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AN APPLICATION OF JÓNSSON MODULES TO SOME QUESTIONS CONCERNING PROPER SUBRINGS

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Abstract.

Given a commutative ring S and the proper subring R of S, it is shown that if each proper subring of S containing R is Artinian, then S is Artinian. Examples are given to show that the corresponding statement for Noetherian rings fails. The structure of pairs (R, S), where S is not Noetherian but each proper subring of S containing R is Noetherian, is determined.

All rings considered in this paper are assumed to be commutative and to contain a unity element. If R is a subring of S, we assume that the unity of S is contained in R, and hence is the unity of R; we use the term *S-overring* of R to mean a subring of S containing R, a *proper S-overring* of R is an S-overring of R distinct from S. We use $\mathbf{Z}$ to denote the ring of integers, and $\mathbf{Q}$ to denote the field of rational numbers.

Several papers (for example, [G_1], [W_1], [W_2], [GH_1], [GH_4], [GH_5]) have dealt with what are called Noetherian and zero-dimensional pairs of rings, hereditarily Noetherian rings, and hereditarily zero-dimensional rings. In a general context, the definitions are as follows. If E is a ring-theoretic property and if R is a subring of S, then (R, S) is said to be an *E-pair* if each S-overring of R has property E, and S has property E *hereditarily* if each subring of S has property E – that is, if (π, S) is an E-pair, where π is the prime subring of S. In considering a pair (R, S) of rings and a class $\mathscr{C}$ of intermediate rings, one question that naturally arises asks whether (R, S) is an E-pair if each element of $\mathscr{C}$ has property E. In this paper we consider the particular case of this problem where $\mathscr{C}$ is the class of all proper S-overrings of R and E is one of the three properties (1) Noetherian, (2) Artinian or (3) being of dimension at most n, where n is a fixed nonnegative integer. In the case of properties (3) and (2), we show in Theorems 1 and 2 that (R, S) is an E-pair, but even in the absolute case $R = \pi$, S need not be Noetherian if each proper subring of S is Noetherian. Theorem 4 and Corollary

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4 are the definitive results we obtain in the Noetherian case. In our work on the Artinian and Noetherian properties, we use the theory of Jónsson ω_0-generated modules developed in [GH$_2$]. The definition is as follows. If α is an infinite cardinal (ω_0 denotes the first infinite cardinal) and if M is a unitary module over the ring R, then M is a *Jónsson α-generated R-module* if M has a generating set of cardinality α, no generating set of cardinality less than α, and each proper submodule of M has a generating set of cardinality less than α. Thus a Jónsson ω_0-generated module is a module which is not finitely generated, while each of its proper submodules is finitely generated.

We begin with property (3), concerning dimension, mentioned above.

THEOREM 1. *Suppose R is a proper subring of a ring S, n is a nonnegative integer, and that each proper S-overring of R that is finitely generated as an R-algebra has dimension at most n. Then S has dimension at most n.*

PROOF. We use induction on n, considering first the case where $n = 0$. Assume that $\dim S > 0$, choose proper primes P and Q of S such that $P < Q$, and choose $t \in Q - P$. Then $R[t]$ and $R[t^2]$ have positive dimension since the contractions of P and Q to these rings are distinct. Hence $S = R[t] = R[t^2]$. The expression of t as an element of $R[t^2]$ shows that t satisfies a polynomial $f(X) \in R[X]$ whose coefficient of X is -1. Because $f(X)$ is a regular element of $R[X]$ [G$_2$, Prop. 28.7], it follows that the kernel I of the canonical R-homomorphism $R[X] \to R[t]$ is contained in no minimal prime of $R[X]$. Therefore $\dim S = \dim R[t] < \dim R[X] = 1$ [G$_2$, Cor. 30.3]. This contradiction yields the desired conclusion in the case where $n = 0$.

At the inductive step, we again assume that $\dim S > n$, where n is positive, and we seek a contradiction. Thus, take a chain $P_0 < P_1 < \ldots < P_{n+1}$ of proper prime ideals of S. Replacing the rings between R and S by their factors modulo the contraction of P_0, we assume without loss of generality that S is an integral domain and $P_0 = (0)$. Again choosing $t_i \in P_{i+1} - P_i$ for $0 \leqq i \leqq n$, we conclude as in the case where $n = 0$ that $S = R[t_0, t_1, \ldots, t_n]$. In particular, S is affine over R, and by replacing R by an appropriate affine S-overring of R, we may assume without loss of generality that $S = R[t]$ is a simple extension of R. Now t is not transcendential over R, for otherwise $R[t^2]$ would be a proper S-overring of R of dimension greater than n. Hence t is algebraic over R, so P_1 meets each proper S-overring of R nontrivially. Thus, $R^* = R/(P_1 \cap R)$ is a proper subring of $S/P_1 = S^*$ such that each proper affine S^*-overring of R^* has dimension at most $n - 1$, while $\dim S^* \geqq n$. This contradiction of the induction hypothesis for $n - 1$ yields the inductive step, thereby completing the proof of Theorem 1.

Taking R to be the prime subring of S in Theorem 1 yields the following absolute case of that result.

COROLLARY 1. *Suppose n is a nonnegative integer and that the ring S admits a proper subring. If each finitely generated proper subring of S has dimension at most n, then S has dimension at most n.*

We use Theorem 1 in proving Theorem 2, an analogue of Theorem 1 for the Artinian property.

THEOREM 2. *Suppose R is a proper subring of a ring S and that each proper S-overring of R is Artinian. Then S is Artinian.*

PROOF. Theorem 1 shows that S is zero-dimensional. Assume that S is not Artinian. Then S is not Noetherian [ZS, Th. 2, p. 203]. Let I be an ideal of S that is not finitely generated. Replacing S and R by $S/(I \cap R)$ and $R/(I \cap R)$, we may assume without loss of generality that $I \cap R = (0)$; in this connection we note that $S/(I \cap R)S$ is not Noetherian, for since $I \cap R$ is a finitely generated ideal of R, $I/(I \cap R)S$ is not finitely generated as an ideal of $S/(I \cap R)S$. Now I is not finitely generated as an ideal of $R + I$, so $R + I = S$ and the sum $R + I$ is direct as a sum of Abelian groups. If J is an ideal of S properly contained in I, then $R + J < R + I$, so $R + J$ is Noetherian and J is finitely generated both as an ideal of $R + J$ and of S. Therefore I is a Jónsson ω_0-generated module over S, and this contradicts Remark 1.3 of [GH_2], which shows that a zero-dimensional ring does not admit a Jónsson α-module for any infinite cardinal α. Therefore S is Artinian, as asserted.

COROLLARY 2. *If the ring S admits a proper subring, and if each proper subring of S is Artinian, then S is Artinian.*

In Corollary 2, it is not sufficient to assume that each finitely generated proper subring is Artinian. For example, if S is hereditarily zero-dimensional, then each finitely generated subring of S is Artinian, but S need not be Artinian. A specific example of this kind is $F[\{X_i\}_{i=1}^{\infty}]/(\{X_iX_j\})$, where F is a finite field and $\{X_i\}_{i=1}^{\infty}$ is a set of indeterminates over F.

In contrast with the situation indicated by Corollary 2, a ring S need not be Noetherian if each proper subring of S is Noetherian. For example, if $S = \mathbf{Z}(+)G$ is the idealization (see [H, p. 161]) of $\mathbf{Z}$ and a p-quasicyclic group G, then S is not Noetherian (G is the unique ideal of S that is not finitely generated; each other ideal of S is principal), but any proper subring of S is of the form $\mathbf{Z}(+)H$ for some proper, hence finite, subgroup H of G. Therefore each proper subring of S is Noetherian. More generally, if R is a Noetherian ring of positive dimension, then R admits a Jónsson ω_0-generated module M [GH_2, Theorem 2.7], and if $S = R(+)M$, then S is not Noetherian, but each proper S-overring of R is Noetherian (cf. [GH_3, (E3), p. 134]). In general one cannot expect S to be of the form $R(+)N$ for some Jónsson ω_0-generated R-module N; for example, if

$S = \mathbf{Z}(+)G$ as above and if $R = \mathbf{Z}(+)H$, where H is a nonzero proper subgroup of G, then R is not a direct sumand of the additive group $(S, +)$ since H is not a pure subgroup of $(S, +)$ [F, Section 26]. In the preceding example it is true that $S = R + G$, where G is a Jónsson ω_0-generated ideal of S, and in fact, it is easy to show in general that if R_0 is a Noetherian subring of a ring S_0 and if $S_0 = R_0 + I$ for some Jónsson ω_0-generated ideal I of S_0, then S_0 is not Noetherian but each proper S_0-overring of R_0 is Noetherian. We note that I is also a Jónsson ω_0-generated module over R_0 in this latter case, and if I is faithful as an R_0-module, then $R_0 \cap I = (0)$ [GH_2, Corollary 1.2], and hence $S_0 = R_0(+)I$. Theorem 4 indicates more precisely what the situation is in the general case. The proof of Theorem 4 uses Theorem 3, and for reference in the proof of Theorems 3 and 4, we record here a result from [GH_1].

PROPOSITION 1. ([GH_1, Prop. 1.3]) *Assume that R is a subring of S and N is a nilpotent ideal of S such that $S = R + N$. Then N is finitely generated as an ideal of S if and only if N is a finitely generated R-module.*

THEOREM 3. *Suppose R is a proper subring of a ring S, each proper S-overring of R is Noetherian, and S is not Noetherian. Let I be an ideal of S that is not finitely generated.*

(1) *$S = R + I$ and $S/I \simeq R/(I \cap R)$. If $I \cap R = (0)$, then I is a Jónsson ω_0-generated module over both S and R and $S = R(+)I$.*

(2) *$S/(I \cap R)S = [R/(I \cap R)](+)[I/(I \cap R)S]$, where $I/(I \cap R)S$ is a Jónsson ω_0-generated module over $S/(I \cap R)S$ and over $R/(I \cap R)$.*

(3) *$I^2 \subseteq (I \cap R)S$ and S has Noetherian spectrum.*

(4) *If J is an ideal of S contained in $(I \cap R)S$, then $(R + J) < S$, so $R + J$ is Noetherian and J is finitely generated both as an ideal of $R + J$ and as an ideal of S.*

(5) *S is not an integral domain.*

(6) *The nilradical N of S is not finitely generated, and hence $S = R + N$. Moreover, $(N \cap R)S$ is a Noetherian R-module.*

(7) *If R is reduced, then $S = R(+)N$, where N is a Jónsson ω_0-generated module over both S and R.*

PROOF. (1): The equality $S = R + I$ follows because $R + I$ is not Noetherian, and isomorphism of S/I and $R/(I \cap R)$ is a consequence of this equality. If $I \cap R = (0)$ and if J is an ideal of S properly contained in I, then $R + J < R + I = S$. Therefore $R + J$ is Noetherian and J is finitely generated both as an ideal of $R + J$ and as an ideal of S. It follows that I is a Jónsson ω_0-generated ideal of S, so $I^2 = (0)$ [GH_2, Corollary 1.2] and $S = R(+)I$, as asserted.

(2): The rings $R/(I \cap R)$ and $S/(I \cap R)S$ satisfy the conditions on R and S in the statement of Theorem 3, and $I/(I \cap R)S$ is a non-finitely generated ideal of $S/(I \cap R)S$ that meets $R/(I \cap R)$ in (0). Thus, (2) follows from (1).

(3): It follows from (2) that $I^2 \subseteq (I \cap R)S$, so rad($I$) is the radical of the finitely generated ideal $(I \cap R)S$. Because I is a arbitrary non-finitely generated ideal of S, each radical ideal of S is the radical of a finitely generated ideal, so S has Noetherian spectrum [OP, Prop. 2.1].

(4): Suppose $S = R + J$, where J is an ideal of S contained in $(I \cap R)S$. Then $I = I \cap S = I \cap (R + J) = (I \cap R) + J$. Thus I/J is finitely generated, $I/(I \cap R)S$ is finitely generated, and hence I is finitely generated. This contradiction shows that $(R + J) < S$ for each J.

(5): Suppose, to the contrary, that S is an integral domain. Then (1) implies that xI is finitely generated as an ideal of S. Because x is not a zero divisor in S, it follows that I is also finitely generated, a contradiction.

(6): If P is a proper prime ideal of S, then each proper (S/P)-overring of $R/P \cap R)$ is Noetherian, so (5) implies that S/P is Noetherian. Since S has Noetherian spectrum, there are only finitely many minimal primes $P_1, P_2, \ldots, P_n$ of S[K, Prop. 4.9, p. 25], and since each S/P_i is Noetherian, $S/(\cap_{i=1}^n P_i) = S/N$ is also Noetherian [N, (3.16)]. Because S is not Noetherian, there exists a prime ideal Q of S that is not finitely generated. Since Q/N is finitely generated, it follows that N is not finitely generated, and (1) shows that $S = R + N$. Now (4) implies that $R + (N \cap R)S$ is Noetherian, and since $(N \cap R)S$ is nilpotent, Proposition 1 implies that $(N \cap R)S$ is a finitely generated, hence Noetherian, R-module.

(7) follows immediately from (1) and (6).

In connection with part (5) of Theorem 3, we remark that a proof similar to that of Theorem 2.1 of [GH_3] shows that under the notation and hypothesis of Theorem 3, $S = R + K$ for each regular ideal K of S, and $K = (K \cap R)S$ if $(K \cap R)S$ is a regular ideal of S. An alternate proof of (5) can be obtained from this assertion.

THEOREM 4. *Suppose R is a Noetherian subring of a non-Noetherian ring T, and let N and M denote the nilradicals of T and R, respectively. The following conditions are equivalent.*

(1) *Each proper T-overring of R is Noetherian.*

(2) *Each proper T-overring of R is a finitely generated R-module.*

(3) *N/MT is a Jónsson ω_0-generated module over R/M.*

PROOF. (1) $\Leftrightarrow$ (2): It is clear that (2) implies (1). If (1) holds, then part (6) of Theorem 3 shows that $T = R + N$. Hence if S is a proper T-overring of R, then $S = R + (S \cap N)$, where $S \cap N$ is nilpotent. Proposition 1 then implies that $S \cap N$ is a finitely generated R-module, so $S = R + (S \cap N)$ is also a finitely generated module over R.

(1) $\Leftrightarrow$ (3): Parts (6) and (2) of Theorem 3 show that (1) implies (3). We show that (2) follows from (3). Thus, assume that (3) is satisfied and let S be a proper T-overring of R. As in the preceeding paragraph, we have $S = R + (S \cap N)$, where $M \subseteq (S \cap N) < N$. Suppose $N = (S \cap N) + MT = (S \cap N) + MR + MN = (S \cap N) + MN$. Then $N = (S \cap N) + M[(S \cap N) + MN] = (S \cap N) + M^2N$, and by an easy induction argument, $N = (S \cap N) + M^kN$ for each positive integer k. Since $M^k = (0)$ for some k, it follows that $N = S \cap N$, a contradiction. Therefore $(S \cap N) + MT < N$, so by hypothesis, $[(S \cap N) + MT]/MT \simeq (S \cap N)/(S \cap MT)$ is a finitely generated R-module. Part (6) of Theorem 3 shows that $S \cap MT$ is also a finitely generated R-module. Therefore $S \cap N$ and $S = R + (S \cap N)$ are finitely generated R-modules, (2) is satisfied, and this completes the proof of Theorem 4.

As previously stated, a zero-dimensional ring does not admit a Jónsson ω_0-generated module [GH$_2$, Remark 1.3]. This fact and Theorem 4 (or part (2) of Theorem 3) yield the next result.

COROLLARY 3. *Suppose R is a proper subring of a ring S and that each proper S-overring of R is Noetherian. Then S is Noetherian if either R or S is zero-dimensional.*

Part (7) of Theorem 3 and Corollary 3 settle the absolute case of the problem under consideration, as follows:

COROLLARY 4. *Suppose each proper subring of the ring S is Noetherian. Then either S is Noetherian or else $S = \mathbf{Z}(+)G$, where G is a p-quasicyclic group.*

PROOF. Suppose S is not Noetherian. Then Corollary 3 implies that $\mathbf{Z}$ is the prime subring of S, and part (7) of Theorem 3 shows that $S = \mathbf{Z}(+)M$, where M is a Jónsson ω_0-generated $\mathbf{Z}$-module – that is, a non-finitely generated abelian group, each of whose proper subgroups is finitely generated. Since the additive group of $\mathbf{Q}$ admits proper subgroups that are not finitely generated, the group M is not torsion-free [GH$_2$, Theorem 1.4]. Hence M is a torsion group, and in this case it is known that, in fact, M must be p-quasicyclic [F, Exercise 4, p. 105], [GH$_2$, page 46].

If the notation and hypothesis are as in the statement of Theorem 4, we remark that N itself need not be a Jónsson ω_0-generated module over T. For an example to establish this assertion, let G be the direct sum of an infinite cyclic group C and a quasicyclic group H. Let $T = \mathbf{Z}(+)G$ and let $R = \mathbf{Z} + C$. The conditions of Theorem 4 are satisfied, but the nilradical of T is G, and G is not a Jónsson ω_0-generated module over R.

We remark that in the paper [GO], Gilmer and O'Malley impose no conditions concerning existence of a unity element in their use of the terms ring and subring. In this more general sense, they show that the p-quasicyclic groups, under trivial multiplication, are the only non-Noetherian rings for which each proper subring is Noetherian; they also show that this conclusion remains valid even for rings that need not be commutative, provided the Noetherian condition is replaced by the ascending chain condition for left (or right) ideals. If one restricts to commutative rings with unity, it is easily seen that if S is a ring with unity element e and if each proper subring of S containing e is Noetherian, then any proper subring R of S containing any unity element is Noetherian; this follows since R is a direct summand, and hence a homomorphic image, of the subring $R[e]$ of S.

REMARK. If R is a subring of a ring T, then T is said to be a J-*algebra* over R (for *Jónsson ω_0-generated algebra*) if T is not finitely generated as an algebra over R, but each proper T-overring of R is finitely generated as an algebra over R. Theorem 4 shows that if R is Noetherian, T is non-Noetherian, and if each proper T-overring of R is Noetherian, then T is a J-algebra over R. The theory of J-algebras is developed in [GH_3]. If R and T satisfy the equivalent conditions of Theorem 4, then N/MT is also a Jónsson ω_0-generated module over T/MT by part (7) of Theorem 3, and therefore also over T. (Since the structure of N/MT as a module over T/MT is essentially the same as its structure as a module over T [G_2, p. 8].) By [GH_2, Prop. 1.1] the annihilator in T of N/MT is a prime ideal Q of T. The prime ideal Q is nonmaximal, but need not be a minimal prime of T. For example, let R be the polynomial ring $Z[X]$, let G be a p-quasicyclic group, and define G as a $Z[X]$-module by defining $Xg = 0$ for each $g \in G$. Then G is a Jónsson ω_0-generated R-module. Hence if $T = R(+)G$, then R and T satisfy the equivalent conditions of Theorem 4, but the prime ideal $Q = XR + G$ is non-minimal and is the annihilator of the Jónsson ω_0-generated T-module G.

If R is a subring of a ring T and if R and T satisfy the equivalent conditions of Theorem 4, then the following result describes finite generation of certain ideals of T.

PROPOSITION 2. *Suppose R is a Noetherian subring of a non-Noetherian ring T and that T is a* J-*algebra over R. Let M and N denote the nilradicals of R and T, respectively, and let Q be the annihilator in T of the module N/MT. Then Q is a prime ideal in T. If I is an ideal of T that is not contained in Q, then I is finitely generated. On the other hand, each prime ideal of T contained in Q is not finitely generated.*

PROOF. By Theorem 4, N/MT is a Jónsson ω_0-generated module over R/M, and as remarked above, it follows that $N/MT = G$ is also a Jónsson ω_0-generated module over T, so that by [GH_2, Prop. 1.1], Q is a prime ideal. Moreover, by this same reference, for each $t \in T$, either $tG = G$, or $tG = (0)$. Since Cohen's Theorem

[N, (3.3)] implies that any ideal in a ring maximal with respect to not being finitely generated is a prime ideal, to show that an ideal I of T is finitely generated, it suffices to show that every prime ideal containing I is finitely generated. A prime ideal P of T is finitely generated if and only if P/MT is finitely generated. Therefore by passing to R/M and T/MT, we are reduced to the case where R is a reduced Noetherian ring and T is a J-algebra over R. By part (7) of Theorem 3, it follows that $T = \mathrm{R}(+)G$, where G is the nilradical of T. If P is not contained in Q, choose $y \in P - Q$. Then $yG = G$ so that yT contains G, and hence $T/yT = R/(yT \cap R)$. Therefore T/yT is Noetherian and P is finitely generated. For any prime P of T, we have $G \subseteq P$, so that $P = (P \cap R) + G$. If $P \subseteq Q$, then $PG = (0)$ and any set of generators of P as an ideal of $T = \mathrm{R}(+)G$ has the property that the components in G of the generators generate G. Since G is not finitely generated, it follows that P is not finitely generated if $P \subseteq Q$.

COROLLARY 5. *Suppose R is a proper subring of the ring T and that each proper T-overring of R is a principal ideal ring* (PIR). *Then T is Noetherian, but T need not be a* PIR, *even in the absolute case where R is the prime subring of T.*

PROOF. Assume that T is not Noetherian. If N and M denote the nilradicals of T and R, respectively, then Theorem 4 and part (7) of Theorem 3 show that we can replace T and R by T/MT and R/M, thereby assuming that $T = R(+)N$, where R is reduced and N is a Jónsson ω_0-generated module over both R and T. Let Q and P denote the annihilators of N in T and R, respectively. As in the proof of Proposition 2, Q and P are prime ideals of T and R, respectively, and P is also an ideal of T. Since P is principal, Proposition 2 shows that Q/P is not finitely generated. Thus, by replacing T and R by T/P and R/P, we further assume without loss of generality that $T = R(+)N$, where R is a PID that is not a field and N is a faithful Jónsson ω_0-generated module over R. In this case we obtain the contradiction that there exists a proper T-overring of R that is not a PIR as follows. If H is a proper R-submodule of N, if $S = R + H$, if $h \in H$, and if r is a nonzero element of R, then it is straightforward to show that the ideal $I = Sr + Sh$ of S is principal if and only if h is divisible by r *in* S. However, N is either a torsion module over T, or else N is torsion-free [GH_2, Prop. 2.1]. If N is a torsion module, let H be any nonzero proper submodule of N. Since H is finitely generated, the annihilator dR of H is nonzero; taking $r = d$ and h to be any nonzero element of H, we conclude that the ideal $Sr + Sh$ of $S = R + H$ is not principal in this case. On the other hand, if N is torsion-free, then to within isomorphism, N is the quotient field of R [GH_2, Theorem 1.4]. Then if $H = R$, $h = 1$, and r is any nonzero element of R, then $Sr + Sh$ is not principal in the proper T-overring $S = R + H$ of R. We conclude that T is Noetherian, as asserted.

If R is any prime ring, it is straightforward to show that there exists an

R-module G such that if $T = R(+)G$, then T is not a PIR, but each proper T-overring of R (that is, each proper subring of T) is a PIR. To wit, if $R = \mathbf{Z}$, then we take G to be a cyclic group of prime order; if $R = \mathbf{Z}/n\mathbf{Z}$, where $n > 1$ is divisible by p^2 for some prime p, then again we take G to be a cyclic group of order p and $T = R(+)G$; if, on the other hand, n is divisible by the prime p, but is not divisible by p^2, let G be $H \oplus H$, where H is the cyclic group of order p (the R-multiplication on G in each case is that induced by considering G as a $\mathbf{Z}$-module).

REFERENCES

[F] L. Fuchs, *Infinite Abelian Groups, Vol.* I (Pure and Applied Mathematics 36), Academic Press, New York-London, 1970.

[G_1] R. Gilmer, *Integral domains with Noetherian subrings*, Comment. Math. Helv. 45(1970), 129–134.

[G_2] R. Gilmer, *Multiplicative Ideal Theory*, Marcel-Dekker, New York, 1972.

[GH_1] R. Gilmer and W. Heinzer, *Noetherian pairs and hereditarily Noetherian rings*, Arch. Math. 41(1983), 131–138

[GH_2] R. Gilmer and W. Heinzer, *Cardinality of generating sets for modules over a commutative ring*, Math. Scand. 52 (1983), 41–57.

[GH_3] R. Gilmer and W. Heinzer, *On Jónsson algebras over a commutative ring*, J. Pure Appl. Algebra 49(1987), 133–159.

[GH_4] R. Gilmer and W. Heinzer, *Products of commutative rings and zero-dimensionally*, Trans. Amer. Math. Soc. (to appear).

[GH_5] R. Gilmer and W. Heinzer, *Artinian subrings of a commutative ring*, Trans. Amer. Math. Soc. (to appear).

[GO] R. Gilmer and M. O'Malley, *Non-Noetherian rings for which each proper subring is Noetherian*, Math. Scand. 31 (1972), 118–122.

[H] J. Huckaba, *Commutative Rings with Zero Divisors*, Marcel-Dekker, New York, 1988.

[K] E. Kunz, *Introduction to Commutative Algebra and Algebraic Geometry*, Birkhäuser, Boston-Basel-Stuttgart, 1985.

[N] M. Nagata, *Local Rings*, Interscience Publishers Interscience Tracts in Pure and Applied Mathematics 13, New York, 1962.

[OP] J. Ohm and R. Pendleton, *Rings with Noetherian spectrum*, Duke Math. J. 35 (1968), 631–639.

[W_1] A. Wadsworth, *Pairs of domains where all intermediate domains are Noetherian*, Trans. Amer. Math. Soc. 195 (1974), 201–211.

[W_2] A. Wadsworth, *Hilbert subalgebras of finitely generated algebras*, J. Algebra 43 (1976), 298–304.

[ZS] O. Zariski and P. Samuel, *Commutative Algebra, Vol.* I, Springer, Heidelberg-New York, 1975.

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