MATH. SCAND. 69 (1991), 291–306

SEMICLASSICAL LIMIT OF THE SPECTRAL DECOMPOSITION OF A SCHRÖDINGER OPERATOR IN ONE DIMENSION

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0. Introduction.

Let $L = L_h = -h^2(d/dx)^2 + V(x)$ be a real differential operator in one variable with a continuous potential $V(x)$ and a parameter $1 \geqq h > 0$. Let H be $L^2(\mathrm{R})$ with the canonical inner product and let $A: D \mapsto H$ be a selfadjoint extension of L operating on $C_0^2(\mathrm{R})$. Let $A = \int \lambda dE(\lambda)$ be the spectral decomposition of A and $E(\Delta)$ a corresponding projection belonging to a bounded interval Δ. We are going to investigate the semiclassical limit of the projections $E(\Delta)$.

Our basic result (Theorem 1) says that if $V(x) \geqq \sup \Delta + c^2$ in some interval I, if J is a subinterval of I whose distance to the boundary of I is $d > 0$ and χ is the characteristic function of J, then

$$\|\chi E(\Delta)u\| \leqq 8\sqrt{5}\, e^{-cd/2h}\, \|u\|$$

for all $u \in H$ provided that either $|J| < d/2$ and $0 < h \leqq 1$ or $e^{-cd/h} \leqq 1/2$. The elementary proof combines the spectral decomposition with the fact that real solutions of the equation $f'' = Vf$ are very convex in intervals where $V > 0$ is large (see Lemma 2). The inequality means in particular that $E(\Delta)$ is a small operator when restricted to intervals where $V(x) > \sup \Delta$, i.e. the classically forbidden regions for a particle of energy not exceeding the least upper bound of Δ.

The inequality above may be improved if we suppose that $d > |J|$ and introduce Agmon distance $\varrho(a, x)$ between two points a and x relative to the potential $U(x) = V(x) - \sup \Delta$, i.e. the distance relative to the metric $U_+(x)\, dx^2$. Then for $0 < h \leqq 1$,

$$\|\chi E(\Delta)u\| \leqq 8e^{-\gamma\varrho(\partial I(-|J|), J)/h}\|u\|$$

if $U(x) \geqq 0$ in J and > 0 in $\overline{I \backslash J} \backslash \partial I$ and $\gamma > 0$ is so small that

$$1 - \gamma^2 - \mathrm{h}\gamma\, \mathrm{sgn}(x - a)V'(x) \tanh(\gamma\varrho(a, x)/h)/2U(x)^{3/2} \geqq 0$$

Received November 12, 1990.

where a is an endpoint of J and x belongs to the corresponding component of $I\backslash J$ and $0 < h \leqq 1$. Here $I(-s)$ denotes the set of points in I whose distance to the boundary of I are at least s. (Below we shall also use $I(s)$ for the set of points whose distance to I are at most s.) Note that $0 \leqq \tanh(\gamma\varrho(a, x)/h) \leqq 1$. Note also that the inequality holds with $\gamma = 1$ and $0 < h \leqq 1$ when $U(x)$ decreases away from J. We can take $1 \geqq \gamma > 0$ arbitrarily close to 1, when h is sufficiently small and $V'(x)/U(x)^{3/2}$ is bounded in $I\backslash J$.

Theorem 1 permits an approximate decomposition of $E(\Delta)$ into a sum of projections $F(\Delta)$ belonging to selfadjoint realizations $A(I)$ of L in intervals I which cover potential wells of $V(x)$ with respect to Δ, i.e. intervals where $V(x) \leqq \sup \Delta$. The domain of $A(I)$ consists of the restrictions to I of all $u \in D$ which vanish at the boundary of I. We shall also say that $A(I)$ is the restriction of A to I.

The precise results are given in Theorem 2 and Theorem 3. Choose a number $c > 0$ and let I be an interval where $V(x) \leqq c^2 + \sup \Delta$ and assume that $V(x) \geqq c^2 + \sup \Delta$ in $I(2d + b)\backslash I$ for some $b > 0, d > 0$. By the basic result a function u such that $E(\Delta)u = u$ is very small in $I(d + b)\backslash I(d)$ and hence it ought to be very close to some eigenspace of $B = A(I(2d + b))$ with spectrum close to Δ. This is indeed the case.

Let F be the spectral measure of B and let $\varphi \in C^2(I(2d + b))$ be 1 in $I(d)$ and 0 in $I(2d + b)\backslash I(d + b)$ and chosen so that $\varphi' = O(1/b)$ and $\varphi'' = O(1/b^2)$ in $I(d + b)\backslash I(d)$. Then Theorem 2 says that if $\Omega \subset \Delta$ is an interval and, $v = E(\Omega)u$ and $E(\Delta)u = u$, then

$$\|F(\Omega(\zeta))(\varphi v) - \varphi v\| = O(M(\log 1/\zeta)^2/b^2\zeta)e^{-cd/2h}\|v\|$$

where $d > 2b, 0 < h \leqq 1, \zeta > 0$ and M depends on a bound of V where $\varphi \neq 0, 1$. Under the same conditions, the same inequality holds with v replaced by u and Ω by Δ. To get the sense of this inequality and the next one, imagine that $\zeta > 0$ is very small so that $\Omega(\pm\zeta)$ is very close to Ω.

This inequality has a counterpart (Theorem 3) with the roles of A and B reversed,

$$\|E(\Omega(\zeta))u - u\| = O(M(\log 1/\zeta)^2/(b^2\zeta))e^{-cd/2h}\|u\| \tag{1}$$

where $\Omega \subset \Delta$, $F(\Omega)u = u$, u is continued by zero outside $I(2d + b)$, $\zeta > 0$, $d > 2b$ and $1 \geqq h > 0$. When $F(\Delta)u = u$ and $F(\Omega)u = 0$ on the other hand, the same right side majorizes $\|E(\Omega(-\zeta))u\|$.

The two inequalities above hold also with Agmon distances in the exponent under conditions indicated above and made precise in Remarks to Theorem 2 and 3 in the text.

If I is bounded, the spectrum of B is discrete with a separation of the order of h and we can let Ω cover just one eigenvalue of B and get two inequalities depending on whether $F(\Omega)u$ is zero not. The result is then that the eigenvalues of

B in Δ give rise to resonances in the sense (1) of the global operator A which get sharper as $h \to 0$.

When $V(x) \leqq c^2 + \sup \Delta$ defines a finite number of potential wells and unbounded wells appear, the physical significance of Theorem 2 and Theorem 3 is that every eigenvalue of A in Δ produces resonances in the bounded wells and that the eigenvalues in Δ comming from the bounded wells produce resonances of A. In the general case, these discrete eigenvalues appear as imbedded in the continuous spectra of the restrictions of A to the unbounded wells.

It follows from (1) that every bounded well gives rise to bound states or resonances of the global operator. For Stark effect with potential wells, $V(x) = x + W(x)$ with $W(x)$ periodic and analytic (Wannier's ladder), existence of resonances of A was proved before by dilation analytic method (see Herbst and Howland, Agler and Froese). In this case every bounded well is paired to an unbounded well on the same level with a continuous spectrum.

When there are no unbounded wells, our results are more or less included into those of Helffer-Sjöstrand 1984 which hold in several dimensions. Our paper should also be compared with that of Briet, Combes and Duclos from 1988 (see the references) where the emphasis is on an infinite number of wells and unbounded wells are permitted. They work in several dimensions with some restrictions on the potential and use Agmon distance to show that the resolvent $R(z)$ has the property that $R(z)u$ is exponentially small in the regions outside the support of u where V exceeds the real part of z. When the minimal Agmon distance between the wells is large enough, they are able to relate the resolvent sets of L and its localizations to wells. Hence their basic Theorem 1 applies only when there are gaps in the spectra of the latter.

The author thanks the referee for some suggestions which improve the paper.

1. Solutions of the equation $u'' = Vu$ where $V > 0$.

LEMMA 1. *If $u'' = Vu$ in an interval I and $V \geqq c^2$ ($c > 0$) is a continuous function there and $u > 0$ in the interval, then*

$$u(x)/u(a) \geqq \cosh c(x - a)$$

for $I \ni x \geqq a$ and $u'(a) \geqq 0$ or for $I \ni x \leqq a$ and $u'(a) \leqq 0$. If $u < 0$ in the interval, the same inequality holds with reversed signs of the derivative.

PROOF. Put $v = \cosh c(x-a)$. Since $(u'v - uv')' = u''v - uv'' = (V - c^2)uv \geqq 0$, the Wronskian $u'v - uv'$ increases. Its value at $x = a$ is $u'(a) \geqq 0$. Hence

$$(u/v)' = (u'v - uv')/v^2 \geqq 0$$

when $x \geqq a$ so that

$$u(x)/v(x) \geqq u(a)/v(a)$$

when $x \geqq a$ which is the first case of Lemma 1. The other cases follow in the same way.

REMARK. Let $\varrho(a, x)$ be Agmon distance between a and x relative to the potential V, i.e. the distance relative to the metric $V_+(x)\,dx^2$. If we use

$$v(x) = \cosh \gamma\varrho(a, x), \quad 1 \geqq \gamma > 0$$

as a new comparison function instead of $\cosh c(x - a)$, we can obtain a better estimate of $u(x)/u(a)$ but it will involve a condition on the derivative of the potential.

The estimate is

$$u(x)/u(a) \geqq \cosh \gamma\varrho(a, x)$$

provided $V(x) > 0$ in I and γ is so small that the quantity

$$\Gamma(a, t; \gamma) = 1 - \gamma^2 - \gamma \operatorname{sgn}(t - a)V'(t) \tanh \gamma\varrho(a, t)/2V(t)^{3/2}$$

is $\geqq 0$ for all t between a and x. In fact, the derivative of the new Wronskian is ΓVuv. Note that if V decreases away from a, we can take $\gamma = 1$.

LEMMA 2. *Let I be an interval and $V(x) \geqq c^2, c > 0$ be a continuous function in I and let u be a real solution of the equation*

$$u'' = Vu$$

in the interval. Let J be a bounded subinterval of I and $f \geqq 0$ a square integrable function with support in J. Then the integral

$$\left(\int u(x) f(x)\,dx\right)^2$$

is not greater than

$$4e^{-2c(y - |J|)}\left(\int u(x) f(x - \varepsilon y)\,dx\right)^2$$

where $y \geqq |J|$ and $\varepsilon y + J \subset I$.
Here $\varepsilon = \pm 1$ and the signs are determined by the following situations.
(a) *$u(x) \neq 0$ in I.*
(a.1) *If $|u(x)|$ has its maximum in J at right endpoint of J, then $\varepsilon = 1$.*
(a.2) *If $|u(x)|$ has its maximum in J at left endpoint of J, then $\varepsilon = -1$.*
(b) *u has a zero in I. Set*

$$M = \int u_+(x) f(x)\,dx, \quad m = \int u_-(x) f(x)\,dx,$$

where $u_+ = \max(u, 0)$, $u_- = \max(-u, 0)$.
(b.1) *If* u *increases in* I, *then* $\varepsilon = 1$ *if* $M \geqq m$, *and* $\varepsilon = -1$ *otherwise.*
(b.2) *If* u *decreases in* I, *then* $\varepsilon = -1$ *if* $M \geqq m$, *and* $\varepsilon = 1$ *otherwise.*
(c) $u = 0$ *on the boundary of* I.
(c.1) *If* $u = 0$ *at the right endpoint of* I, *then* $\varepsilon = -1$.
(c.2) *If* $u = 0$ *at the left endpoint of* I, *then* $\varepsilon = 1$.

REMARK. Since $u'' = Vu$ and $V > 0$ in I, u is convex towards the x-axis, and hence the cases (a) and (b) cover all situations that can occur.

Using Remark of Lemma 1, we can replace the exponent $-2c(y - |J|)$ in Lemma 2 by $-2\gamma\varrho(\partial I(-|J|), J)$ provided $V(x) \geqq 0$ in J and > 0 in $\overline{I\backslash J}\backslash\partial I$, the distance from J to the boundary of I is greater than $|J|$ and $\Gamma(a, x; \gamma) \geqq 0$ when a is an endpoint of J and x is in the corresponding component of $I\backslash J$. When $V(x)$ decreases away from J, we can take $\gamma = 1$. We can take γ arbitrarily close to 1, when h is sufficiently small and $V'(x)/V(x)^{3/2}$ is bounded in $I\backslash J$.

PROOF OF LEMMA 2. Part (a).

When $u > 0$ in I, $u'' = Vu > 0$ in J so that u is convex in J. Hence it attains its maximum in J at one of the endpoints. If it is the right endpoint β, by Lemma 1 (since $u'(\beta) \geqq 0$),

$$u(x + y)/u(x) \geqq u(x + y)/u(\beta) \geqq \cosh c(x + y - \beta)$$

where $x \in J$, $y \geqq |J|$ and $y + J \subset I$. Since $\cosh c(x + y - \beta) > e^{c(y + x - \beta)}/2 \geqq e^{c(y - |J|)}/2$ when $x \in J$, this proves that

$$u(x) \leqq 2e^{-c(y - |J|)}u(x + y)$$

where $x \in J$, $y \geqq |J|$ and $y + J \subset I$. If we multiply this by $f(x)$, integrate, square and change variables we get the required inequality.

When the maximum is attained at the left endpoint, we get the same result with $\varepsilon = -1$ and the case $u < 0$ is similar.

Part (b). Since $|u|$ is convex in I, u has at most one zero in I. In this case, either u increases in I or u decreases in I. When u is increasing in I and $M \geqq$ m, then

$$\left(\int u(x)f(x)\,dx\right)^2 = (M - m)^2 \leqq M^2 = \left(\int u_+(x)f(x)\,dx\right)^2$$

where u_+ is convex and attains its maximum in J at its right endpoint. Hence we can apply the part (a) with $\varepsilon = 1$. When u is decreasing in I and $M \geqq m$, u_+ attains its maximum in J at its left endpoint and we can apply the part (a) with $\varepsilon = -1$. The case $M < m$ is similar and (c) is proved in the same way.

LEMMA 3. *If u'' and u are square integrable in a bounded interval K, then*

$$|u'(x)| = O(b^{-3/2}\|u\| + b^{1/2}\|u''\|),$$
$$\|u'\| = O(b^{-1}\|u\| + b\|u''\|)$$

and

$$|u(x)| = O(b^{-1/2}\|u\| + b^{1/2}\|u'\|)$$

where b is the length of K and $\|u\|^2$ is the square integral of u over K.

PROOF. Integrating the equality

$$u'(x) = u'(y) + \int_y^x u''(t)\,dt$$

with respect to y between y_1 and y_2 and taking absolute value, we get

$$|y_2 - y_1\|u'(x)| \leqq |u(y_1)| + |u(y_2)| + |y_2 - y_1|\|u''\|_1.$$

where $\|u''\|_1$ is the L^1-norm of u'' over K. Integrating with respect to y_1, y_2 over K gives

$$b^3|u'(x)|/3 \leqq 2b\|u\|_1 + b^3\|u''\|_1/3$$

so that, after passing to L^2-norm,

$$|u'(x)| \leqq 6b^{-3/2}\|u\| + b^{1/2}\|u''\|$$

which is the first formula. Squaring and integrating gives the second one.

Starting with the equality

$$u(x) = u(y) + \int_y^x u'(t)\,dt,$$

taking absolute value, integrating with respect to y over K and passing to L^2-norm gives the third formula.

2. Estimates of projections.

Let $L, A, E(\Delta)$ be as in the introduction. Since A is represented by a differential operator, A has a concrete diagonalization (see Appendix). There are two Radon measures $d\mu_j \geqq 0$ on the real axis and a unitary mapping

$$L^2(\mathsf{R}) \ni f \mapsto F = (F_1(\lambda), F_2(\lambda)) \in L^2(\mathsf{R}, \mu_1) \oplus L^2(\mathsf{R}, \mu_2)$$

which diagonalizes A. Here F is defined as

$$F_j(\lambda) = \int e_j(\lambda, x) f(x)\, dx$$

when f has compact support and $e_j(\lambda, x)$ are real eigenfunctions of L,

$$Le_j(\lambda, x) = \lambda e_j(\lambda, x), \quad j = 1, 2$$

which are of class C^2 in x, measurable with respect to the product measure $dx \times d\mu_j$ and measurable with respect to the measure $d\mu_j$ for fixed x. The inverse of the mapping $f \mapsto F$ is given by

$$f(x) = \sum_{j=1}^{2} \int F_j(\lambda) e_j(\lambda, x)\, d\mu_j(\lambda)$$

when F has compact support. In particular we get Parseval's identity

$$\int f(x)^2\, dx = \int F_1(\lambda)^2\, d\mu_1(\lambda) + \int F_2(\lambda)^2\, d\mu_2(\lambda)$$

where $f \in L^2(\mathsf{R})$ or $F \in L^2(\mathsf{R}, \mu_1) \oplus L^2(\mathsf{R}, \mu_2)$.

In terms of the diagonalizing mapping, we have

$$E(\Delta) f(x) = \sum_{j=1}^{2} \int_{\Delta} F_j(\lambda) e_j(\lambda, x)\, d\mu_j(\lambda).$$

LEMMA 4. *Assume that Δ is an interval on the λ-axis and that I is an interval on the x-axis where*

$$V(x) - \sup \Delta \geqq c^2$$

with $c > 0$ a fixed number. If J is a subinterval of I and f a square integrable function with support in J,

$$\|E(\Delta) f\| \leqq 8 e^{-c(d - |J|)/h} \|f\| \tag{2}$$

where d is the distance from J to the boundary of I and $d \geqq |J|$ and $0 < h \leqq 1$.

REMARK. Using Remark of Lemma 2 applied to an equation $h^2 u'' = (V - \sup \Delta) u$, we may replace the exponent $-c(d - |J|)/h$ by $-\gamma \varrho(\partial I(-|J|), J)/h$ provided $U(x) = V(x) - \sup \Delta \geqq 0$ in J and > 0 in $\overline{I \backslash J} \backslash \partial I$ and the distance from J to the boundary of I is greater than $|J|$ and

$$1 - \gamma^2 - h\gamma \operatorname{sgn}(x - a) V'(x) \tanh(\gamma \varrho(a, x)/h)/2U(x)^{3/2}$$

is $\geqq 0$ when a is an endpoint of J and x belongs to the corresponding component of $I \backslash J$ and $0 < h \leqq 1$ where ϱ refers to U. When U decreases away from J, we can take $\gamma = 1$. Note that we can take γ arbitrarily close to 1, when h is sufficiently small and $V'(x)/U(x)^{3/2}$ is bounded in $I\backslash J$.

PROOF: By Parseval's formula

$$\|E(\Delta)f\|^2 = \int_\Delta F_1(\lambda)^2\, d\mu_1(\lambda) + \int_\Delta F_2(\lambda)^2\, d\mu_2(\lambda)$$

where

$$F_j(\lambda) = \int e_j(\lambda, x) f(x)\, dx.$$

Assume first that $f \geqq 0$. We can then apply Lemma 2 with $u(x) = e_j(\lambda, x)$, $\lambda \in \Delta$ and $y = d$. If we write

$$T = e^{-2c(d-|J|)/h},$$

the result is

$$F_j(\lambda)^2 \leqq 4T\left(\int e_j(\lambda, x) f(x - \varepsilon_j(\lambda) y)\, dx\right)^2$$

where $\varepsilon_j(\lambda) = \pm 1$. Hence

$$\int_\Delta F_j(\lambda)^2\, d\mu_j(\lambda) \leqq 4T \int\left(\int e_j(\lambda, x) f(x - y)\, dx\right)^2 d\mu_j(\lambda)$$
$$+ 4T \int\left(\int e_j(\lambda, x) f(x + y)\, dx\right)^2 d\mu_j(\lambda).$$

Summing for $j = 1, 2$ and by Parseval's formula again

$$\|E(\Delta)f\|^2 \leqq 4T \int f(x - y)^2\, dx + 4T \int f(x + y)^2\, dx$$

so that

$$\|E(\Delta)f\| \leqq 2\sqrt{2}T^{1/2}\|f\|.$$

If we write $f = f_+ - f_-$ as a sum of positive and negative parts, then

$$\|E(\Delta)f\| \leqq \|E(\Delta)f_+\| + \|E(\Delta)f_-\| \leqq 4\sqrt{2}T^{1/2}\|f\|.$$

For complex f we have

$$\|E(\Delta)f\|^2 = \|E(\Delta)\Re f\|^2 + \|E(\Delta)\Im f\|^2 \leqq 64T\|f\|^2.$$

This proves Lemma 4.

REMARK. Since the spectral theory applies also to the restrictions $A(K)$ to intervals K, the proof above applies also to this case.

THEOREM 1. *Assume that* $V(x) - \sup \Delta \geqq c^2, c > 0$, *in an interval* I *and that* J *is a subinterval of* I *whose distance to the boundary of* I *is* $d > 0$, *then*

$$\|E(\Delta)f\| \leqq 8\sqrt{5}\, e^{-cd/2h} \|f\| \tag{3}$$

if either $|J| \leqq d/2$ *and* $0 < h \leqq 1$ *or* $e^{-cd/h} \leqq 1/2$ *and in both cases* f *is a square integrable function with support in* J. *Under the same conditions,*

$$\|\chi_J E(\Delta)f\| \leqq 8\sqrt{5}\, e^{-cd/2h} \|f\| \tag{4}$$

where $f \in H$ *and* χ_J *is the characteristic function of* J.

REMARK. When $E(\Delta)$ refers to a restriction $A(K)$ of A to an interval K, (3) and (4) hold when the distance of J to the boundary of $I \subset K$ is $\geqq d$. This follows from the previous remark.

PROOF. If $|J| \leqq d/2$, the theorem follows from Lemma 4. If $|J| > d/2$, we can write $|J| = Nd/2 + l$ where $0 \leqq l < d/2$ and N is an integer. With a the left endpoint of J, put $J_k = [a + (k-1)d/2, a + kd/2]$ where $1 \leqq k \leqq N$ and $J_{N+1} = [a + Nd/2, a + Nd/2 + l]$. Let χ_k be the characteristic function of J_k where $1 \leqq k \leqq N + 1$ and let $f_k = \chi_k f$. By Lemma 4, we have

$$\|E(\Delta)f_k\| \leqq 8e^{-c(d_k - |J_k|)/h} \|f_k\| \tag{5}$$

where d_k is the distance from J_k to the boundary of I. The definition of J_k gives $|J_k| \leqq d/2$ and $d_k \geqq \min((k+1)d/2, (N-k+2)d/2)$ where $k \leqq N$ so that

$$d_k - |J_k| \geqq \min(kd/2, (N-k+1)d/2) \tag{6}$$

where $k \leqq N$ and

$$d_{N+1} - |J_{N+1}| \geqq d/2. \tag{7}$$

Now

$$\|E(\Delta)f\| \leqq \sum_{k=0}^{N} \|E(\Delta)f_k\|$$

which, by Schwarz' inequality and (5), does not exceed $8\|f\|$ times the square root of the sum

$$\sum_{k=0}^{N} e^{-2c(d_k - |J_k|)/h}$$

which, by virtue of (6) and (7), is not greater than

$$2e^{-cd/h}\sum_{k=0}^{\infty} e^{-ckd/h} + e^{-cd/h} = 2e^{-cd/h}/(1 - e^{-cd/h}) + e^{-cd/h}$$

which proves the inequality (3).

To prove (4) write

$$\|\chi_J E(\Delta)f\|^2 = (\chi_J E(\Delta)f, \chi_J E(\Delta)f) = (f, E(\Delta)(\chi_J E(\Delta)f)).$$

Hence (4) follows by applying Schwarz' inequality to the last term and using (3). This proves Theorem 1.

LEMMA 5. *Under the same conditions as in Theorem* 1, *let* $K \subset I$ *be an interval of length* b *whose distance to the boundary of* I *is* $d > 0$, *then for* $0 < h \leqq 1$

$$\|\chi_K(E(\Delta)f)'\| = O(1)(Mbh^{-2} + b^{-1})\|\chi_K E(\Delta)f\|$$

if Δ *is bounded*, $M = M(K) = \max(\sup_{\lambda\in\Delta}|\lambda|, \sup_{x\in K}|V(x)|)$ *and* $f \in H$.

PROOF. We have for $f \in H$,

$$LE(\Delta)f(x) = \sum_{j=1}^{2}\int_{\Delta} F_j(\lambda)\lambda e_j(\lambda, x)\, d\mu_j(\lambda).$$

Since Δ is bounded, $LE(\Delta)f \in D$ and hence $(E(\Delta)f)''$ is square integrable on K. Hence Lemma 3 gives with $\chi = \chi_K$

$$\|\chi(E(\Delta)f)'\| = O(b\|\chi(E(\Delta)f)''\| + b^{-1}\|\chi E(\Delta)f\|).$$

Now

$$\chi(E(\Delta)f(x))'' = \chi(-LE(\Delta)f + V(x)E(\Delta)f)/h^2$$

so that, by Parseval's formula,

$$\|\chi(E(\Delta)f)''\| \leqq \left(\sup_{\lambda\in\Delta}|\lambda|\|\chi E(\Delta)f\| + \sup_{x\in K}|V(x)|\|\chi E(\Delta)f\|\right)/h^2$$

which gives

$$\|\chi(E(\Delta)f)'\| \leqq O(1)(Mbh^{-2} + b^{-1})\|\chi E(\Delta)f\|.$$

3. Approximation of $E(\Delta)H$ in a potential well.

Let Δ be a bounded spectral interval, suppose that $E(\Delta)u = u$, let $\Omega \subset \Delta$ and consider the function $v = E(\Omega)u$. By Theorem 1, v is small in intervals where $V(x) > \Delta$. We shall see that v, when multiplied by a suitable cut-off function, is close to being an eigenfunction of $F(\Omega')$ where $\Omega' \supset \Omega$ is close to Ω and $F(\Omega')$ is a spectral projection belonging to the restriction of A to an interval such that $V(x) > \Delta$ at its boundary. The precise statement is given below.

THEOREM 2. *Let Δ be a bounded spectral interval and $c > 0$ a positive number. Let $I \neq \mathsf{R}$ be an interval and assume $V(x) \geqq c^2 + \sup \Delta$ in $I(2d + b)\backslash I$ for some $b > 0$, $d > 0$. Put $B = A(I(2d + b))$, let Ω be a spectral interval and let $F(\Omega)$ be a corresponding projection relative to B. Let $\varphi \in C^2(I(2d + b))$ be one in $I(d)$ and zero in $I(2d + b)\backslash I(d + b)$ and chosen so that $\varphi' = O(1/b)$, $\varphi'' = O(1/b^2)$ in $I(d + b)\backslash I(d)$.*

Assume also that $E(\Delta)u = u$ and that $v = E(\Omega)u$ where Ω is an interval contained in Δ. Then

$$\|F(\Omega(\zeta))(\varphi v) - \varphi v\| = O(M(\log 1/\zeta)^2/(b^2\zeta))e^{-cd/2h}\|v\| \tag{8}$$

for all $\zeta > 0$, $M = M(I(d + b)\backslash I(d))$ and $0 < h \leqq 1$ provided $d > 2b$.

REMARK. When $V(x) > \sup \Delta$ in I, $F(\Delta)$ vanishes. The interval I may contain one or several potential wells where $V(x) < \sup \Delta$. To get a good estimate, $I(2d + b)$ should be large and b small relative to d. When $V(x) \geqq c^2 + \sup \Delta$ on the complement of $I, I(2d + b)$ tends to R and $A(I(2d + b))$ to A as $d \to \infty$.

We can apply Remark of Lemma 4 to replace the exponent $-cd/2h$ by a better one. To do this let I_1 and J_1 be the right parts of $I(2d + b)\backslash I$ and $I(d + b)\backslash I(d)$ respectively and let I_{-1} and J_{-1} the left parts. The new exponent is

$$\min_{\varepsilon = \pm 1} \gamma\varrho(\partial I_\varepsilon(-|J_\varepsilon|), J_\varepsilon)/h$$

provided $d > b$, $0 < h \leqq 1$, $U(x) = V(x) - \sup \Delta \geqq 0$ in $J_{-1} \cup J_1$ and > 0 in $\overline{(I_{-1}\backslash J_{-1}) \cup (I_1\backslash J_1)}\backslash \partial(I_{-1} \cup I_1)$ and

$$1 - \gamma^2 - h\gamma \operatorname{sgn}(x - a)V'(x)\tanh(\gamma\varrho(a, x)/h)/2U(x)^{3/2} \geqq 0$$

when a is an endpoint of J_ε and x belongs to the corresponding component of $I_\varepsilon\backslash J_\varepsilon$ and $0 < h \leqq 1$. When $U(x)$ decreases away from J_{-1} and J_1 respectively, we can take $\gamma = 1$. We can also take γ arbitrarily close to 1, when h is sufficiently small and $V'(x)/U(x)^{3/2}$ is bounded in $(I_{-1}\backslash J_{-1}) \cup (I_1\backslash J_1)$.

In order to prove Theorem 2 we need a preliminary result.

LEMMA 6. *Let Q be a rectangle symmetric with respect to the real axis and with the projection $\Lambda = [\alpha, \beta]$ on the real axis. Let $f(z)$ be positive on Λ, analytic in Q and on the boundary except at α, β and assume that*

$$T(f) = \int_Q |f(z)|\, d|z|/|\Im z|$$

is finite. Assume also that the distance δ from Λ to Δ is positive and that $E(\Delta)u = u$. Then

$$\|f(B)(\varphi u)\| = O(T(f)M/(b^2\delta))e^{-cd/2h}\|u\| \tag{9}$$

when $0 < h \leqq 1$ and a suitable choice of φ.

PROOF OF LEMMA 6. Let $R(z)$ and $S(z)$ be the resolvents of A and B respectively. Put

$$w(z) = f(z)(S(z)(\varphi u) - \varphi R(z)u).$$

Spectral theory shows that

$$(2\pi i)^{-1}\int_Q w(z)\,dz = f(B)(\varphi u) - \varphi f(A)u$$

where f is supposed to be continued by zero outside Λ on the right. Since $E(\Delta)u = u, f(A)u$ vanishes.

The function $w(z)$ belongs to $D(B)$ and hence it can be written as $S(z)(L - z)w(z)$ which turns out to be

$$f(z)S(z)[\varphi, L]R(z)u = f(z)S(z)[\varphi, L]E(\Delta)R(z)u$$

since $E(\Delta)$ commutes with $R(z)$. Hence we can write the integral of $w(z)$ as

$$(2\pi i)^{-1}\int_Q f(z)S(z)[\varphi, L]E(\Delta)R(z)u\,dz.$$

Here $[\varphi, L] = h^2(2\varphi'(x)(d/dx) + \varphi''(x))$ is supported in $I(d + b)\backslash I(d)$ where we can choose φ so that $\varphi' = O(1/b)$ and $\varphi'' = O(1/b^2)$. Hence by Theorem 1 and Lemma 5,

$$\|[\varphi, L]E(\Delta)R(z)u\| \leqq O(1)Mb^{-2}e^{-cd/2h}\|R(z)E(\Delta)u\|.$$

Here

$$\|R(z)E(\Delta)u\| = O(\delta^{-1})\|u\|$$

uniformly on Q and this proves Lemma 6 since $\|S(z)v\| = O(\|v\|/|\Im z|)$.

PROOF OF THEOREM 2. Let us first note that the left side of (8) equals to $F(\Omega(\zeta)^c)(\varphi v)$ where $\Omega(\zeta)^c$ is the complement of $\Omega(\zeta)$.

Next, choose the function f be as in Lemma 6 to be

$$f(z) = \left(\log\frac{1}{(z-\alpha)(\beta - z)}\right)^{-2}$$

where $\beta - \alpha = 1$ which makes $f(x) > 0$ when $\alpha < x < \beta$. A simple computation shows that $T(f) = O(1)$. Let

$$f_1, f_2, \ldots$$

be one way translates of f away from Ω on one side of the interval. We assume that the overlap is constant equal to $|\Lambda|/3$ which makes the sum

$$\sum f_k(x)^2$$

positive, bounded and bounded from below away from then the endpoint of the support Λ_1 of f_1 which is closest to Ω. In particular, by chosing Λ_1 at a distance $\zeta/2$ from Ω we are sure that for some $C > 0$,

$$\sum f_k(x)^2 \geqq C(\log 1/\zeta)^{-4} \tag{10}$$

in points not closer than ζ to Ω. By Lemma 6 and since $T(f_i) = T(f) = O(1)$,

$$\|f_j(B)(\varphi v)\| = O(M/(b^2\delta_j))e^{-cd/2h}\|v\| \tag{11}$$

where δ_j is the distance of the support of f_j to Ω. By construction, the square sum

$$\sum 1/\delta_j^2$$

converges and is $O(\zeta^{-2})$. Hence

$$(\sum f_j(B)^2(\varphi v), \varphi v) = \sum \|f_j(B)(\varphi v)\|^2 = O(M^2/(\zeta^2 b^4))e^{-cd/h}\|v\|^2.$$

By the spectral theorem and (10), the left side is larger than a positive constant times

$$(\log 1/\zeta)^{-4}\|F(K)(\varphi v)\|^2$$

where K is an unbounded interval on one side of Ω with the distance ζ to Ω. If we add two such estimates, one for each side of Ω, Theorem 2 follows.

The point of Theorem 2 is that a function with spectrum relative to the global operator in an interval which is dominated by a potential well localizes by multiplication to a function with about the same spectrum relative to the operator localized to the well.

Our next result is a converse of Theorem 2. We shall see among other things that the eigenfunctions of the local operator when extended by zero turn into resonances of the global operator.

THEOREM 3. *Assume that the first hypotheses of Theorem 2 hold, let Ω be a subinterval of Δ and $d > 2b$.*

1) *If $F(\Omega)u = u$ and u is continued by zero outside of $I(2d + b)$ then*

$$\|E(\Omega(\zeta))u - u\| = O(M(\log 1/\zeta)^2/(b^2\zeta))e^{-cd/2h}\|u\| \tag{12}$$

for $\zeta > 0$ *and* $0 < h \leqq 1$.

2) *If* $F(\Delta)u = u$ *and* $F(\Omega)u = 0$, *then*

$$\|E(\Omega(-\zeta))u\| = O(M(\log 1/\zeta)^2/(b^2\zeta))e^{-cd/2h}\|u\| \tag{13}$$

for $\zeta > 0$ *and* $0 < h \leqq 1$.

REMARK. When I is bounded, B has a discrete spectrum and the formulas (12) and (13) say that an eigenvalue of B in Δ gives a resonance of A and that an interval between two such eigenvalues gives a non-resonant spectral interval of A. The estimates are rather good. By the mini-max principle, the number of eigenvalues $\leqq t$ of B is $O(1/h)$ when t is bounded. Hence the average distance between two eigenvalues of B in a bounded interval has the order h.

The exponent $-cd/2h$ may be improved as in Remark after Theorem 2.

PROOF OF THEOREM 3. By Remark of Theorem 1, $u - \varphi u = O(e^{-cd/2h})\|u\|$. This inequality is used to prove the following counterpiece to Lemma 6.

LEMMA 6′. *Let* $f(z)$ *be as in Lemma* 6 *and assume also that the distance* δ *from* Λ *to* Ω *is positive and that* $F(\Omega)u = u$. *Then*

$$\|f(A)u\| = O(MT(f)/(b^2\delta))e^{-cd/2h}\|u\| \tag{14}$$

when $0 < h \leqq 1$.

PROOF OF LEMMA 6′. By the inequality above and the boundedness of $f(A)$, to prove the inequality (14), it suffices to prove the same inequality with left side replaced by $\|f(A)(\varphi u)\|$ where φ is as in Theorem 2. Now the proof goes as the proof of Lemma 6 with the roles of $R(z)$ and $S(z)$ interchanged.

The proof of the first part of Theorem 3 runs as the proof of Theorem 2 with the same choice of f.

To prove (13), note that $F(\Omega^c)u = u$ and use f based on $\Omega(-\zeta/2)$. It is then easy to verify that

$$(f(A)^2u, u) = \|f(A)u\|^2$$

exceeds a positive constant times

$$(\log 1/\zeta)^{-4}(E(\Omega(-\zeta))u, u)$$

and a similar Lemma 6′ holds. This proves the (13).

Appendix.

When $a(\lambda) = (a_{jk}(\lambda))$ is a family of non-decreasing 2×2 real symmetric matrices whose elements are of bounded variation on every bounded interval, let $L^2(\mathsf{R}, a)$

be the space of functions $u(\lambda) = (u_1(\lambda), u_2(\lambda))$ measurable with respect to $da(\lambda)$ such that

$$(u, u)_{da} = \int \sum_{j,k=1}^{2} \overline{u_j(\lambda)} u_k(\lambda)\, da_{jk}(\lambda)$$

is finite.

As in the introduction, let $A : D \mapsto H$ be a selfadjoint extension of L operating on $C_0^2(\mathsf{R})$. Also, let $u_1(\lambda, x)$ and $u_2(\lambda, x)$ be real basis of the space of solutions of $Lu = \lambda u$ with real λ. By the spectral theory (see Coddington-Levinson 1955, pp. 246–252) there is such a basis which is analytic in λ and a map $U : f \mapsto Uf$ given by

$$(U f)_j(\lambda) = u_j(\lambda, f) = \int u_j(\lambda, x) f(x)\, dx, \quad j = 1, 2$$

which is unitary from $L^2(\mathsf{R})$ to some $L^2(\mathsf{R}, a)$ and defined pointwise when f has compact support. We shall prove

LEMMA 7. *There is a basis $e_1(\lambda, x)$ and $e_2(\lambda, x)$ of the solutions of the $Lu = \lambda u$ which is measurable with respect to $db(\lambda) = d\mathrm{tr}a(\lambda)$ and such that the map*

$$(u_1(\lambda, f), u_2(\lambda, f)) \mapsto (e_1(\lambda, f), e_2(\lambda, f))$$

is unitary from $L^2(\mathsf{R}, a)$ to a space $L^2(\mathsf{R}, b)$ with the inner product

$$\int (|e_1(\lambda, f)|^2 + |e_2(\lambda, f)|^2)\, db(\lambda).$$

PROOF. Let I be the unit 2×2 matrix and let a be as above. It is immediately verified that $Idb - da \geqq 0$ since

$$Ib(\mu) - a(\mu) - (Ib(\lambda) - a(\lambda)) = \mathrm{tr}(a(\mu) - a(\lambda)) - (a(\mu) - a(\lambda)) \geqq 0$$

when $\mu \geqq \lambda$ and $j, k = 1, 2$. It follows from Radon-Nikodym theorem that there is a measurable family $c(\lambda)$ of 2×2 matrices such that $da(\lambda) = c(\lambda)\, db(\lambda)$. It is clear that $c(\lambda) \geqq 0$ almost everywhere with respect to db.

To prove Lemma 7 we only have to write the quadratic forms

$$\sum c_{jk}(\lambda) \overline{u_j(\lambda, f)} u_k(\lambda, f)$$

as sums of squares in a measurable way. To do this put $c(\lambda) = c$ to simplify and

$$e_1(\lambda, x) = c_{11}^{1/2} u_1(\lambda, x) + c_{12} c_{11}^{-1/2} u_2(\lambda, x)$$

and

$$e_2(\lambda, x) = (\det c / c_{11})^{1/2} u_2(\lambda, x)$$

when $c_{11} > 0$, put

$$e_1(\lambda, x) = 0, \quad e_2(\lambda, x) = c_{22}^{1/2} u_2(\lambda, x)$$

when $c_{11} = 0$, $c_{22} > 0$ and $e_1(\lambda, x) = e_2(\lambda, x) = 0$ when $c = 0$. We have then

$$\sum c_{jk}(\lambda) \overline{u_j(\lambda, x)} u_k(\lambda, x) = \sum |e_j(\lambda, x)|^2.$$

The same equality holds for $e_j(\lambda, f) = \int e_j(\lambda, x) f(x)\, dx$ and $u_j(\lambda, f) = \int u_j(\lambda, x) f(x)\, dx$. This proves Lemma 7.

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