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# MEAN VALUES OF SUBHARMONIC FUNCTIONS OVER GREEN SPHERES

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## 1. Introduction.

If $w$ is subharmonic on $\mathsf{R}^n$ for any $n \geqq 2$, and $\mathscr{L}(w, x_0, r)$ denotes the integral mean value of $w$ over the sphere of centre $x_0$ and radius $r$, it is well-known that $\mathscr{L}(w, x_0, r)$ is a real-valued, increasing function of $r$, and a convex function of $\tau(r)$, where $\tau(r) = -\log r$ if $n = 2$, and $\tau(r) = r^{2-n}$ if $n \geqq 3$. Furthermore, $w$ has a harmonic majorant on $\mathsf{R}^n$ if and only if $\mathscr{L}(w, x_0, \cdot)$ is bounded above. For certain other special domains, such as a half-space, integral means have been found that have analogous properties (see [7, 8] for references). It is the purpose of this paper to present such means for arbitrary Dirichlet regular Greenian domains.

Some work on wide classes of domains has already been carried out. In [13], Wu introduced a class of integral means on subdomains of $\mathsf{R}^2$, which had convexity properties and were linked to the shape of the domains. The integrals were taken over level curves of certain harmonic functions. In [7, 8], Gardiner produced integral means that were completely analogous to $\mathscr{L}(w, x_0, r)$, on unbounded, locally Lipschitz domains in $\mathsf{R}^n$ for any $n \geqq 2$. Those means directly generalized known means on certain special domains, and the integrals were defined in terms of level surfaces of certain functions, but were unrelated to those of Wu.

Let $D$ be an arbitrary Dirichlet regular Greenian domain in $\mathsf{R}^n$, and let $x_0 \in D$. We define integral means $\mathscr{L}_D(w, x_0, r)$ over level surfaces $G_D(x_0, \cdot)^{-1}(\{\tau(r)\})$ of the Green function $G_D(x_0, \cdot)$ for $D$ with pole at $x_0$, an idea suggested by two papers of Brelot and Choquet [2, 3], where such surfaces are called Green spheres. Since the gradient of $G_D(x_0, \cdot)$ may vanish at some points, we cannot assume that the level sets are all smooth regular manifolds, but (Lebesgue) almost all of them are. For the convexity theorem, we prove that there is a convex function $\phi$ such that $\mathscr{L}_D(w, x_0, r) = \phi(\tau(r))$ for every $r$ such that $G_D(x_0, \cdot)^{-1}(\{\tau(t)\})$ is a smooth regular manifold. These means are completely analogous to, and direct generalizations of, the mean $\mathscr{L}(w, x_0, r)$, but do not directly generalize other known means for

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special domains, and are unrelated to those of Wu and Gardiner. For the convexity theorem, our methods are based on the elementary approach to the classical result for $\mathscr{L}(w, x_0, r)$ given by Dinghas [4], but because $\mathscr{L}_D(w, x_0, r)$ may not be defined for all $r$, we have to use more sophisticated calculus.

Theorem 1 establishes the properties of $\mathscr{L}_D$, with the exception of the convexity, which is left to Theorem 2. Convexity results are proved not only for $\mathscr{L}_D(w, x_0, r)$, but also for $\mathscr{L}_D((w^+)^p, x_0, r)^{1/p}$ whenever $1 < p < \infty$, and for $\log \mathscr{L}_D(e^w, x_0, r)$. Theorem 3 extends most of the properties of $\mathscr{L}_D$ to the associated volume means $\mathscr{A}_D$, and also establishes inequalities between $\mathscr{L}_D$ and $\mathscr{A}_D$.

Our final result, Theorem 4, is a generalization of the classical three spheres theorem, in which the spheres are replaced by level sets of $G_E(x_0, \cdot)$ for an arbitrary Greenian subdomain $E$ of $\mathsf{R}^n$. For this result, we do not require any smoothness of the level sets, so that we do not need to assume Dirichlet regularity, or to avoid exceptional values of $r$. The level sets may meet the boundary of $E$, but only in a harmonic measure null set, which creates no difficulties.

We put $\nabla u = (D_1 u, \ldots, D_n u)$, $\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$, and $\|x\| = \langle x, x \rangle^{1/2}$. On any smooth surface, we use $\sigma$ to denote surface area measure, $\nu$ to denote the outward unit normal, and $D_\nu u$ to denote $\langle \nabla u, \nu \rangle$. The surface area of the unit ball in $\mathsf{R}^n$ is $\sigma_n$, and the volume is $\upsilon_n$. Where no measure is specified, the term 'almost everywhere' is to be interpreted with respect to Lebesgue measure, as is the term 'measurable'. The words 'increasing' and 'decreasing' are used in the wide sense. The characteristic function of a set $S$ is denoted by $\chi_S$. By a 'smooth' function, we mean one which is twice continuously differentiable. For potential theoretic generalities, we refer the reader to Doob's book [5].

## 2. Preliminary discussion of surfaces and means.

Throughout this paper, $G(x_0, \cdot)$ denotes the fundamental superharmonic function with pole at $x_0$, and $\tau, D, G_D(x_0, \cdot)$ are as described above. For any $x_0 \in D$, we have

$$G_D(x_0, \cdot) = G(x_0, \cdot) - h \tag{1}$$

on $D$, where $h$ is the PWB solution of the Dirichlet problem for $D$ with boundary function $G(x_0, \cdot)$ [5, pp. 104, 136], since all boundary points of $D$ are Dirichlet regular (including the point at infinity if $D$ is unbounded). For the same reason, given any $r$ such that $\tau(r) > 0$, the set

$$B_D(x_0, r) = \{x \in D : G_D(x_0, x) > \tau(r)\}$$

is bounded and its closure is contained in $D$; it is also a domain [2, p. 118]. Since

(1) shows that $G_D(x_0,\cdot)$ is analytic on $D\backslash\{x_0\}$, for almost every $r$ such that $\tau(r)>0$ the set

$$\{x\in D: G_D(x_0,x)=\tau(r)\} \tag{2}$$

is a smooth, regular, $(n-1)$-dimensional manifold, by Sard's theorem [12, p. 45]. (According to Kellogg [9, p. 276], in any bounded interval there are only finitely many values of $r$ such that $\nabla G_D(x_0,\cdot)$ has a zero at some point of the set (2), but this does not seem to make any difference to our arguments.) We call such a value of $r$ a *regular value*. (To avoid confusion, the potential theoretic notion of regularity is referred to as Dirichlet regularity.) For any regular value of $r$, the set (2) is $\partial B_D(x_0,r)$, and the outward unit normal to it is given by the standard formula $\nu=-\nabla G_D(x_0,\cdot)\|\nabla G_D(x_0,\cdot)\|^{-1}$.

We now define the surface means. Let $\kappa_2=\sigma_2$, and $\kappa_n=(n-2)\sigma_n$ if $n\geqq 3$. Given $x_0\in D$ and a regular value of $r$, we put

$$\mathscr{L}_D(u,x_0,r)=\kappa_n^{-1}\int_{\partial B_D(x_0,r)}\|\nabla G_D(x_0,\cdot)\|u\,d\sigma$$

whenever the integral exists.

THEOREM 1. *Let $w$ be subharmonic on an open superset $E$ of $\bar{B}_D(x_0,r_0)$.*

(i) *The function $S\mapsto\mathscr{L}_D(\chi_S,x_0,r)$ on the Borel subsets of $\partial B_D(x_0,r)$, where $r$ is a regular value in $]0,r_0]$, is the harmonic measure relative to $x_0$. Therefore $w(x_0)\leqq\mathscr{L}_D(w,x_0,r)$ for all such $r$, and the function $\mathscr{L}_D(w,x_0,\cdot)$ is increasing and finite-valued.*

(ii) *If $D\subset E$, then $w$ has a harmonic majorant on $D$ if and only if $\mathscr{L}_D(w,x_0,\cdot)$ is bounded above.*

PROOF. Let $r$ be a regular value in $]0,r_0]$, and let $B=B_D(x_0,r)$. We have two representations of $G_B(x_0,\cdot)$, namely

$$G_B(x_0,\cdot)=G(x_0,\cdot)-H, \tag{3}$$

where $H$ is the PWB solution of the Dirichlet problem for $B$ with boundary function $G(x_0,\cdot)$, and

$$G_B(x_0,\cdot)=G_D(x_0,\cdot)-\tau(r), \tag{4}$$

which is given in [2, p. 118]. The representation (3) shows that $G_B(x_0,\cdot)-G(x_0,\cdot)$ can be extended to a smooth function on $B$, and the identities (4) and (1) then show that $G_B(x_0,\cdot)-G(x_0,\cdot)$ can be extended to a smooth function on $D$, which is equal on $\partial B$ to $-G(x_0,\cdot)$ because $B$ is Dirichlet regular. It follows that the function

$$S \mapsto -\kappa_n^{-1} \int_S D_\nu G_B(x_0,\cdot)d\sigma$$

on the Borel subsets of $\partial B$ is the harmonic measure relative to $x_0$ [5, p. 13]. However, in view of (4), on $\partial B$ we have

$$-D_\nu G_B(x_0,\cdot) = \langle \nabla G_B(x_0,\cdot), \nabla G_D(x_0,\cdot)\|\nabla G_D(x_0,\cdot)\|^{-1}\rangle = \|\nabla G_D(x_0,\cdot)\|,$$

so that the first part of (i) is proved. The remainder of the theorem now follows from [5, pp. 122–3].

We need a lemma on the transformation of integrals. For generality, we sometimes work with domains bounded by two surfaces of the form $\partial B_D(x_0, r)$, where $x_0$ is fixed and $r$ is regular. Such domains are direct generalizations of an annulus. If $x_0 \in D$, and $r_1, r_2$ are regular values with $r_1 < r_2$, we put

$$A_D(x_0, r_1, r_2) = B_D(x_0, r_2)\backslash \bar{B}_D(x_0, r_1).$$

It is sometimes convenient to write $A_D(x_0, 0, r_2)$ for $B_D(x_0, r_2)$.

LEMMA 1. *Let $F$ be a measurable function on $A_D(x_0, r_1, r_2)$, where $0 \leqq r_1 < r_2$. If*

$$\int_{A_D(x_0,r_1,r_2)} F\,dx$$

*exists as a Lebesgue integral, then it is equal to*

$$-\int_{r_1}^{r_2} \tau'(\rho)d\rho \int_{\partial B_D(x_0,\rho)} F\|\nabla G_D(x_0,\cdot)\|^{-1}d\sigma.$$

PROOF. Define $g$ on $\mathsf{R}^n$ by putting $g(x) = G_D(x_0, x)$ if $x \in A_D(x_0, r_1, r_2)$, $g(x) = \tau(r_1)$ if $x \in \bar{B}_D(x_0, r_1)$ (if $r_1 > 0$), and $g(x) = \tau(r_2)$ otherwise. Then $g$ is a Lipschitz function on $\mathsf{R}^n$. Define $F_0$ and $f$ on $\mathsf{R}^n$ by putting $F_0(x) = F(x)$ if $x \in A_D(x_0, r_1, r_2)$, $F_0(x) = 0$ otherwise, and $f = F_0\|\nabla g\|^{-1}$. Suppose first that $F \geqq 0$. Noting that Federer uses the term 'integrable' to mean 'has a well-defined integral', we can apply the coarea formula [6, p. 249] to $f$, and obtain

$$\int_{\mathsf{R}^n} f\|\nabla g\|\,dx = \int_{\mathsf{R}} dt \int_{g^{-1}(\{t\})} f\,d\sigma.$$

Hence

$$\int_{A_D(x_0,r_1,r_2)} F\,dx = \int_{\tau(r_2)}^{\tau(r_1)} dt \int_{g^{-1}(\{t\})} f\,d\sigma$$

$$= -\int_{r_1}^{r_2} \tau'(\rho)d\rho \int_{\partial B_D(x_0,\rho)} F\|\nabla G_D(x_0,\cdot)\|^{-1}d\sigma.$$

If now $F$ is arbitrary, the result follows when the above formula is applied to the positive and negative parts of $F$.

## 3. An essential lemma.

The next result forms the basis of our proof of the convexity theorem.

LEMMA 2. *Let $w$ be a smooth function on an open superset $E$ of $\bar{A}_D(x_0, r_1, r_2)$, where $0 < r_1 < r_2$. Then there is an absolutely continuous function $f$ on $[r_1, r_2]$ such that $f(r) = \mathscr{L}_D(w, x_0, r)$ for all regular values of $r \in [r_1, r_2]$, and*

$$\sigma_n r^{n-1} f'(r) = \int_{\partial B_D(x_0,r)} D_\nu w\,d\sigma \tag{5}$$

*whenever $f'(r)$ exists (hence* a.e.*). If, in addition, $\Delta w \geqq 0$ on $E$, then $f'$ exists everywhere on $[r_1, r_2]$, is absolutely continuous there, and satisfies*

$$\varepsilon_n (r^{n-1} f'(r))' = r^{1-n} \int_{\partial B_D(x_0,r)} \|\nabla G_D(x_0,\cdot)\|^{-1}\,\Delta w\,d\sigma \tag{6}$$

*for almost all $r \in [r_1, r_2]$, where $\varepsilon_2 = \sigma_2$, and $\varepsilon_n = \sigma_n/(n-2)$ if $n \geqq 3$.*

PROOF. If $v$ is a smooth function on $E$, and $A$ is a domain, with closure in $E$, for which the divergence theorem is applicable, the identity

$$w\,\Delta\,v = \sum_{i=1}^{n} D_i(wD_i v) - \langle \nabla w, \nabla v \rangle$$

implies that

$$\int_A (w\,\Delta\,v + \langle \nabla w, \nabla v \rangle)dx = \int_{\partial A} wD_\nu v\,d\sigma. \tag{7}$$

Interchanging $v$ and $w$, then taking $v = 1$, we obtain

$$\int_A \Delta w\, dx = \int_{\partial A} D_\nu w\, d\sigma. \tag{8}$$

By using the notation $A_D(x_0, r_1, r_2)$ in the statement of the lemma, we have implicitly assumed that $r_1$ and $r_2$ are regular values. Let $r$ be a regular value in $]r_1, r_2]$, and put $A_D = A_D(x_0, r_1, r)$. We take $A = A_D$ in (7), and apply Lemma 1 to obtain

$$\left(\int_{\partial B_D(r)} - \int_{\partial B_D(r_1)}\right) w D_\nu v\, d\sigma = -\int_{r_1}^{r} \tau'(\rho)d\rho \int_{\partial B_D(\rho)} (w\,\Delta\, v + \langle \nabla w, \nabla v\rangle)\|\nabla G_D\|^{-1} d\sigma, \tag{9}$$

where $B_D(r) = B_D(x_0, r)$ and $G_D = G_D(x_0, \cdot)$. In (9) we take $v = \tau^{-1}(G_D)$, so that on $A_D$ we have $\nabla v = (\tau^{-1})'(G_D)\nabla G_D$ and $\Delta v = (\tau^{-1})''(G_D)\|\nabla G_D\|^2$. Therefore, on $\partial B_D(\rho)$ where $G_D = \tau(\rho)$, we have $-\tau'(\rho)\nabla v = -\nabla G_D$ and $-\tau'(\rho)\,\Delta\, v = \tau''(\rho)\tau'(\rho)^{-2}\|\nabla G_D\|^2$. Therefore the right side of (9) divided by $\sigma_n$ is

$$\sigma_n^{-1}\int_{r_1}^{r} d\rho \int_{\partial B_D(\rho)} (w\tau''(\rho)\tau'(\rho)^{-2}\|\nabla G_D\| - \langle \nabla w, \nabla G_D\rangle \|\nabla G_D\|^{-1})d\sigma$$

$$= (n-1)\kappa_n^{-1}\int_{r_1}^{r} d\rho \int_{\partial B_D(\rho)} \rho^{n-2}\|\nabla G_D\| w\, d\sigma + \sigma_n^{-1}\int_{r_1}^{r} d\rho \int_{\partial B_D(\rho)} D_\nu w\, d\sigma$$

$$= (n-1)\int_{r_1}^{r} \rho^{n-2}\mathscr{L}_D(w, x_0, \rho)d\rho + \sigma_n^{-1}\int_{r_1}^{r}\int_{\partial B_D(\rho)} D_\nu w\, d\sigma$$

for all $n \geqq 2$. Furthermore, on $\partial B_D(\rho)$ we have $\sigma_n^{-1}(\tau^{-1})'(G_D) = \sigma_n^{-1}\tau'(\rho)^{-1} = -\kappa_n^{-1}\rho^{n-1}$, so that the left side of (9) divided by $\sigma_n$ is

$$\left(\int_{\partial B_D(r)} - \int_{\partial B_D(r_1)}\right) w\sigma_n^{-1}(\tau^{-1})'(G_D)D_\nu G_D\, d\sigma$$

$$= \kappa_n^{-1}\left(r^{n-1}\int_{\partial B_D(r)} - r_1^{n-1}\int_{\partial B_D(r_1)}\right)\|\nabla G_D\| w\, d\sigma$$

$$= r^{n-1}\mathscr{L}_D(w, x_0, r) - r_1^{n-1}\mathscr{L}_D(w, x_0, r_1)$$

for all $n \geqq 2$. Hence

$$r^{n-1}\mathscr{L}_D(w,x_0,r) - r_1^{n-1}\mathscr{L}_D(w,x_0,r_1) = (n-1)\int_{r_1}^{r} \rho^{n-2}\mathscr{L}_D(w,x_0,\rho)d\rho + \sigma_n^{-1}\int_{r_1}^{r} d\rho \int_{\partial B_D(\rho)} D_\nu w\, d\sigma. \tag{10}$$

The right side of (10) defines an absolutely continuous function of $r$ on $[r_1,r_2]$, so that (10) enables us to extend $\mathscr{L}_D(w,x_0,\cdot)$ to an absolutely continuous function $f$ on $[r_1,r_2]$ such that

$$r^{n-1}f(r) - r_1^{n-1}f(r_1) = (n-1)\int_{r_1}^{r} \rho^{n-2}f(\rho)d\rho + \sigma_n^{-1}\int_{r_1}^{r} d\rho \int_{\partial B_D(\rho)} D_\nu w\, d\sigma$$

for all $r \in [r_1,r_2]$. The function $f$ is differentiable a.e., with

$$(r^{n-1}f(r))' = (n-1)r^{n-2}f(r) + \sigma_n^{-1}\int_{\partial B_D(r)} D_\nu w\, d\sigma,$$

so that (5) holds.

We now suppose that $\Delta w \geqq 0$, apply (8) with $A = A_D$, and use Lemma 1 with $F = \Delta w$. Thus

$$\int_{\partial A_D} D_\nu w\, d\sigma = -\int_{r_1}^{r} \tau'(\rho)d\rho \int_{\partial B_D(\rho)} \|\nabla G_D\|^{-1}\Delta w\, d\sigma.$$

It now follows from (5) that, for almost all $r \in [r_1,r_2]$,

$$\sigma_n(r^{n-1}f'(r) - r_1^{n-1}f'(r_1)) = -\int_{r_1}^{r} \tau'(\rho)d\rho \int_{\partial B_D(\rho)} \|\nabla G_D\|^{-1}\Delta w\, d\sigma. \tag{11}$$

(Since we can vary $r_1$ and $r_2$ slightly without affecting the statement of the lemma, we can assume that (5) holds when $r \in \{r_1,r_2\}$, and then (11) holds with $r = r_2$.) The right side of (11) defines an absolutely continuous function of $r$ on $[r_1,r_2]$, so that (11) enables us to extend $f'$ to an absolutely continuous function $g$ on $[r_1,r_2]$. Then $f$ is the indefinite integral of $g$, so that $f'(r)$ exists and (11) holds for all $r \in [r_1,r_2]$. Since $f' = g$, it is differentiable a.e., and (6) follows from (11).

## 4. Convexity of surface means of subharmonic functions.

We are now able to complete the list of desirable properties of $\mathscr{L}_D$.

THEOREM 2. *Let $w$ be subharmonic on an open superset $E$ of $\bar{A}_D(x_0, r_1, r_2)$, where $0 < r_1 < r_2$. Then there are convex functions $\phi_p$, $1 \leqq p \leqq \infty$, such that for all regular values of $r \in [r_1, r_2]$,*

$$\mathscr{L}_D(w, x_0, r) = \phi_1(\tau(r)),$$

$$\mathscr{L}_D((w^+)^p, x_0, r)^{1/p} = \phi_p(\tau(r)) \text{ if } 1 < p < \infty,$$

$$\log \mathscr{L}_D(e^w, x_0, r) = \phi_\infty(\tau(r)).$$

PROOF. Suppose first that $w$ is smooth and has a positive lower bound. Put $\Psi(t) = t^p$ if $1 \leqq p < \infty$, and $\Psi(t) = e^t$ if $p = \infty$. Then $\Psi$ satisfies the conditions

$$\Psi(t) > 0, \quad \Psi'(t) > 0, \quad \delta\Psi'(t)^2 = \Psi(t)\Psi''(t) \tag{12}$$

for all $t > 0$, where $\delta = 1 - p^{-1}$ if $1 \leqq p < \infty$, and $\delta = 1$ if $p = \infty$. By Lemma 2, there is an absolutely continuous function $f$ on $[r_1, r_2]$ such that $f(r) = \mathscr{L}_D(\Psi(w), x_0, r)$ for all regular values of $r$, $f'$ exists and is absolutely continuous on $[r_1, r_2]$,

$$\sigma_n r^{n-1} f'(r) = \int_{\partial B_D(r)} \Psi'(w) D_\nu w \, d\sigma \tag{13}$$

for all $r$, and

$$\varepsilon_n (r^{n-1} f'(r))' = r^{1-n} \int_{\partial B_D(r)} \|\nabla G_D(x_0, \cdot)\|^{-1} \Delta(\Psi(w)) \, d\sigma \tag{14}$$

for almost all $r$. Since $\Delta(\Psi(w)) = \Psi''(w)\|\nabla w\|^2 + \Psi'(w)\Delta w$, it follows from (14) that

$$\varepsilon_n (r^{n-1} f'(r))' \geqq r^{1-n} \int_{\partial B_D(r)} \Psi''(w)\|\nabla w\|^2 \|\nabla G_D(x_0, \cdot)\|^{-1} \, d\sigma. \tag{15}$$

Putting $d\lambda = \kappa_n^{-1} \|\nabla G_D(x_0, \cdot)\| d\sigma$, so that $f(r) = \int_{\partial B_D(r)} \Psi(w) d\lambda$ for all regular values of $r$, we deduce from (13) that

$$f'(r) = \sigma_n^{-1} r^{1-n} \kappa_n \int_{\partial B_D(r)} \Psi(w)^{1/2} (\Psi'(w)\Psi(w)^{-1/2} D_\nu w \|\nabla G_D(x_0, \cdot)\|^{-1}) d\lambda$$

$$\leqq \kappa_n \sigma_n^{-1} r^{1-n} f(r)^{1/2} \left( \int_{\partial B_D(r)} \Psi'(w)^2 \Psi(w)^{-1} (D_\nu w)^2 \|\nabla G_D(x_0, \cdot)\|^{-2} d\lambda \right)^{1/2},$$

so that

$$f'(r)^2 \leqq \kappa_n \sigma_n^{-2} r^{2-2n} f(r) \int_{\partial B_D(r)} \Psi'(w)^2 \Psi(w)^{-1} \|\nabla w\|^2 \|\nabla G_D(x_0, \cdot)\|^{-1} d\sigma.$$

It now follows from (12) and (15) that

$$f'(r)^2 \leqq \kappa_n \sigma_n^{-2} r^{2-2n} f(r) \int_{\partial B_D(r)} \delta^{-1} \Psi''(w) \|\nabla w\|^2 \|\nabla G_D(x_0, \cdot)\|^{-1} d\sigma$$

$$\leqq \delta^{-1} r^{1-n} f(r) (r^{n-1} f'(r))',$$

so that

$$rf(r)f''(r) + (n-1)f(r)f'(r) - \delta r f'(r)^2 \geqq 0. \tag{16}$$

Let $\Phi = \Psi^{-1}$, so that

$$\Phi(t) > 0, \quad \Phi'(t) > 0, \quad \Phi''(t) = -\delta t^{-1} \Phi'(t), \tag{17}$$

in view of (12). The function $g$ on $[r_1, r_2]$, given by $g(r) = \Phi(f(r))\tau(r)^{-1}$, is differentiable, with

$$g'(r) = \Phi'(f(r))f'(r)\tau(r)^{-1} - \Phi(f(r))\tau(r)^{-2}\tau'(r). \tag{18}$$

Since $w$ has a positive lower bound, so does $f$. Therefore the right side of (18) is an absolutely continuous function of $r$, so that $g''$ exists a.e. and

$$\begin{aligned} g''(r) = {} & \Phi''(f(r))f'(r)^2\tau(r)^{-1} + \Phi'(f(r))f''(r)\tau(r)^{-1} - 2\Phi'(f(r))f'(r)\tau(r)^{-2}\tau'(r) \\ & + 2\Phi(f(r))\tau(r)^{-3}\tau'(r)^2 - \Phi(f(r))\tau(r)^{-2}\tau''(r). \end{aligned} \tag{19}$$

We now put $h(s) = g(r)$, where $s = \tau(r)^{-1}$, so that $h'(s) = -g'(r)(\tau^{-1})'(s^{-1})s^{-2}$ for all $s$, and

$$\begin{aligned} h''(s) &= g''(r)(\tau^{-1})'(s^{-1})^2 s^{-4} + g'(r)(\tau^{-1})''(s^{-1})s^{-4} + 2g'(r)(\tau^{-1})'(s^{-1})s^{-3} \\ &= \tau'(r)^{-3} s^{-4} (g''(r)\tau'(r) - g'(r)\tau''(r) + 2g'(r)\tau'(r)^2\tau(r)^{-1}) \end{aligned}$$

for almost all $s$. It now follows from (17), (18), (19) and the identity $\tau''(r) = (1-n)\tau'(r)r^{-1}$ that

$$\begin{aligned} h''(s) &= \tau'(r)^{-3}s^{-4}(\Phi''(f(r))f'(r)^2\tau(r)^{-1}\tau'(r) + \Phi'(f(r))f''(r)\tau(r)^{-1}\tau'(r) \\ &\qquad - \Phi'(f(r))f'(r)\tau(r)^{-1}\tau''(r)) \\ &= \Phi'(f(r))\tau'(r)^{-3}s^{-3}f(r)^{-1}(-\delta f'(r)^2\tau'(r) + f(r)f''(r)\tau'(r) - f(r)f'(r)\tau''(r)) \\ &= \Phi'(f(r))\tau'(r)^{-2}s^{-3}f(r)^{-1}r^{-1}(-\delta rf'(r)^2 + rf(r)f''(r) + (n-1)f(r)f'(r)). \end{aligned}$$

Therefore (16) implies that $h'' \geqq 0$ for almost all $s$. Since $g'$ is absolutely continuous, the same is true of $h'$. Therefore $h'$ is increasing, and so $h$ is convex. Thus $\Phi(f(r))\tau(r)^{-1}$ is a convex function of $\tau(r)^{-1}$, which means that $\Phi(f(r))$ is a convex function of $\tau(r)$.

If $w$ is now an arbitrary non-negative subharmonic function, we take a decreasing sequence $\{w_j\}$ of smooth subharmonic functions which converges to $w$ on a neighbourhood of $\bar{A}_D(x_0, r_1, r_2)$. Then $\{w_j + j^{-1}\}$ has similar properties, and each $v_j = w_j + j^{-1}$ has a positive lower bound, so that each $\Phi(\mathscr{L}(\Psi(v_j), x_0, r))$ is equal at the regular values of $r$ to a convex function $\psi_j$ of $\tau(r)$. Hence, at such points,

$$\Phi(\mathscr{L}(\Psi(w), x_0, r)) = \lim_{j\to\infty} \Phi(\mathscr{L}(\Psi(v_j), x_0, r)) = \lim_{j\to\infty} \psi_j(\tau(r)),$$

and $\lim \psi_j$ is convex.

If $p \in \{1, \infty\}$, the result is easily extended to arbitrary lower bounded subharmonic functions, and then to arbitrary subharmonic functions by another approximation argument.

COROLLARY. *Let $w$ be subharmonic on an open superset $E$ of $\bar{A}_D(x_0, r_1, r_2)$, where $0 < r_1 < r_2$. If $v$ is defined on $\partial B_D(x_0, r)$ for all regular values of $r \in ]r_1, r_2[$ by*

$$v(x) = \mathscr{L}_D(w, x_0, \tau^{-1}(G_D(x_0, x))),$$

*then $v$ can be extended to a subharmonic function on $A_D(x_0, r_1, r_2)$.*

PROOF. By Theorem 2, there is a convex function $\phi$ on $[\tau(r_2), \tau(r_1)]$ such that $v(x) = \phi(G_D(x_0, x))$ wherever $v(x)$ is defined. Since $G_D(x_0, \cdot)$ is harmonic on $A_D(x_0, r_1, r_2)$, the function $\phi \circ G_D(x_0, \cdot)$ is subharmonic there.

## 5. Volume means of subharmonic functions.

Let $\lambda_2 = v_2$, and $\lambda_n = (n-2)^2 v_n$ if $n \geqq 3$. For any $r$ (such that $\tau(r) > 0$), and any $x_0 \in D$, we put

$$\mathscr{A}_D(u, x_0, r) = \lambda_n^{-1} r^{-n} \int\limits_{B_D(x_0,r)} \tau^{-1}(G_D(x_0, \cdot))^{2n-2} \|\nabla G_D(x_0, \cdot)\|^2 u \, dx$$

whenever the integral exists, in which case it follows from Lemma 1 that

$$\mathscr{A}_D(u, x_0, r) = nr^{-n} \int_0^r \rho^{n-1} \mathscr{L}_D(u, x_0, \rho) d\rho. \tag{20}$$

Since Theorem 1 implies that $\mathscr{L}_D(1, x_0, r) = 1$ for all $x_0$ and regular values of $r$, it follows that $\mathscr{A}_D(1, x_0, r) = 1$ for all $x_0$ and $r$. Since subharmonic functions are locally bounded above, it follows that $\mathscr{A}_D(w, x_0, r)$ is defined whenever $w$ is subharmonic on an open superset of $\bar{B}_D(x_0, r)$.

THEOREM 3. *Let $w$ be subharmonic on an open superset of $\bar{B}_D(x_0, r)$. Then:*
(i) $w(x_0) \leqq \mathscr{A}_D(w, x_0, \cdot)$ *on* $]0, r_0]$;
(ii) $\mathscr{A}_D(w, x_0, \cdot)$ *is finite-valued and increasing on* $]0, r_0]$;
(iii) *there is a convex function $\psi$ such that for all $r \in ]0, r_0]$*

$$\mathscr{A}_D(w, x_0, r) = \psi(\tau(r));$$

(iv) *for all regular values of $r \in ]0, r_0]$,*

$$\mathscr{A}_D(w, x_0, r) \leqq \mathscr{L}_D(w, x_0, r);$$

(v) *if $\kappa = e^{-1/2}$ when $n = 2$, and $\kappa = (2/n)^{1/(n-2)}$ when $n \geqq 3$, then for all regular values of $\kappa r \in ]0, \kappa r_0]$ we have*

$$\mathscr{L}_D(w, x_0, \kappa r) \leqq \mathscr{A}_D(w, x_0, r),$$

*and the constant $\kappa$ is the best possible.*

PROOF. Properties (i) and (iv) follow from (20) and the inequalities

$$w(x_0) \leqq \mathscr{L}_D(w, x_0, \rho) \leqq \mathscr{L}_D(w, x_0, r)$$

of Theorem 1. The proofs of (ii) and (iii) follow those in [11, p. 9]. Thus, by Theorem 2 there is a convex function $\phi$ such that $\mathscr{L}_D(w, x_0, \rho) = \phi(\tau(\rho))$ for all regular values of $\rho \in ]0, r_0]$. The convexity of $\phi$ ensures that $t^{-1}\phi(t) \to \alpha$ as $t \to \infty$, for some $\alpha > -\infty$. Since $\mathscr{L}_D(w, x_0, \cdot)$ is increasing by Theorem 1, $\phi$ is decreasing and hence $\alpha \leqq 0$. In particular $\alpha \in \mathsf{R}$, so that

$$\rho^{n-1} \mathscr{L}_D(w, x_0, \rho) = (\rho^{n-1}\tau(\rho))\tau(\rho)^{-1}\phi(\tau(\rho)) \to 0$$

as $\rho \to 0$. Thus the integrand in (20) is equal a.e. to a real continuous function on $[0, r_0]$. Therefore $\mathscr{A}_D(w, x_0, \cdot)$ is real-valued, and we can approximate the integral in (20) by Riemann sums. Thus

$$\mathscr{A}_D(w,x_0,r) = \lim_{k\to\infty}\left(n\sum_{j=1}^{k} j^{n-1}k^{-n}\phi(\tau(jr/k))\right),$$

and properties (ii) and (iii) follow easily.

The inequality in (v) is proved by an application of Jensen's inequality, following Beardon's proof [1] for the case $D = \mathsf{R}^n$. Thus, if $\kappa r$ is a regular value in $]0,\kappa r_0]$, then by Theorem 2 there is a convex function $\phi_1$ such that

$$\begin{aligned}\mathscr{L}_D(w,x_0,\kappa r) &= \phi_1(\tau(\kappa r))\\ &= \phi_1\left(nr^{-n}\int_0^r \rho^{n-1}\tau(\rho)d\rho\right)\\ &\leqq nr^{-n}\int_0^r \rho^{n-1}\phi_1(\tau(\rho))d\rho\\ &= \mathscr{A}_D(w,x_0,r)\end{aligned}$$

in view of (20). Finally, if $w = -G_D(x_0,\cdot)$ then $\mathscr{L}_D(w,x_0,\rho) = -\tau(\rho)$, so that, by (20),

$$\mathscr{A}_D(w,x_0,r) = -nr^{-n}\int_0^r \rho^{n-1}\tau(\rho)d\rho = -\tau(\kappa r).$$

Therefore if $\lambda > \kappa$ we have $\mathscr{A}_D(w,x_0,r) < \mathscr{L}_D(w,x_0,\lambda r)$.

## 6. The generalized three spheres theorem.

In Theorem 2, we could have made $p\to\infty$ and concluded that the maximum of a non-negative subharmonic function over $\partial B_D(x_0,r)$ is equal a.e. to a convex function of $\tau(r)$, and subsequently used an approximation argument to remove the non-negativity hypothesis. However, we can obtain a better result under less stringent conditions on the domain, and now proceed to do so.

THEOREM 4. *Let $E$ be an arbitrary Greenian subdomain of $\mathsf{R}^n$. Let $w$ be subharmonic and bounded above on $A_0 = \{x\in E:\tau(r_2) < G_E(x_0,x) < \tau(r_1)\}$ (where $\tau(r_2) > 0$), and define $w$ on $\partial A_0$ to make it upper semicontinuous on $\bar{A}_0$. If, for each $r\in[r_1,r_2]$, we put*

$$\mathscr{S}_E(r) = \mathscr{S}_E(w,x_0,r) = \sup\{w(x): G_E(x_0,x) = \tau(r)\},$$

*then there is a finite-valued, convex function $\phi$ such that $\mathscr{S}_E(r) = \phi(\tau(r))$ for all $r\in[r_1,r_2]$.*

PROOF. Let $r_1 \leqq s_1 < s_2 \leqq r_2$, and put $A = \{x \in E : \tau(s_2) < G_E(x_0, x) < \tau(s_1))\}$. If $\mathscr{S}_E(s_1) = -\infty$, then $\{x \in E : G_E(x_0, x) = \tau(s_1)\}$ is a harmonic measure null set relative to $A$. Since $\partial A \cap \partial E$ is also a harmonic measure null set (by [2, p. 119] or [3, p. 228]), the whole of $\partial A \backslash \{x \in E : G_E(x_0, x) = \tau(s_2)\}$ is negligible, contrary to the fact that $G_E(x_0, \cdot)$ is constant on the remainder of $\partial A$ but not on $A$. A similar argument works if $\mathscr{S}_E(s_2) = -\infty$, so that $\mathscr{S}_E$ is real-valued.

The rest of the proof is a slight modification of the standard proof for spheres in [10, p. 131]. The function $u$, defined for all $x \in E$ by

$$u(x) = \frac{\tau(s_1)\mathscr{S}_E(s_2) - \tau(s_2)\mathscr{S}_E(s_1) + (\mathscr{S}_E(s_1) - \mathscr{S}_E(s_2))G_E(x_0, x)}{\tau(s_1) - \tau(s_2)},$$

is harmonic on the open superset $E \backslash \{x_0\}$ of $\bar{A}$. Therefore $w - u$ is subharmonic on $A$, and is upper semicontinuous and bounded above on $\bar{A}$. Furthermore, whenever $i \in \{1, 2\}$ and $G_E(x_0, y) = \tau(s_i)$, we have $w(y) - u(y) = w(y) - \mathscr{S}_E(s_i) \leqq 0$. Since $\partial A \cap \partial E$ is a harmonic measure null set, it follows that $w \leqq u$ on $\bar{A}$, so that

$$\mathscr{S}_E(s) \leqq \frac{(\tau(s) - \tau(s_2))\mathscr{S}_E(s_1) + (\tau(s_1) - \tau(s))\mathscr{S}_E(s_2)}{\tau(s_1) - \tau(s_2)}$$

whenever $s \in [s_1, s_2]$.

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