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# THE REAL RANK OF INDUCTIVE LIMIT $C^*$-ALGEBRAS

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There has lately been a renewed interest in $C^*$-algebras $A$ that are inductive limits of direct sums of $C^*$-algebras of the form $C(\Omega, M_n)$, where $\Omega$ is a (connected, compact) Hausdorff space. The theory gained new momentum with G. Elliott's classification [5] of those algebras $A$ that have real rank zero and stable rank one in the case where the base spaces $\Omega$ have a special form.

In [2] it is proved that if the inductive limit $C^*$-algebra $A$ is simple and all the base spaces $\Omega$ are of dimension at most two, then the real rank of $A$ is zero if and only if projections in $A$ separate the traces on $A$. It is proved in [4] that if $A$ is simple and the dimensions of the base spaces $\Omega$ are bounded, then $A$ has stable rank one. Combining techniques from both papers, we improve both results (the real rank result most significantly) to the case where the dimension restrictions are replaced with a weaker one: slow dimension growth.

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Throughout this paper let $A = \varinjlim(A_n, \Phi_{m,n})$ be the $C^*$-algebra inductive limit of $C^*$-algebras $A_n$ of the form

$$A_n = \bigoplus_{j=1}^{r_n} C(\Omega_{n,j}, M_{[n,j]}),$$

with $\Omega_{n,j}$ connected compact Hausdorff spaces, $[n,j]$ positive integers, and $M_{[n,j]}$ the $[n,j] \times [n,j]$ matrices, and with a system of connecting unital homomorphisms $\Phi_{m,n}: A_n \to A_m$ $(m \geq n)$. Let $\Phi_{m,n}^{(j)}$ denote the homomorphism from $A_n$ to $C(\Omega_{m,j}, M_{[m,j]})$.

DEFINITION. $A$ has *slow dimension growth* if $A$ can be written as an inductive limit as above sucht that

$$\lim_{n\to\infty} \max_j \left\{ \frac{\dim \Omega_{n,j}}{[n,j]} \right\} = 0.$$

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If $A$ is simple, then $\lim_{n\to\infty} \min_j\{[n,j]\} = \infty$, and so $A$ has slow dimension growth if $\{\dim \Omega_{n,j}\}$ is bounded. Note also that if $A$ is simple, each $\Omega_{n,j}$ is of finite dimension and $B$ is a UHF-algebra, then $A \otimes B$ can be written as an inductive limit with slow dimension growth.

THEOREM 1 (cf. [4]). *If $A$ is simple and has slow dimension growth, then the stable rank of $A$ is one.*

Recall that $A$ has stable rank one, written $\mathrm{sr}(A) = 1$, if the invertible elements in $A$ are norm-dense in $A$.

THEOREM 2. *If $A$ is simple and has slow dimension growth, then the real rank of $A$ is zero if and only if projections in $A$ separate traces on $A$.*

Recall here that $A$ has real rank zero, written $\mathrm{RR}(A) = 0$, if every self-adjoint element of $A$ can be approximated by one with finite spectrum. In [2] the reader will find examples of simple $C^*$-algebras $A$ where the projections do not separate traces. We remark also that if $A$ is simple with slow dimension growth, and projections in $A$ do not separate traces, then Theorems 1 and 2 yield $\mathrm{RR}(A) \neq 0$ and $\mathrm{sr}(A) = 1$, and from [3] it then follows that $\mathrm{RR}(A) = 1$.

We remark further that Theorems 1 and 2 also hold in the case where the connecting maps $\Phi_{m,n}$ are not necessarily unital, provided the slow dimension growth condition be changed accordingly. In more detail, let $e$ be the unit of $A_1$, and let $f$ be the image of $e$ in $A$. Then $fAf$ is unital and is stably isomorphic to $A$ if $A$ is simple. Moreover, $fAf$ is the inductive limit of $\Phi_{n,1}(e)A_n\Phi_{n,1}(e)$ with unital connecting homomorphisms. The greater generality here lies in the possibility that the projections $\Phi_{n,1}(e)$ are not trivial in $A_n$. Upon making a slight refinement of Lemma D below, and replacing $[n,j]$ by $\dim \Phi_{n,1}^{(j)}(e)$ in the definition of slow dimension growth, the argument below can be amended to cover the situation of non-unital connecting maps.

NOTATION. If $B$ is a $C^*$-algebra and $a, b \in B^+$, then write $a \prec b$ if there is a $c \in B^+$ such that $ac = a$ and $bc = c$.

It is easily seen that if $a \prec b$ and $n \in \mathrm{N}$, then there are $c_j \in B^+$ such that $a \prec c_1 \prec \ldots \prec c_n \prec b$.

LEMMA A. *Let $B = \varinjlim B_n$ be an inductive limit of $C^*$-algebras $B_n$. If for every $n$ and every $a, b \in B_n^+$ with $a \prec b$ there is a projection $p$ in $B$ with $a \prec p \prec b$, then* $\mathrm{RR}(B) = 0$.

PROOF. A standard argument (cf. [3]) shows that under the conditions of Lemma A, every self-adjoint element of $B_n$ can be approximated with an invertible self-adjoint element of $B$. Since every self-adjoint element of $B$ can be approximated with a self-adjoint element of some $B_n$, we conclude that $\mathrm{RR}(B) = 0$.

A key step in finding a projection that interpolates $a \prec b$, and also in establishing Theorem 1, is the selection Principle from [4] restated here for the convenience of the reader.

In the following four lemmas $\Omega$ denotes a connected compact Hausdorff space of dimension $d$.

LEMMA B [4, Prop. 3.2]. *Let $\mathscr{W}$ be an open cover of $\Omega$ such that for every $W \in \mathscr{W}$ there is a continuous projection valued map $p_W: W \to M_n$ with* $\dim p_W(\omega) \geqq k$ *for each $\omega \in W$. Then there is a projection $p \in C(\Omega, M_n)$ so that for every $\omega \in \Omega$ the range of $p(\omega)$ is contained in the span of the ranges of $p_W(\omega)$, $\omega \in W$, and* $\dim p \geqq k - \frac{1}{2}(d+1)$.

LEMMA C. *Let $f, g \in C(\Omega, M_n)^+$ and $k, k' \in \mathrm{N}$ be given satisfying $f \prec g$,* rank $f(\omega) \geqq k$, *and* rank $g(\omega) \leqq k'$ *for all $\omega \in \Omega$. Then there are projections $p$ and $p'$ in $C(\Omega, M_n)$ so that*

(i) *$p \prec g$ and* $\dim p \geqq k - \frac{1}{2}(d+1)$,
(ii) *$f \prec p'$ and* $\dim p' \leqq k' + \frac{1}{2}(d+1)$.

PROOF. Upon replacing $f$ and $g$ by $\varphi(f)$ and $\varphi(g)$ where $\varphi(t) = \min\{t, 1\}$, we may assume that $f$ and $g$ both have norm at most 1.

(i) Find $h \in C(\Omega, M_n)^+$ with $f \prec h \prec g$. Given $\omega \in \Omega$, choose $\delta \in (0,1)$ not in the spectrum of $h(\omega)$, and let $W$ be an open subset of $\Omega$ containing $\omega$ so that none of $h(\omega')$, $\omega' \in W$, have $\delta$ in their spectrum. Then

$$p_W(\omega') = \chi_{[\delta,\infty)}(h(\omega')), \ \omega' \in W,$$

defines a continuous projection valued map with $\dim p_W(\omega') \geqq k$ and the range of $p_W(\omega')$ is contained in the range of $h(\omega')$. The existence of $p$ now follows from Lemma B.

(ii) Apply (i) to $1 - g \prec 1 - f$ and let $p'$ be the complement of the resulting projection.

A projection in $C(\Omega, M_n)$ is said to be trivial if it is unitarily equivalent to a constant projection.

LEMMA D (cf. [1, 6.10.3]). *Let $q$ and $q'$ be projections in $C(\Omega, M_n)$. Then*

(i) *$q$ contains a trivial subprojections $p \in C(\Omega, M_n)$ with* $\dim p \geqq q - \frac{1}{2}d$,

(ii) *$q'$ is a subprojection of a trivial projection $p' \in C(\Omega, M_n)$ with* $\dim p' \leqq \dim q' + \frac{1}{2}d$, *and*

(iii) *if* $\dim q - \dim q' \geqq d$, *then $q'$ is equivalent to a subprojection of $q$.*

PROOF. (i) If $\dim q \leqq \frac{1}{2}d$, then we may take $p = 0$. Otherwise, from [6, 8.1.2], $q$ contains a subprojection $p' \in C(\Omega, M_n)$ which is equivalent to a trivial projection $r$, and with $\dim p' = \dim q - \langle \frac{d-1}{2} \rangle$ ($\langle \cdot \rangle$ denotes the least integer $\geqq$). Since $\dim(1 - p') \geqq \langle \frac{d-1}{2} \rangle$, it follows from [6, 8.1.5] that $1 - p'$ and $1 - r$ are equivalent.

(ii) Apply (i) to $1 - q'$ and take the complement of the resulting projection.

(iii) is a direct consequence of (i) and (ii).

LEMMA E. *Assume that $C(\Omega, M_n)$ is a unital subalgebra of a $C^*$-algebra $B$ with $sr(B) = 1$. Let $f, g \in C(\Omega, M_n)^+$ and $k, k' \in \mathbb{N}$ be given so that $f \leqq g$, $\operatorname{rank} f(\omega) \leqq k'$ and $\operatorname{rank} g(\omega) \geqq k$ for all $\omega \in \Omega$. If $k - k' \geqq 2d + 1$, $f_0 \prec f$ and $g \prec g_0$ for some $f_0$, $g_0 \in C(\Omega, M_n)^+$, then there is a projection $p \in B$ such that $f_0 \prec p \prec g_0$.*

PROOF. First find $f_1, g_1 \in C(\Omega, M_n)^+$ so that $f_0 \prec f_1 \prec f$ and $g \prec g_1 \prec g_0$. By Lemma C there are projections $q$ and $q'$ in $C(\Omega, M_n)$ satisfying

$$q \prec g_1 \text{ and } \dim q \geqq k - \tfrac{1}{2}(d + 1),$$
$$f_1 \prec q' \text{ and } \dim q' \leqq k' + \tfrac{1}{2}(d + 1).$$

Since $\dim q - \dim q' \geqq d$ it follows from Lemma D (iii) that there is a partial isometry $v \in C(\Omega, M_n)$ with $q' = v^*v$ and $vv^* \leqq q$.

Put $z = vf_1$ and notice that $z$ is in $B_0 = \overline{g_1 B g_1}$, the hereditary subalgebra of $B$ generated by $g_1$. Let $\varphi: \mathbb{R}^+ \to \mathbb{R}^+$ be continuous with $\varphi|[0, \frac{1}{2}] \equiv 0$ and $\varphi(1) = 1$. Because $\operatorname{sr}(\tilde{B_0}) = 1$, $B_0$ being Morita equivalent to an ideal of $B$ (see [8]), $\varphi(|z|) = w\varphi(|z^*|)w^*$ for some unitary $w \in \tilde{B_0}$ (see [9, Thm. 2.2] or [7]). Put $p = wqw^*$, and notice that $p \in B_0$ and so $p \prec g_0$. Moreover,

$$p\varphi(|z|) = wq\varphi(|z^*|)w^* = w\varphi(|z^*|)w^* = \varphi(|z|).$$

Since $|z| = f_1$ and $\varphi(1) = 1$, we have $f_0 \prec \varphi(|z|)$ and hence $f_0 \prec p$.

LEMMA F. *Assume that $A$ is simple with slow dimension growth, let $h \in A_n^+$ be non-zero and let $\alpha, \beta \in \mathbb{R}^+$ be given. Then there is $m_0 \geqq n$ so that for all $m \geqq m_0$, every $j = 1, \dots, r_m$ and all $\omega \in \Omega_{m,j}$ we have*

$$\operatorname{rank} \Phi_{m,n}^{(j)}(h)(\omega) \geqq \alpha \dim \Omega_{m,j} + \beta.$$

PROOF. Because $A$ is simple there are $m_1 \geqq n$, $N \in \mathbb{N}$ and $x_j \in A_{m_1}$ so that

$$\sum_{j=1}^{N} x_j \Phi_{m_1,n}(h) x_j^* = 1.$$

It follows that for $m \geqq m_1$, all $j$ and $\omega \in \Omega_{m,j}$,

$$\operatorname{rank} \Phi_{m,n}^{(j)}(h)(\omega) \geqq N^{-1}[m,j].$$

Simplicity and slow dimension growth of $A$ imply that there is $m_0 \geqq m_1$ so that

$$N^{-1}[m,j] \geqq \alpha \dim \Omega_{m,j} + \beta$$

for all $m \geqq m_0$ and all $j$.

PROOF OF THEOREM 1. Take an arbitrary non-invertible $a \in A_n$. Following [4, Lemma 2.4] (see also [10, Lemma 3.4]) $a$ can be approximated by $ubv$, where $u, v \in \mathrm{GL}(A_n), b \in A_n$, and $bh = 0 = hb$ for some non-zero $h \in A_n^+$. From [4, Lemma 3.3], $\Phi_{m,n}(b)$ is in the closure of $\mathrm{GL}(A_m)$ if for all $j$ there are trivial projections $p_j \in C(\Omega_{m,j}, M_{[m,j]})$ with $\dim p_j \geqq \frac{1}{2}\dim \Omega_{m,j} + 2$, and $p_j \Phi_{m,n}^{(j)}(b) = 0 = \Phi_{m,n}^{(j)}(b) p_j$.

Now, let $h_0 \in A_n^+$ be non-zero with $h_0 \prec h$, and use Lemma F to find $m \geqq n$ so that

$$\operatorname{rank} \Phi_{m,n}^{(j)}(h_0)(\omega) \geqq \frac{3}{2} \dim \Omega_{m,j} + 3$$

for all $j$ and $\omega \in \Omega_{m,j}$, Then use Lemma C (i) and Lemma D (ii) to find trivial projections $p_j$ as above with $p_j \prec \Phi_{m,n}^{(j)}(h)$.

PROOF OF THEOREM 2. It is trivial that $\mathrm{RR}(A) = 0$ implies that projections separate traces. For the other implication we use Lemma A, and let $a, b \in A_n^+$ be given with $a \prec b$. Find $a_0, b_0, c \in A_n^+$ so that $a \prec a_0 \prec c \prec b_0 \prec b$. If $\delta \in (0,1)$ is not in the spectrum of $c$, then $p = \chi_{[\delta,\infty]}(c) \in A_n$ is a projection with $a \prec p \prec b$. If $(0,1) \cap \mathrm{sp}(c) \neq \emptyset$ and $\varphi(t) = \max\{t(1-t), 0)$ then $h = \varphi(c)$ is non-zero, $h \perp a_0$ and $a_0 + h \prec b_0$. Let $m_0 \geqq n$ be as in Lemma F corresponding to $\alpha = 2$ and $\beta = 1$. From [2, Thm. 1.3] there are $m \geqq m_0$ and $k_j$ so that

$$\begin{aligned} &\operatorname{rank} \Phi_{m,n}^{(j)}(a_0)(\omega) + \operatorname{rank} \Phi_{m,n}^{(j)}(h)(\omega) \\ &= \operatorname{rank} \Phi_{m,n}^{(j)}(a_0 + h)(\omega) \leqq k_j \leqq \operatorname{rank} \Phi_{m,n}^{(j)}(b_0)(\omega) \end{aligned}$$

for all $j$ and $\omega \in \Omega_{m,j}$. Lemma E together with Theorem 1 now yield the existence of a projection $p \in A$ with $a \prec p \prec b$.

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