\mathfrak{g}$ is the Lie algebra of the "$ax + b$ − group" whose defining basis relation is $[e_1, e_2] = e_2$. One might expect that symmetric cocycles are always coboundaries. We shall see next that in the analytic case this is indeed so. However, it is crucial that A be connected, we give below a counter-example with A discrete.

PROPOSITION. *Let G be a connected and simply connected Lie group, ω be an analytic 2-cocycle on G, ω cohomologous to ω_C where C is an alternating bilinear map on* $\mathfrak{g}$ *as in Theorem* 1(a). *Then*

(a) *ω_C is symmetric if and only if $\omega_C = 0$.*
(b) *If ω is symmetric then ω is trivial.*

PROOF. If ω_C is symmetric we have $\hat{\omega}_C \equiv 0$, hence $C = 0$ by Corollary 1.8. Consequently, ω_C is equal to 0. The converse is obvious.

(b) We write $\omega = \omega_C + \alpha$, in which α is trivial. Consequently we can find an analytic $f: G \to A$ such that $\alpha(x, y) = f(xy) - f(x) - f(y)\,(x, y \in G)$. Now, assuming ω is symmetric, the antisymmetric part $(\omega_C)_a$ is cancelled by an antisymmetric part α_a of α, $(\omega_C)_a = -\alpha_a$. In view of this,

$$\frac{d}{dt}\hat{\alpha}(\exp X, \exp tY)|_{t=0} = \frac{d}{dt}\hat{\alpha}_a(\exp X, \exp tY)|_{t=0}$$

$$= -\frac{d}{dt}(\hat{\omega}_C)_a(\exp X, \exp tY)|_{t=0} = -C(X, Y)$$

On the other hand, arguing as in the proof of Theorem 1 (b) (Sec. 1.10) we have,

$$\frac{d}{dt}\hat{\alpha}(\exp X, \exp tY)|_{t=0} = Df(0) \circ [X, Y].$$

Accordingly, $C = -Df(0) \circ [.,.]$ is trivial, and this implies ω_C and ω are trivial (Theorem 1 (b)).

1.14. EXAMPLE. Generally speaking, symmetric cocycles ned not be coboundaries. For an example in which the group G is connected whereas A is discrete, we may simply take for $G(\omega)$ the additive group R of real numbers which is a central extension of the additive group $[0, 1)$ (addition modulo 1) by the integers, given by the symmetric cocycle mapping (x, y) in $[0, 1) \times [0, 1)$ to the integral part of $x + y$.

1.15. Recall that $x \in G$ is ω-regular if $\omega(x, y) = \omega(y, x)$ for all y in the centralizer of x. We let R_ω denote the set of all ω-regular elements in G. Suppose G is discrete. Then the left regular ω-representation of G is primary if and only if $\{e\}$ is the ony finite ω-regular conjugacy class in G, [Kl]. For torsion free nilpotent lattices the following result holds.

PROPOSITION. *Let Γ be a finitely generated torsion free nilpotent lattice group. Assume $\omega = \omega_C$ is a multiplier on Γ which corresponds to an alternating bicharacter C on the lattice $L = \log \Gamma$ as in Theorem* 1 (a). *Then the left regular ω-representation of Γ is type* II_1 *primary if and only if $S_\omega \cap Z_\Gamma = \{e\}$* ($Z_\Gamma$ = center of Γ).

We give only a sketch of the proof, complete details together with more applications will appear in a subsequent article. Observe first that $S_\omega \cap Z_\Gamma = R_\omega \cap Z_\Gamma$ which implies the "only if" part. Next, assume $R_\omega \cap Z_\Gamma = \{e\}$. If $x \in R_\omega$ and $x \neq e$, the main difficulty consists in proving that Γ mod the centralizer of x in Γ is torsion free which follows by an inductive argument.

2. Almost fibered extensions.

Our main result in the present section is that any extension of locally compact groups is almost fibered. This was proved by Nagao, [N, Theorem 2] for first countable groups. Our argument is independent of the one given in [N]. Instead we are using a result due to A. Borel on groups having Lie quotients, [B].

2.1. DEFINITION. Let H and N be locally compact topological groups. By an extension $E(H, N)$ of H by N we shall understand a locally compact group G, for which N is a closed normal subgroup and, in addition, there is an isomorphism of the factor group G/N onto H.

2.2. DEFINITION. An extension $G = E(H, N)$ is said to be almost fibered (resp. fibered) if there is a section $u: G/N \to G$ that is continuous at the identity element of G/N (resp. locally continuous).

In view of [Ca, Definition 3.2 and Proposition 3.5], the above condition (defining an almost fibered extension) is equivalent to continuity of the map $h: N \times H \to G, (n, h) \mapsto nu(h)$ and its inverse at the identity element of $N \times H$ and G, respectively. Moreover, it is also equivalent to require the extension to be associated with a cocycle that is continuous at the identity; cf. [Ca, Definition 4.1] and the arguments following it. According to this, the almost fibered group extensions are exactly the ones associated to extension cocycles that are continuous at the identity element.

2.3. LEMMA. *Assume K and N are closed normal subgroups of G, $K \subset N$. Further assume that K is compact and G/K is Lie. If the extension $G/K = E(G/N, N/K)$ is almost fibered then the extension $G = E(G/N, N)$ is.*

PROOF. In view of a result due to A. Borel, [B, Théorème] $E(G/K, K)$ is fibered. Let $u: G/K \to G$ and $v: G/N \to G/K$ be sections, both continuous at the identity. We define a section $w: G/N \to G$ by composition, $w = u \circ v$. Clearly, w is continuous at the identity in G/N. The situation is illustrated by the following diagram.

$$\begin{tikzcd}
 & G/N \arrow[dl, "\cong"'] \arrow[dr, "w = u \circ v"] & \\
(G/K)/(N/K) \arrow[r, "v"] & G/K \arrow[r, "u"] & G
\end{tikzcd}$$

2.4. Suppose next $G = E(G/N, N)$ is an extension of locally compact groups and K is a compact normal subgroup of G with Lie quotient G/K. We let $K_1 = K \cap N$, $G_1 = G/K_1$, $N_1 = N/K_1$. Then both G_1 and N_1 are Lie, and we have a fibered extension $G_1 = E(G_1/N_1, N_1) \cong E(G/N, N/K_1)$. Consequently, by the above lemma, the extension $G = E(G/N, N)$ is almost fibered. We have shown

LEMMA. *Any extension $G = E(G/N, N)$, in which G is a projective limit of Lie groups, is almost fibered.*

2.5. As a consequence we derive the following theorem which does not seem to have appeared in the litterature before. However, for first countable locally compact groups it has been proved by Nagao, [N, Theorem 2].

THEOREM 2. *Let G be a locally compact group, N be a closed normal subgroup of G. Then the group extension $E(G/N, N)$ is almost fibered.*

PROOF. We fix an open subgroup H of G which is a projective limit of Lie groups, [MZ]. Then $N_1 = N \cap H$ is a closed normal subgroup of H. Now, by Lemma 2.4 the extension $E(H/N_1, N_1)$ is almost fibered, hence there exists a section $u: H/N_1 \to H$, continuous at the identity in H/N_1. We proceed to construct a section $G/N \to G$, continuous at the neutral element N of G/N. Choose any section $v: G/H \to G$ of the discrete space of left cosets G/H. We denote by $q_N: G \to G/N$ the quotient map. The situation is indicated by the following schemes,

$$G/H \xrightarrow{v} G \xrightarrow{q_N} G/N; \quad H/N_1 \xrightarrow{u} H \subset G \xrightarrow{q_N} G/N$$

It is readily seen that $q_N \circ v(G/H) \cdot q_N \circ u(H/N_1) = G/N$. We define $w: G/N \to G$ by letting

$$w(q_N \circ v(gH) \cdot q_N \circ u(hN_1)) = v(gH)u(hN_1)$$

Then w is a well defined section (since q_N is a homomorphism) and is continuous at the identity of G/N. As a consequence, the extension $E(G/N, N)$ is almost fibered.

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