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CONTINUOUS TRACE GROUPOID C*-ALGEBRAS, II

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Introduction.

In this note we continue our investigation of the properties that guarantee that the C*-algebra of a locally compact groupoid is a continuous trace C*-algebra. We follow the notation of [7] which is consistent with [11] except that we write s (for "source") for the map Renault writes d. We also assume that all our groupoids are locally compact, Hausdorff, second countable, and admit a Haar system. In [7] we showed that if $\mathfrak{G}$ is a principal groupoid in the sense that the map π from $\mathfrak{G}$ to $\mathfrak{G}^0 \times \mathfrak{G}^0$, given by the formula $\pi(\gamma) = (r(\gamma), s(\gamma))$, is one-to-one, then for any Haar system λ on $\mathfrak{G}$, $C^*(\mathfrak{G}, \lambda)$ is a continuous trace C*-algebra if and only if π is a proper map; i.e., if and only if π is a homeomorphism onto a closed subset of $\mathfrak{G}^0 \times \mathfrak{G}^0$. In this case $\mathfrak{G}$ is called a *proper* principal groupoid. In fact, we showed that when $C^*(\mathfrak{G}, \lambda)$ has continuous trace, then $C^*(\mathfrak{G}, \lambda)$ is isomorphic to the C*-algebra of a continuous field of elementary C-algebras defined by a continuous field of Hilbert spaces over $\mathfrak{G}^0/\mathfrak{G}$, where $\mathfrak{G}^0/\mathfrak{G}$ is the quotient obtained from the natural action of $\mathfrak{G}$ on $\mathfrak{G}^0$ on the right. (The properness of π is equivalent to the assertion that this action is proper [7; Lemma 2.1]; and this, in turn, implies that $\mathfrak{G}^0/\mathfrak{G}$ is locally compact and Hausdorff.) Thus, in particular, we showed that if $C^*(\mathfrak{G}, \lambda)$ is a continuous trace C*-algebra, then its Dixmier-Douady invariant is trivial.

Our work in [7], then, leads naturally to the question: Assuming $\mathfrak{G}$ is principal, but adding in a 2-cocycle $\omega \in Z^2(\mathfrak{G}, \mathsf{T})$, when does $C^*(\mathfrak{G}, \lambda, \omega)$ have continuous trace and what effect does the addition of ω have on the Dixmier-Douady invariant? We answer the first half of this question in the present paper postponing the second half for future consideration. Actually, the second half has received attention in the literature under hypotheses more restrictive than those assumed here (e.g., [2, 9, 10]). In fact it follows from [9] that, if X is a paracompact locally compact space, then any class in $\check{H}^3(X; \mathsf{Z})$ may arise as the Dixmier-Douady class of $C^*(\mathfrak{G}, \lambda, \omega)$ for suitable $\mathfrak{G}$ and ω. However, the precise

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relationship between the cohomology of $\mathfrak{G}$ and Dixmier-Douady invariants is complex.

It is well known [11] that an element ω in $Z^2(\mathfrak{G}, \mathrm{T})$ gives rise to a certain type of extension $\mathfrak{E}$ of $\mathfrak{G}$ which, in turn, is an example of what Kumjian [3] calls a T-groupoid. Roughly, this is a groupoid $\mathfrak{E}$ endowed with a T-action such that the quotient $\mathfrak{E}/\mathrm{T}$ is isomorphic to $\mathfrak{G}$; the precise definition will be given in the next section. The algebra $C^*(\mathfrak{G}, \lambda, \omega)$ is generalized by an algebra denoted $C^*(\mathfrak{G}; \mathfrak{E}, \lambda)$. As with $C^*(\mathfrak{G}, \lambda, \omega)$, it is a quotient of the C^*-algebra $C^*(\mathfrak{E}, \sigma)$ where σ is a Haar system on $\mathfrak{E}$ induced naturally by the Haar system λ on $\mathfrak{G}$.

The primary goal, then, of the present note is the following result which we divide into two theorems below, Theorem 4.2 and Theorem 4.3.

MAIN THEOREM. *Let $\mathfrak{E}$ be a second countable, locally compact* T*-groupoid and assume that $\mathfrak{G} = \mathfrak{E}/\mathrm{T}$ is a principal groupoid admitting a Haar system λ. Then $C^*(\mathfrak{G}; \mathfrak{E}, \lambda)$ is a continuous trace C^*-algebra if and only if $\mathfrak{G}$ is proper.*

In particular, when $\mathfrak{G}$ is principal and $\omega \in Z^2(\mathfrak{G}, \mathrm{T})$ then $C^*(\mathfrak{G}, \lambda, \omega)$ is a continuous trace C^*-algebra if and only if $\mathfrak{G}$ is proper. The proof resembles the proof of Theorem 2.3 of [7] in broad outline, but the details are non-trivial, more complex, and require additional finesse. Section 2 establishes notation and helps to delimit the generality of our presentation. In particular, we show that the notion of T-groupoid does indeed generalize properly the notion of an extension determined by a 2-cocycle. In section 3, we investigate the structure of certain irreducible represenations of $C^*(\mathfrak{G}; \mathfrak{E}, \lambda)$ and show in Proposition 3.3 that when $C^*(\mathfrak{G}; \mathfrak{E}, \lambda)$ is a continuous trace C^*-algebra, then these representations exhaust $C^*(\mathfrak{G}; \mathfrak{E}, \lambda)^\wedge$. Finally, in section 4, we prove our main Theorem.

§2. Groupoid Extensions.

Throughout this note, $\mathfrak{G}$ will be a locally compact principal groupoid with unit space $\mathfrak{G}^0 = \mathfrak{X}$. As our eventual goal is to study groupoid C^*-algebras which are twisted by a (groupoid) 2-cocycle, it will be convenient to recall the notion of a groupoid extension. As we shall see in Example 2.1, unlike the situation for groups, this approach is more general than one which is specific to cocycles.

We let $\mathfrak{S} = \mathfrak{X} \times \mathrm{T}$, and consider a locally compact groupoid extension $\mathfrak{E}$ of $\mathfrak{G}$ by $\mathfrak{S}$. That is, we have an exact sequence of locally compact groupoids

$$\mathfrak{X} \longrightarrow \mathfrak{S} \xrightarrow{\ i\ } \mathfrak{E} \xrightarrow{\ j\ } \mathfrak{G}, \tag{2.1}$$

where i is a homeomorphism onto a closed subgroupoid of $\mathfrak{E}$ (which we often identify with $\mathfrak{S}$) satisfying $i(x) = x$, and j is an open, continuous surjection. Notice that $\mathfrak{E}$ admits a free T-action defined for $t \in \mathrm{T}$ by

$$t \cdot \gamma = i(r(\gamma), t)\gamma,$$

and that the quotient space $\mathfrak{E}/\mathrm{T}$ may be identified with $\mathfrak{G}$.

Conversely, suppose that $\mathfrak{E}$ is a locally compact groupoid which is also a T-bundle (i.e., $\mathfrak{E}$ is a free T-space and $\mathfrak{E}/\mathrm{T}$ is Hausdorff). We will also require that

$$r(t \cdot \gamma) = r(\gamma), \text{ and} \tag{2.2}$$
$$s(t \cdot \gamma) = s(\gamma),$$

so that $t, s \in \mathrm{T}$ and $(\gamma, \eta) \in \mathfrak{E}^{(2)}$ imply that $(t \cdot \gamma, s \cdot \eta) \in \mathfrak{E}^{(2)}$. If in addition, we always have

$$(t \cdot \gamma)(s \cdot \eta) = (ts) \cdot (\gamma\eta), \tag{2.3}$$

and if $\mathfrak{E}/\mathrm{T}$ is a principal groupoid, then we follow [3; Definition 2.2] and call $\mathfrak{E}$ a T-groupoid. If in this case, we let

$$\mathfrak{S} = \{\gamma \in \mathfrak{E} : r(\gamma) = s(\gamma)\},$$

then $\mathfrak{E}^0 \subset \mathfrak{S}$ and $(u, t) \mapsto t \cdot u$ is easily seen to be a (topological) groupoid isomorphism of $\mathfrak{E}^0 \times \mathrm{T}$ onto $\mathfrak{S}$. Therefore, if $\mathfrak{E}$ is a T-groupoid, then we have an exact sequence of locally compact groupoids

$$\mathfrak{E}^0 \longrightarrow \mathfrak{E}^0 \times \mathrm{T} \xrightarrow{\ i\ } \mathfrak{E} \xrightarrow{\ j\ } \mathfrak{E}/\mathrm{T},$$

where j is an open, continuous surjection. One should think of such an extension as the groupoid analogue of a central circle extension of groups. *From now on, we'll assume that Sequence* (2.1) *is an extension which is determined by a second countable, locally compact* T*-groupoid* $\mathfrak{E}$. In particular, in the sequel $\mathfrak{G}$ will always denote the principal groupoid $\mathfrak{E}/\mathrm{T}$. Since $u \mapsto i(u, 1)$ is a homeomorphism of $\mathfrak{X}$ onto $\mathfrak{E}^0$, we shall often identify $\mathfrak{E}^0$ and $\mathfrak{X}$.

Let $\{\lambda^u\}_{u \in \mathfrak{X}}$ be a Haar system for $\mathfrak{G}$. Since we wish to study the twisted groupoid C*-algebra $C^*(\mathfrak{G}; \mathfrak{E}, \lambda)$ introduced in [12; §3], we review some of the basic constructions in our rather special setting. We let $C_c(\mathfrak{G}, \mathfrak{E})$ denote the collection of $f \in C_c(\mathfrak{E})$ which also satisfy

$$f(t \cdot \gamma) = tf(\gamma). \tag{2.4}$$

Given $g, f \in C_c(\mathfrak{G}, \mathfrak{E})$, we define

$$f * g(\gamma) = \int_{\mathfrak{G}} f(\gamma\eta)g(\eta^{-1})d\lambda^{s(\gamma)}(\dot{\eta}), \tag{2.5}$$

and

$$f^*(\gamma) = \overline{f(\gamma^{-1})}.$$

Here we have written $\dot{\eta}$ in place of $j(\eta)$. Of course the integrand in Equation (2.5) is a function of $\dot{\eta}$ in view of Equation (2.4) and Equation (2.3). We define Rep($\mathfrak{G}$; $\mathfrak{E}$) to be the collection of non-degenerate $*$-representations $L: C_c(\mathfrak{G}; \mathfrak{E}) \to \mathfrak{B}(\mathfrak{H}_L)$ which are continuous when $C_c(\mathfrak{G}; \mathfrak{E})$ is given the inductive limit topology and $\mathfrak{B}(\mathfrak{H}_L)$ is given the weak operator topology. It follows from [12; Thèoréme 4.1 and Proposition 3.5] that

$$\|f\| = \sup\{\|L(f)\| : L \in \mathrm{Rep}(\mathfrak{G}; \mathfrak{E})\}$$

is finite and defines a pre-C*-norm on $C_c(\mathfrak{G}; \mathfrak{E})$. We denote the completion by $C^*(\mathfrak{G}; \mathfrak{E}, \lambda)$. Alternatively, one can define

$$\|f\|_I = \max\left[\sup_{u \in \mathfrak{X}} \int_{\mathfrak{G}} |f(\gamma)| d\lambda^u(\dot{\gamma}), \sup_{u \in \mathfrak{X}} \int_{\mathfrak{G}} |f(\gamma)| d\lambda_u(\dot{\gamma})\right].$$

It is a consequence of [12; Proposition 3.5 and Thèoréme 4.1] that Rep($\mathfrak{G}$; $\mathfrak{E}$) consists exactly of non-degenerate I-norm bounded represenations of $C_c(\mathfrak{G}; \mathfrak{E})$.

It will be very convenient to realize that $\mathfrak{E}$ has a naturally defined Haar system of its own. For each $u \in \mathfrak{X} \cong \mathfrak{E}^0$, let σ^u be defined by

$$\int_{\mathfrak{E}} f(\gamma) d\sigma^u(\gamma) = \int_{\mathfrak{G}} \int_{\mathrm{T}} f(t \cdot \gamma) dt \, d\lambda^u(\dot{\gamma}). \quad (f \in C_c(\mathfrak{E}))$$

It is perhaps comforting to know that when using this Haar system on $\mathfrak{E}$, the right hand side of Equation (2.5) becomes

$$\int_{\mathfrak{E}} f(\gamma\eta) g(\eta^{-1}) \, d\sigma^{s(\gamma)}(\eta).$$

The point is: no dot! Another immediate consequence of this is the fact that $C^*(\mathfrak{G}; \mathfrak{E}, \lambda)$ is a quotient of the ordinary groupoid C*-algebra $C^*(\mathfrak{E}; \sigma)$; this follows from [12; Lemma 3.3] (in our case, $\chi(t) = \bar{t}$).

We conclude this section with a discussion designed to place in perspective the extent to which T-groupoids are more general than groupoid extensions associated with *continuous* 2-cocycles as developed in [11]. We begin with an analysis that will prove useful later. It shows that at the *measure theoretic* level, T-groupoids do reduce to extensions and it helps to identify where the topological problems lie.

Give a T-groupoid $\mathfrak{E}$, one may apply a lemma of Mackey [4; Lemma 1.1] to find a *Borel* cross function c for the quotient map $j: \mathfrak{E} \to \mathfrak{G}$ in Sequence (2.1). (One may even choose c so that $c(K)$ has compact closure for each compact set K in $\mathfrak{G}$.

This will be useful later. We call such cross sections *regular*.) The element $\gamma c(j(\gamma))^{-1}$ lies in the image of i and so, effectively, defines a function $t: \mathfrak{E} \to \mathsf{T}$ such that $\gamma = t(\gamma) \cdot c(\dot{\gamma})$ where $\dot{\gamma} = j(\gamma)$. Evidently t is Borel since c is. Note, too, that t is continuous if and only if c is, because j is open. If for $(x, y) \in \mathfrak{G}^{(2)}$, we set $\omega(x, y) = t(c(xy)^{-1}c(x)c(y))$, then ω is a Borel function from $\mathfrak{G}^{(2)}$ to T such that $\omega(x, y) \cdot c(xy) = c(x)c(y)$. On the one hand $c(x)c(y)c(z) = \omega(x, y)c(xy)c(z) = \omega(x, y)\omega(xy, z)c(xyz)$, while on the other $c(x)c(y)c(z) = \omega(y, z)c(x)c(yz) = \omega(y, z)\omega(x, yz)c(xyz)$. Consequently, $\omega(x, y)\omega(xy, z) = \omega(y, z)\omega(x, yz)$. Thus ω is a Borel 2-cocycle on $\mathfrak{G}$. (This cocycle is not *normalized* in the sense that $\omega(x, y) = 1$, if either x or y is a unit, but that need not concern us here.)

Form $\mathfrak{E}' = \mathsf{T} \times \mathfrak{G}$ with product defined by

$$(s_1, x_1)(s_2, x_2) = (s_1 s_2 \omega(x_1, x_2), x_1 x_2),$$

if $(x_1, x_2) \in \mathfrak{G}^{(2)}$, and inverse defined by

$$(s, x)^{-1} = (\overline{s\omega(x^{-1}, x)}, x^{-1})$$

Then $\mathfrak{E}'$ is a Borel groupoid, called the *extension of* $\mathfrak{G}$ *by* T *determined by* ω, which is Borel isomorphic to $\mathfrak{E}$ via the map $(s, x) \mapsto s \cdot c(x)$, $(s, x) \in \mathfrak{E}'$. The inverse is $\gamma \mapsto (t(\gamma), (j(\gamma)), \gamma \in \mathfrak{E}$. Note that this isomorphism is actually T-equivariant. Note, too, that if c were continuous then ω would be also, and if $\mathfrak{E}'$ were endowed with the product topology, then $\mathfrak{E}'$ would be a T-groupoid topologically isomorphic to $\mathfrak{E}$ as a T-groupoid.

Conversely, suppose ω is a continuous 2-cocycle on $\mathfrak{G}$, not coming a priori from a cross section, and form the associated extension of $\mathfrak{G}$ by T, $\mathfrak{E}'$. If $\mathfrak{E}'$ is topologically isomorphic to $\mathfrak{E}$ as a T-groupoid, then, in fact j has a *continuous* cross section. Indeed, if $\alpha: \mathfrak{E}' \to \mathfrak{E}$ is such an isomorphism, and if $\dot{\alpha}: \mathfrak{E}'/\mathsf{T} (= \mathfrak{G}) \to \mathfrak{E}/\mathsf{T} (= \mathfrak{G})$ is the induced map at the level of quotients, then c, defined by the formula

$$c(x) = \alpha(1, \dot{\alpha}^{-1}(x)),$$

is a continuous cross section. Thus, a T-groupoid comes from an extension of $\mathfrak{G}$ by T via a continuous cocycle if and only if the map j admits a continuous cross section.

Consequently, one way to show that T-groupoids are more general than extensions is to produce a (principal) groupoid $\mathfrak{G}$ carrying a Borel cocycle ω yielding an extension $\mathfrak{E}'$ with a new topology (inducing the original topology on $\mathfrak{G}$) that is *not* homeomorphic to the product topology. This can be done using a theorem of Mackey [5] which shows that for locally compact *groups* there is a one-to-one correspondence between locally compact group extensions by T and *Borel* cocycles. (One applies this result to a suitable group and then builds a suitable transformation group.) In general, it is very difficult to see when a Borel cocycle on a groupoid gives rise to a topological extension by T, but it is often easy to discern when continuous cross sections fail to exist. This is illustrated in the following example from [3] and is more in spirit with our earlier work. We are greatful to the referee for calling it to our attention.

EXAMPLE 2.1. Let P be a principal T-bundle over a second countable locally compact Hausdorff space X. In [3; Proposition 4.4], Kumjian shows that if $\mathfrak{E}(P)$ is the quotient of $P \times P$ by the (skew) diagonal action of T $((p_1, p_2) \sim (tp_1, \bar{t}p_2))$ on P, then $\mathfrak{E}(P)$ carries the structure of a T-groupoid over the trivial groupoid $X \times X$. In fact, P is an equivalence between $\mathfrak{E}(P)$ and T in the sense of [6] and, moreover, two bundles give rise to (topologically) isomorphic T-groupoids if and only if the bundles are isomorphic. The map $j: \mathfrak{E}(P) \to X \times X$ is given by the formula $j([p_1, p_2]) = (\pi(p_1), \pi(p_2))$ where $[p_1, p_2]$ denotes the equivalence class of (p_1, p_2) in $\mathfrak{E}(P)$ and π denotes the bundle map from $P \to X$. Thus, it is clear that j admits a continuous cross section if and only if π does; i.e., j admits a section if and only if P is trivial. Since spaces X exist that carry nontrivial P's, the example is complete.

We note, finally, that if the T-groupoid $\mathfrak{E}$ is a groupoid extension of $\mathfrak{G}$ by T coming from a continuous 2-cocycle ω, then the map $\Phi: C_c(\mathfrak{G}, \mathfrak{E}) \to C_c(\mathfrak{G})$ defined by $\Phi(f)(x) = f(1, x)$, implements an isomorphism between $C_c(\mathfrak{G}; \mathfrak{E})$ and $C_c(\mathfrak{G})$ that extends to one between $C^*(\mathfrak{G}; \mathfrak{E}, \lambda)$ and Renault's $C^*(\mathfrak{G}, \omega, \lambda)$ [11].

§3. Irreducible representations.

Now for each $u \in \mathfrak{X}$, we want to define a representation L^u of $C_c(\mathfrak{G}; \mathfrak{E})$. First, let $\mathscr{H}_u^0$ be the collection of bounded Borel functions with compact support on $\mathfrak{E}_u = s^{-1}(u)$, and with the property that $f(t \cdot \gamma) = tf(\gamma)$ for all $t \in$ T and $\gamma \in \mathfrak{E}$. Then for each $\xi, \eta \in \mathscr{H}_u^0$ define

$$\langle \xi, \eta \rangle_u = \int_{\mathfrak{G}} \xi(\gamma)\overline{\eta(\gamma)}\, d\lambda_u(\dot{\gamma})$$

$$= \int_{\mathfrak{E}} \xi(\gamma)\overline{\eta(\gamma)}\, d\sigma_u(\gamma).$$

The Hilbert space completion of $\mathscr{H}_u^0$ will be denoted by $\mathscr{H}_u$, and is clearly a subspace of $L^2(\mathfrak{E}_u, \sigma_u)$. Furthermore, we claim that the functions obtained by restricting elements in $C_c(\mathfrak{G}; \mathfrak{E})$ to $s^{-1}(u)$ form a dense subset of $\mathscr{H}_u$. To see this, recall that by [12; Lemma 3.3], there is a surjective $*$-homomorphism

$\chi: C_c(\mathfrak{E}) \to C_c(\mathfrak{G}; \mathfrak{E})$ defined by

$$\chi(f)(\gamma) = \int_{\mathrm{T}} f(t \cdot \gamma)\bar{t}\, dt.$$

Suppose that $f \in \mathscr{H}_u^0$, and that $\langle f, f_0 \rangle_u = 0$ for all $f_0 \in C_c(\mathfrak{G}; \mathfrak{E})$. Therefore, for all $g \in C_c(\mathfrak{E})$,

$$\begin{aligned} 0 = \langle f, \chi(g) \rangle_u &= \int_{\mathfrak{G}} f(\gamma) \overline{\int_{\mathrm{T}} g(t \cdot \gamma)\bar{t}\, dt}\, d\lambda_u(\dot{\gamma}) \\ &= \int_{\mathfrak{G}} \int_{\mathrm{T}} f(\gamma) \overline{g(t \cdot \gamma)} t\, dt\, d\lambda_u(\dot{\gamma}) \\ &= \int_{\mathfrak{G}} \int_{\mathrm{T}} f(t \cdot \gamma) \overline{g(t \cdot \gamma)}\, dt\, d\lambda_u(\dot{\gamma}) \\ &= \int_{\mathfrak{E}} f(\gamma) \overline{g(\gamma)}\, d\sigma_u(\gamma). \end{aligned}$$

Since g is arbitrary in $C_c(\mathfrak{E})$, it follows that f is the zero element of $\mathscr{H}_u$. This proves the claim.

We define L^u on vectors in $C_c(\mathfrak{G}; \mathfrak{E})$ by the formula

$$\begin{aligned} L^u(f)\xi(\gamma) &= f * \xi(\gamma) \\ &= \int_{\mathfrak{G}} f(\gamma\alpha)\xi(\alpha^{-1})\, d\lambda^u(\dot{\alpha}) \\ &= \int_{\mathfrak{E}} f(\gamma\alpha)\xi(\alpha^{-1})\, d\sigma^u(\alpha) \end{aligned}$$

for $f, \xi \in C_c(\mathfrak{G}; \mathfrak{E})$. Of course, this only defines L^u on a dense subspace of $\mathscr{H}_u$. However, L^u is the restriction to $C_c(\mathfrak{G}; \mathfrak{E}) \subset C_c(\mathfrak{E})$ of the representation R^u of $C_c(\mathfrak{E})$ in $\mathscr{H}_u^0$ defined by

$$R^u(f)\xi(\gamma) = \chi(f) * \xi.$$

But R^u is a representation of $C_c(\mathfrak{E})$ in $\mathscr{H}_u^0$ which is continuous in the inductive limit topology. So by [12, Proposition 4.2], R^u extends to a representation of $C^*(\mathfrak{E}, \sigma)$ on $\mathscr{H}_u$. Since $C^*(\mathfrak{G}; \mathfrak{E}, \lambda)$ is a quotient of $C^*(\mathfrak{E}, \sigma)$, it follows that each $L^u(f)$ is bounded – and hence extends to all of $\mathscr{H}_u$ – and that $f \mapsto L^u(f)$ is in $\mathrm{Rep}(\mathfrak{G}; \mathfrak{E})$.

LEMMA 3.1. *If* $\phi \in C_0(\mathfrak{X})$ *and* $f \in C_c(\mathfrak{G}; \mathfrak{E})$, *then the equation*

$$R(\phi)f(\gamma) = \phi(r(\gamma))f(\gamma)$$

defines a homomorphism R of $C_0(\mathfrak{X})$ *into* $\mathscr{M}(C^*(\mathfrak{G}; \mathfrak{E}, \lambda))$ (*the corresponding right multipication is given by* $S(\phi)f(\gamma) = \phi(s(\gamma))f(\gamma)$.)

PROOF. The only nonobvious fact to check is that $R(\phi)$ can be extended to an operator on all of $C^*(\mathfrak{G}; \mathfrak{E}, \lambda)$. However by [6; Proposition 2.10 and Lemma 2.12], $C_c(\mathfrak{G}; \mathfrak{E})$ has an approximate identity $\{e_\alpha\}$ for the inductive limit topology such that $\|e_\alpha\|_I$ tends to 1 (simply choose such an approximate identity for $C_c(\mathfrak{E})$, and notice that χ is I-norm decreasing as well as continuous in the inductive limit topology [12; Lemma 3.3]). Since one clearly has

$$\|R(\phi)f\|_I \leqq \|\phi\|_\infty \|f\|_I,$$

one computes that

$$\begin{aligned}\|R(\phi)f\| &= \lim_\alpha \|R(\phi)e_\alpha f\| \\ &\leqq \lim_\alpha \|R(\phi)e_\alpha\|_I \|f\| \\ &\leqq \|\phi\|_\infty \|f\|.\end{aligned}$$

The result now follows.

LEMMA 3.2. *If* $\mathfrak{E}$ *is a second countable locally compact* $\mathbf{T}$-*groupoid, then the representations* L^u *defined above are irreducible for each* $u \in \mathfrak{X}$. *Furthermore if* $[u] = [v]$, *then* L^u *and* L^v *are unitarily equivalent.*

PROOF. The idea of the proof is to realize L^u as (equivalent to) a representation on $L^2([u], d\mu_{[u]})$, where $\mu_{[u]}$ is defined as in the proof of [7; Lemma 2.4], and then to follow the proof there practically verbatim. The first step is to establish an isomorphism of $\mathscr{H}_u$ with $L^2(\mathfrak{G}_u, \lambda_u)$.

Let c be a regular cross section to j, as discussed in §2. We claim that we may normalize c so that

(1) if $u \in \mathfrak{X}$, then $c(u) = u$, and
(2) if $x \in \mathfrak{G}$, then $c(x^{-1}) = c(x)^{-1}$.

Indeed, if $\tilde{c}(x) = \overline{t(r(x))} \cdot c(x)$, where $\gamma = t(\gamma) \cdot c(\dot{\gamma})$ as before, then clearly $\tilde{c}$ satisfies (1). Assuming, then, as we may, that c satisfies (1), the function $\delta : \mathfrak{G} \to \mathbf{T}$ defined by the equation

$$c(x)c(x^{-1}) = \delta(x) \cdot c(x)c(x)^{-1}$$
$$= \delta(x)r(c(x)) = \delta(x)r(x),$$

is clearly Borel. Letting $\beta : \mathfrak{G} \to \mathbf{T}$ be any Borel square root of δ and setting $\tilde{c}(x) = \beta(x) \cdot c(x)$, we obtain a regular Borel cross section satisfying both (1) and (2).

Define $U : \mathscr{H}_u \to L^2(\mathfrak{G}_u, \lambda_u)$ by the formula $U(\xi)(x) = \xi(c(x))$, $\xi \in \mathscr{H}_u^0$ and $x \in \mathfrak{G}$. Then a calculation shows that U is isometric and so extends to an isometry from $\mathscr{H}_u$ into $L^2(\mathfrak{G}_u, \lambda_u)$. However, since c is normalized, the function t satisfies $t(s \cdot \gamma) = st(\gamma)$ and so U has a left inverse defined by

$$V(\xi)(\gamma) = t(\gamma)\xi(\dot{\gamma}),$$

$\xi \in L^2(\mathfrak{G}_u, \lambda_u)$, $\gamma \in \mathfrak{E}$. Thus U is unitary.

Let ω be the cocycle associated with c and define $M^u : C_c(\mathfrak{G}; \mathfrak{E}) \to \mathscr{B}(L^2(\mathfrak{G}_u, \lambda_u))$ by

$$M^u(f)\, k(x) = \int_{\mathfrak{G}} f(c(xy))\, k(y^{-1})\omega(x, y)\, d\lambda^u(y)$$

$$= \int_{\mathfrak{G}} f(c(x)c(y))\, k(y^{-1})\, d\lambda^u(y)$$

for $f \in C_c(\mathfrak{G}; \mathfrak{E})$ and $k \in L^2(\mathfrak{G}_u, \lambda_u)$. Then one can compute that

$$M^u(f(U(\xi))(x) = U(L^u(f)(\xi))\,(x);$$

in short, $M^u \cong L^u$.

On the other hand since $\mathfrak{G}$ is a second countable principal groupoid, $\mathfrak{G}_u$ is Borel isomorphic to $[u]$ (via the map $\gamma \mapsto r(\gamma)$). This map carries λ_u to the measure $\mu_{[u]}$ and defines a Hilbert space isomorphism $V_u : L^2(\mathfrak{G}_u, \lambda_u) \to L^2([u], \mu_{[u]})$ which implements an equivalence between M^u and the representation T_u on $L^2([u], \mu_{[u]})$ defined by

$$T^u(f)\,(k)\,(x \cdot u) = \int_{\mathfrak{G}} f(c(xy))\, k(y^{-1} \cdot u)\omega(x, y)\, d\lambda^u(y).$$

Now the proof proceeds exactly along the lines of [7; Lemma 2.4]. The key point is that for $\phi \in C_0(\mathfrak{X})$, $f \in C_c(\mathfrak{G}; \mathfrak{E})$, and $k \in L^2([u], \mu_{[u]})$,

$$T^u(R(\phi)f)\,k(x\cdot u) = \int_{\mathfrak{G}} (R(\phi)f)(c(xy))\,k(y^{-1}\cdot u)\omega(x,y)\,d\lambda^u(y)$$

$$= \int_{\mathfrak{G}} \phi(r(c(xy)))f(c(xy))\,k(y^{-1}\cdot u)\omega(x,y)\,d\lambda^u(y)$$

$$= \int_{\mathfrak{G}} \phi(r(xy))\,f(c(xy))\,k(y^{-1}\cdot u)\omega(x,y)\,d\lambda^u(y)$$

$$= \phi(x\cdot u)(T^u f)\,k(x\dot{u}).$$

Since the functions $\phi(x\cdot u)$, $x \in \mathfrak{G}_u$, $\phi \in C_0(\mathfrak{X})$, are bounded Borel functions on $[u]$ separating points, when viewed as multiplication operators on $L^2([u], \mu_{[u]})$, they generate a masa. The above discussion and the analysis in [7; Lemma 2.4] shows that if a projection commutes with $T^u(C_c(\mathfrak{G};\mathfrak{E}))$, then it is given by a multiplication operator determined by a bounded Borel function ϕ on $[u]$ satisfying $\phi(x\cdot u) = \phi(u)$ almost everywhere with respect to $\mu_{[u]}$; i.e., ϕ is constant $\mu_{[u]}$-almost everywhere. Thus, T^u, and hence L^u, is irreducible.

PROPOSITION 3.3. *Suppose that* $\mathfrak{E}$ *is a second countable* T*-groupoid and that* $C^*(\mathfrak{G};\mathfrak{E},\lambda)$ *has continuous trace. Then* $u \mapsto [L^u]$ *induces a homeomorphism of* $\mathfrak{X}/\mathfrak{G}$ *onto* $C^*(\mathfrak{G};\mathfrak{E},\lambda)^\wedge$.

PROOF. If $f, \xi, \eta \in C_c(\mathfrak{G};\mathfrak{E})$, then $u \mapsto \langle L^u(f)\xi, \eta\rangle_u$ is continuous by virtue of the continuity property of the Haar system. It follows from Lemma 3.2 that $u \mapsto [L^u]$ induces a continuous map $\Psi: \mathfrak{X}/\mathfrak{G} \to C^*(\mathfrak{G};\mathfrak{E},\lambda)^\wedge$. Just as in the proof of [7; Proposition 2.5], it follows that Ψ is an injection, that orbits are closed, and that Ψ is a homeomorphism onto its range.

The difficulty is to see that Ψ is surjective. Towards this end, let L be an irreducible representation of $C^*(\mathfrak{G};\mathfrak{E},\lambda)$, and let N be the associated representation of $C_0(\mathfrak{X})$ (i.e., $L(R(\phi)f) = N(\phi)L(f)$ for $\phi \in C_0(\mathfrak{X})$ and $f \in C_c(\mathfrak{G};\mathfrak{E})$). Also let $\chi: C^*(\mathfrak{E},\sigma) \to C^*(\mathfrak{G};\mathfrak{E},\lambda)$ be the quotient map defined earlier. Standard arguments show that $\ker(N) = J_{[u]}$ for some $u \in \mathfrak{X}$, where $J_{[u]}$ is the ideal of functions in $C_0(\mathfrak{X})$ which vanish on $[u]$. In particular, $L \circ \chi$ factors through $C^*(\mathfrak{E}|_{[u]}, \sigma)$. Since $\mathfrak{E}|_{[u]}$ is a transitive groupoid with isotropy group T, it follows from [6; Theorem 3.1] that

$$C^*(\mathfrak{E}|_{[u]}, \sigma) \cong C_0(\mathrm{Z}, \mathcal{K}), \tag{3.1}$$

where $\mathcal{K}$ denotes the algebra of compact operators on a suitable Hilbert space. For convenience, we will denote the image in $C_0(\mathrm{Z},\mathcal{K})$ of $f \in C^*(\mathfrak{E}|_{[u]}, \sigma)$ by $\hat{f}$. A careful examination of the proof of [6; Theorem 3.1] reveals the fact that if

$\phi \in C(\mathrm{T})$, then the isomorphism in Equation (3.1) carries $\phi * f$ to $\hat{\phi}\hat{f}$, where $\hat{\phi}\hat{f}(n) = \hat{\phi}(n)\hat{f}(n)$ (of course, $\hat{\phi}(n)$ is simply the n^{th} Fourier coefficient of ϕ). On the other hand, $\chi(\phi * f) = \hat{\phi}(1)\chi(f)$ [12; Lemma 3.3]. It follows that $\chi(C^*(\mathfrak{E}|_{[u]}, \sigma))$ is isomorphic to $\mathscr{K}$, and hence has only one distinct class of irreducible representation. Therefore, $L \cong L^u$. This proves that Ψ is surjective.

§4. Proper T-groupoids.

Now it will be profitable to view $L^u(f)$ as a special type of integral operator for each $f \in C_c(\mathfrak{G}; \mathfrak{E})$. In fact, let $K(\gamma, \eta) = f(\gamma\eta^{-1})$. Then, if $\xi \in C_c(\mathfrak{G}; \mathfrak{E})$,

$$(4.1) \qquad L^u(f)(\xi)(\gamma) = \int_{\mathfrak{G}} f(\gamma\alpha)\xi(\alpha^{-1})\, d\lambda^u(\dot{\alpha})$$

$$= \int_{\mathfrak{G}} f(\gamma\alpha^{-1})\xi(\alpha)\, d\lambda_u(\dot{\alpha})$$

$$= \int_{\mathfrak{E}} K(\gamma, \alpha)\xi(\alpha)\, d\sigma_u(\alpha).$$

We assert that

(1) $K(t \cdot \gamma, s \cdot \eta) = t\bar{s}K(\gamma, \eta)$ for all $t, s \in \mathrm{T}$ and $\gamma, \eta \in \mathfrak{E}_u$, and
(2) if $[u]$ is closed, then $K \in C_c(\mathfrak{E}_u \times \mathfrak{E}_u)$.

Property (1) follows immediately from the fact that $f \in C_c(\mathfrak{G}; \mathfrak{E})$. When $[u]$ is closed, $f|_{\mathfrak{E}_{[u]}}$ has compact support. Thus, Property (2) follows from [6; Theorem 2.2B] (here, $H = \mathrm{T}$). Of course, any kernel K which satisfies properties (1) and (2) above defines a Hilbert-Schmidt operator $T_K : \mathscr{H}_u \to \mathscr{H}_u$.

PROPOSITION 4.1. *Let $\mathfrak{E}$ be a T-groupoid. Suppose that $T_K : \mathscr{H}_u \to \mathscr{H}_u$ is a positive integral operator of the type discussed above. Then T_K is trace class and*

$$\mathrm{Tr}(T_K) = \int_{\mathfrak{G}} K(\gamma, \gamma)\, d\lambda_u(\dot{\gamma}).$$

In particular, if points are closed in $\mathfrak{E}^0/\mathfrak{E} = \mathfrak{X}/\mathfrak{G}$ and if f is positive in $C_c(\mathfrak{G}; \mathfrak{E})$, in the sense that $f = g^* * g$ for some $g \in C_c(\mathfrak{G}; \mathfrak{E})$, then $L^u(f)$ is trace class for each $u \in \mathfrak{X}$ and

$$\operatorname{Tr}(L^u(f)) = \int_{\mathfrak{G}} f(r(\gamma))\, d\lambda_u(\dot{\gamma}).$$

PROOF. Let P_u be the orthogonal projection of $L^2(\mathfrak{E}_u, \sigma_u)$ onto $\mathscr{H}_u$. Notice that K defines a Hilbert-Schmidt operator T on $L^2(\mathfrak{E}_u, \sigma_u)$ in the obvious way. Furthermore, $T(\mathscr{H}_u) \subset \mathscr{H}_u$ and it follows from Equation (4.1) that $T(L^2(\mathfrak{E}_u, \sigma_u)) \subset C_c(\mathfrak{G}; \mathfrak{E}) \subset \mathscr{H}_u$. Thus, $T = P_u T = T P_u$, and the restriction of T to $\mathscr{H}_u$, viewed as an operator on $\mathscr{H}_u$, is simply T_K. It follows that T is positive and that

$$\operatorname{Tr}(T_K) = \operatorname{Tr}(T), \tag{4.2}$$

where the trace on the left-hand side is computed in $\mathscr{B}(\mathscr{H}_u)$ and the trace on the right-hand side is computed in $\mathscr{B}(L^2(\mathfrak{E}_u, \sigma_u))$. It follows from Mercer's Theorem (e.g., [13; §98]) that

$$\operatorname{Tr}(T) = \int_{\mathfrak{G}} K(\gamma, \gamma)\, d\lambda_u(\dot{\gamma}).$$

Combining this with Equation (4.2), the first assertion follows. The second assertion follows immediately from the first.

THEOREM 4.2. *If $\mathfrak{E}$ is a second countable* T*-groupoid, and if $\mathfrak{G} = \mathfrak{E}/\mathrm{T}$ is a proper principal groupoid, then* $C^*(\mathfrak{G}; \mathfrak{E}, \lambda)$ *has continuous trace.*

PROOF. Notice that $\mathfrak{G}$ being proper forces orbits to be closed. Suppose that f is positive in $C_c(\mathfrak{G}; \mathfrak{E})$ as in Proposition 4.1. Let $u_0 \in \mathfrak{X}$, and suppose that C is a compact neighborhood of u_0. Since $\mathfrak{G}$ is proper, there is a $\ell \in C_c(\mathfrak{G})$ so that $\ell(x) = f(r(x))$ provided $s(x) \in C$. The point is that if $u \in C$, then

$$\operatorname{Tr}(L^u(f)) = \int_{\mathfrak{G}} f(r(\gamma))\, d\lambda_u(\dot{\gamma}) = \int_{\mathfrak{G}} \ell(x)\, \lambda_u(x)$$

which is continuous in u since $\{\lambda_u\}_{u \in \mathfrak{X}}$ is a (right) Haar system. It follows that the ideal $\mathfrak{m}(C^*(\mathfrak{G}; \mathfrak{E}, \lambda))$ of continuous trace elements in $C^*(\mathfrak{G}; \mathfrak{E}, \lambda)$ is dense, and hence that $C^*(\mathfrak{G}; \mathfrak{E}, \lambda)$ has continuous trace [1; 4.5.2].

At this point we should point out that in sharp contrast with the situation for groupoid C^*-algebras of proper principal groupoids [7; Proposition 2.2], when $C^*(\mathfrak{G}; \mathfrak{E}, \lambda)$ has continuous trace, the Dixmier-Douady invariant can be any class in $\check{H}^3(\mathfrak{X}/\mathfrak{G}; \mathbf{Z})$. This was observed in its essential details by Raeburn and Taylor [9; Remark 3] under the hypothesis that $\mathfrak{X}$ is compact. Only minor changes are necessary when $\mathfrak{X}$ is locally compact. The idea, basically, is given a locally compact second countable (and hence paracompact) space X and a element $\mathfrak{a}$ in $\check{H}^3(X;\mathbf{Z})$, identified with $H^2(X,\mathcal{S})$ where $\mathcal{S}$ is the sheaf of germs of continuous $\mathbf{T}$-valued functions, one may choose a cover $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ and an *alternating* 2-cocycle μ in $Z^2(\mathcal{U},\mathcal{S})$ that represents $\mathfrak{a}$. Using $\mathcal{U}$, one constructs in a natural fashion a principal groupoid $\mathfrak{G}$ with Haar system λ such that $\mathfrak{G}^{(0)}/\mathfrak{G}$ is homeomorphic to X, and using μ, one constructs a 2-cocycle $\omega \in Z^2(\mathfrak{G},\mathbf{T})$ such that the Dixmier-Douady invariant of $C^*(\mathfrak{G},\omega,\lambda)$ is $\mathfrak{a}$. Since $C^*(\mathfrak{G},\omega,\lambda) \cong C^*(\mathfrak{G};\mathfrak{E},\lambda)$, where $\mathfrak{E}$ is the $\mathbf{T}$-groupoid associated to ω, the cohomological richness of $\mathbf{T}$-groupoids is established. It should also be emphasized that it appears quite difficult in the general case to determine the Dixmier-Douady invariant for $C^*(\mathfrak{G};\mathfrak{E},\lambda)$ in terms of the structure of $\mathfrak{E}$ when this algebra has continuous trace. We hope to investigate this in the future.

THEOREM 4.3. *Suppose that $\mathfrak{E}$ is a second countable $\mathbf{T}$-groupoid and that $C^*(\mathfrak{G};\mathfrak{E},\lambda)$ has continuous trace. Then $\mathfrak{G} = \mathfrak{E}/\mathbf{T}$ is a proper principal groupoid.*

Before we begin the proof, we need some preliminary definitions and some technical results. We suppose throughout that $\mathfrak{G}$ is not proper. Then there is a sequence $\{x_n\} \subset \mathfrak{G}$ which is eventually disjoint from every compact set $C \subset \mathfrak{G}$, and is such that $\{r(x_n)\}$ and $\{s(x_n)\}$ converge to some $z \in \mathfrak{X}$ [7; Lemma 2.6]. We fix a function $g \in C_c^+(\mathfrak{X})$ which is identically one on a neighborhood U of z. As in [7: Lemma 2.7], we choose symmetric neighborhoods V_0 and V_1 of z in $\mathfrak{G}$, as well as symmetric conditionally compact neighborhoods W_0 and W_1 of $\mathfrak{X}$ in $\mathfrak{G}$ so that, $\overline{V_0} \subset V_1$, $\overline{W_1}$ is conditionally compact, $\overline{W_0} \subset W_1$, and

$$r^{-1}(\mathfrak{X}\backslash \operatorname{supp} g) \subset \overline{W_1}^7\overline{V_1}\overline{W_1}^7\backslash W_0V_0W_0.$$

There is certainly no loss of generality if we assume that $\{r(x_n)\}$, $\{s(x_n)\} \subset U$. Recall that $j:\mathfrak{E} \to \mathfrak{G}$ is the quotient map. Define

$$g^{(1)}(\gamma) = \begin{cases} g(r(\gamma)) & \text{if } \gamma \in j^{-1}(\overline{W_1}^7\overline{V_1}\overline{W_1}^7), \\ 0 & \text{if } \gamma \notin j^{-1}(W_0V_0W_0). \end{cases}$$

As in [7], $g^{(1)}$ is continuous with compact support on $\mathfrak{E}$. Furthermore, $g^{(1)}$ is constant on $\mathbf{T}$-orbits. Choose $b \in C_c^+(\mathfrak{G})$ so that $0 \leqq b \leqq 1$, that b is identically one on $W_0V_0W_0^2V_0W_0$, and so that b vanishes off $\overline{W_1}^4\overline{V_1}\overline{W_1}^4$. Replacing b by $(b + b^*)/2$, we may assume that $b = b^*$ (in $C_c(\mathfrak{G})$).

Recall that as $\mathfrak{E}$ is a $\mathbf{T}$-groupoid, there is a topological isomorphism i of $\mathfrak{S} = \mathfrak{X} \times \mathbf{T}$ into $\mathfrak{E}$. Furthermore, $s\cdot i(u,t) = i(u,st)$. At this point, it is crucial to recall that elements of $\mathfrak{E}^0$ are of the form $i(u,1)$ where $u \in \mathfrak{X}$. Let C be a compact subset of $\mathfrak{X}$ so that $i(C \times \mathbf{T})$ contains the intersection of $\mathfrak{E}^0$ and $\operatorname{supp} g^{(1)}$. Using Tietze's extension theorem, there is a function $\tilde{h} \in C_c(\mathfrak{E})$ with the property that

$$\tilde{h}(i(u,t)) = t \tag{4.3}$$

for all $u \in C$ and $t \in \mathrm{T}$. Notice that we may replace $\tilde{h}$ by $(\tilde{h} + \tilde{h}^*)/2$ and still retain the property in Equation (4.3). Thus, we may assume that $\tilde{h}$ is self-adjoint in $C^*(\mathfrak{E}, \sigma)$ and that $h = \chi(\tilde{h})$ is self-adjoint in $C^*(\mathfrak{G}; \mathfrak{E}, \lambda)$. The point is that

$$h(i(u, 1)) = 1 \tag{4.4}$$

for all $u \in C$! Define $F \in C_c(\mathfrak{E})$ by

$$F(\gamma) = g(r(\gamma))\, g(s(\gamma))\, b(\dot{\gamma}) h(\gamma).$$

Since $h(t \cdot \gamma) = th(\gamma)$ while the other factor of F is T-invariant, it follows that $F \in C_c(\mathfrak{G}; \mathfrak{E})$. Furthermore, observe that F is self-adjoint in $C_c(\mathfrak{G}; \mathfrak{E})$. Now let $E = j^{-1}(W_0 V_0 W_0)$ and for each $u \in \mathfrak{X}$ let

$$\mathscr{E}_{u,1} = \mathscr{H}_u \cap L^2(\mathfrak{E}_u \cap E, \sigma_u)$$

$$\mathscr{E}_{u,2} = \mathscr{H}_u \cap L^2(\mathfrak{E}_u \backslash E, \sigma_u).$$

Observe that $\mathscr{E}_{u,1}$ and $\mathscr{E}_{u,2}$ are orthogonal complements in $\mathscr{H}_u$.

Now if $\xi \in \mathscr{H}_u$, then

$$\begin{aligned} L^u(F)(\xi)(\gamma) &= \int_{\mathfrak{G}} F(\gamma\alpha)\xi(\alpha^{-1})\, d\lambda^u(\dot{\alpha}) \\ &= \int_{\mathfrak{G}} \int_{\mathrm{T}} F(t \cdot \gamma\alpha)\xi(\bar{t}\alpha^{-1})\, dt\, d\lambda^u(\dot{\alpha}) \\ &= \int_{\mathfrak{E}} F(\gamma\alpha)\xi(\alpha^{-1})\, d\sigma^u(\alpha) \\ &= \int_{\mathfrak{E}} F(\gamma\alpha^{-1})\xi(\alpha)\, d\sigma_u(\alpha) \\ &= g(r(\gamma)) \int_{\mathfrak{E}} g(r(\alpha))\, b(j(\gamma\alpha^{-1}))\, h(\gamma\alpha^{-1})\xi(\alpha)\, d\sigma_u(\alpha) \end{aligned} \tag{4.5}$$

Following the argument of [7; Lemma 2.8], we see that if $\gamma \notin j^{-1}(\overline{W_1{}^7 V_1 W_1{}^7})$ and if $\alpha \in j^{-1}(W_0 V_0 W_0)$, then $b(j(\gamma\alpha^{-1})) = 0$. Using Equation (4.5), we see that if $\xi \in \mathscr{E}_{u,1}$, then

$$L^u(F)(\xi)(\gamma) = g^{(1)}(\gamma) \int_{\mathfrak{G}} g^{(1)}(\alpha) b(j(\gamma\alpha^{-1})) h(\gamma\alpha^{-1}) \xi(\alpha)\, d\sigma_u(\alpha) \tag{4.6}$$

$$= g^{(1)}(\gamma) \int_{\mathfrak{G}} g^{(1)}(\alpha) h(\gamma\alpha^{-1}) \xi(\alpha)\, d\sigma_u(\alpha),$$

since $\alpha, \gamma \in W_0 V_0 W_0$ imply that $b(j(\gamma\alpha^{-1})) = 1$.

It also follows from Equation (4.6) and the fact that $\operatorname{supp} g^{(1)} \subset E$, that $L^u(F)\mathscr{E}_{u,1} \subset \mathscr{E}_{u,1}$; since $F = F^*$, we see that $L^u(F)\mathscr{E}_{u,2} \subset \mathscr{E}_{u,2}$ (recall that $\mathscr{E}_{u,1}^{\perp} = \mathscr{E}_{u,2}$). Let P_i^u be the projection onto $\mathscr{E}_{u,i}$ ($i = 1, 2$).

LEMMA 4.4. *With F defined as above,*

$$u \mapsto \operatorname{Tr}(L^u(F * F)P_1^u),$$

is continuous at z.

PROOF. From the above discussion, it follows that $L^u(F)P_1^u$ is an integral operator T_K of the type considered in Proposition 4.1 with $K(\gamma, \eta) = g^{(1)}(\gamma) g^{(1)}(\eta) h(\gamma\eta^{-1})$. Unfortunately, there is no reason to suspect that $L^u(F)P_1^u$ is a positive operator, so we can't apply Proposition 4.1 to compute the trace. However, since $L^u(F)$ is self-adjoint and commutes with P_1^u,

$$L^u(F * F)P_1^u = L^u(F)P_1^u L^u(F)P_1^u$$

is a positive integral operator T_{K_2}, where

$$K_2(\gamma, \eta) = g^{(1)}(\gamma) g^{(1)}(\eta) H(\gamma, \eta)$$

with

$$H(\gamma, \eta) = \int_{\mathfrak{G}} g^{(1)}(\alpha)^2 h(\gamma\alpha^{-1}) h(\alpha\eta^{-1})\, d\lambda_u(\alpha).$$

Since $g^{(1)}$ and h have compact support, it is not hard to see that H is the type of kernel considered in Proposition 4.1. Consequently,

$$\operatorname{Tr}(L^u(F * F)P_1^u) = \int_{\mathfrak{G}} g^{(1)}(\gamma)^2 H(\gamma, \gamma)\, d\lambda_u(\gamma).$$

The latter is continuous in u, and the result follows.

Now as in [7; Lemma 2.9], we choose a neighborhood V_2 of z in $\mathfrak{G}$ so that $V_2 \subset V_0$, and a conditionally compact neighborhood Y of $\mathfrak{X}$ in $\mathfrak{G}$ so that

$r(Yx) \subset U$ whenever $x \in V_2$. For convenience, we assume that $\{x_n\} \subset V_2$ and $Y \subset W_0$.

LEMMA 4.5. *With F defined as above, there is a positive constant $a > 0$ so that*

$$\|L^{s(x_n)}(F * F)P_2^{s(x_n)}\| \geqq 2a.$$

PROOF. Let $\mathcal{O}_1$ be a neighborhood of $i(C \times 1)$ in $\mathfrak{E}$ so that $\operatorname{Re}(h(\gamma)) > \frac{1}{2}$ for all $\gamma \in \mathcal{O}_1$. An argument similar to [7; Lemma 2.9] shows that there is a conditionally compact neighborhood Y_0 of $\mathfrak{X}$ in $\mathfrak{G}$ so that $CY_0 \subset j(\mathcal{O}_1)$. Of course, we may assume that $Y \subset Y_0$.

Now let $\mathrm{T}_0 = \{t_i\}_{i=1}^{\infty}$ be a countable dense subset of T, and let c be a regular cross section to j as discussed in §2. If $y \in Yx_n$, then

$$j(c(x_n)c(y)^{-1}) \in j(\mathcal{O}_1).$$

Therefore there is a $t \in \mathrm{T}_0$ so that $t \cdot c(x_n)c(y)^{-1} \in \mathcal{O}_1$. Define $\zeta_n(y)$ to be t_i where $i = \min\{k : t_k \cdot c)(x_n)c(y)^{-1} \in \mathcal{O}_1\}$.

LEMMA 4.6. *For each n, ζ_n is a Borel function.*

PROOF. It is clearly suffices to show that $B_i = \zeta_n^{-1}(\{i\})$ is Borel for each i. Define

$$A_i = \{\eta \in \mathfrak{E} : t_i \cdot c(x_n)\eta^{-1} \in \mathcal{O}_1\}.$$

Since A_i is open, $c^{-1}(j(A_i))$ is Borel. The lemma follow as

$$B_1 = c^{-1}(j(A_1)), \text{ and}$$

$$B_k = c^{-1}(j(A_k)) \setminus \bigcup_{i=1}^{k-1} B_i.$$

Now observe that if we define

$$D_n = \{(\zeta_n(y)c(x_n)c(y)^{-1} : y \in Yx_n\},$$

then $\bigcup_{n=1}^{\infty} D_n \subset \mathcal{O}_1$ and $\bigcup_{n=1}^{\infty} D_n \subset j^{-1}(CY)$. Since the last set is compact in $\mathfrak{E}$, the closure of $\bigcup_{n=1}^{\infty} D_n$ is compact and contained in $\overline{\mathcal{O}_1}$. Let $\mathcal{O}_2$ be the neighborhood of $i(C \times 1)$ containing those $\gamma \in \mathfrak{E}$ such that $\operatorname{Re}(h(\gamma)) > \frac{1}{4}$. Of course, $\overline{\mathcal{O}_1} \subset \mathcal{O}_2$. As above, there is a conditionally compact neighborhood $\tilde{Y}_1$ of $\mathfrak{E}^0$ in $\mathfrak{E}$ so that *for all n*,

$$\tilde{Y}_1\zeta_n(y)c(x_n)c(y)^{-1} \subset \mathcal{O}_2$$

for all $y \in Yx_n$. Since $j(\tilde{Y}_1)$ is a conditionally compact neighborhood of $\mathfrak{X}$ in $\mathfrak{G}$, we may assume that $j(\tilde{Y}_1) = Y$.

Let f_n be the characteristic function of Yx_n, and define

$$\xi_n(y) = t(y)\zeta_n(\dot{y})f_n(\dot{y}).$$

where t is defined as before. Then $\xi_n \in \mathfrak{H}_u$ (since $t(s\cdot y) = s\cdot t(y)$ for all $s \in \mathrm{T}$ and $y \in \mathfrak{E}$, and since ζ_n is measurable by Lemma 4.6). In fact, we may choose N so that $\{x_n\}_{n \geqq N}$ is disjoint from the compact set $\overline{W_1}^2\overline{V_1}\,\overline{W_1}^2$. Notice that if $n \geqq N$ and $x \in Yx_n$, then $x \notin W_0V_0W_0$ (since $Y \subset W_0$). Therefore, $\xi_n \in \mathscr{E}_{u,2}$ whenever $n \geqq N$. In the sequel, we'll assume that $N = 1$.

Notice next that if $\gamma, \eta \in j^{-1}(Yx_n)$, then $j(\gamma\eta^{-1}) \subset Yx_nx_n^{-1}Y \subset YV_0Y \subset W_0V_0W_0$. Thus, $b(j(\gamma\eta^{-1})) = 1$. We compute, using Equation (4.5), that if $\gamma \in \tilde{Y}_1c(x_n)$, then

$$L^{s(x_n)}(F)(\xi_n)(\gamma) = g(r(\gamma))\int_{\mathfrak{G}} g(r(\alpha))\,b(j(\gamma\alpha^{-1}))\,h(\gamma\alpha^{-1})\xi_n(\alpha)\,d\lambda_{s(x_n)}(\dot{\alpha}) \tag{4.7}$$

$$= g(r(\gamma))\int_{Yx_n} g(r(y))\,h(\gamma c(y)^{-1})\,\xi_n(c(y))\,d\lambda_{s(x_n)}(y),$$

which, since $\gamma \in \tilde{Y}_1c(x_n)$ and $y \in Yx_n$ implies that both $r(\gamma), r(y) \in U$,

$$= \int_{Yx_n} h(\gamma c(y)^{-1})\,\xi_n(c(y))\,d\lambda_{s(x_n)}(y),$$

and since $t(c(y)) = 1$,

$$= \int_{Yx_n} h(\gamma\zeta_n(y)c(y)^{-1})\,d\lambda_{s(x_n)}(y).$$

Our constructions imply that, if $\gamma \in \tilde{Y}_1c(x_n)$, then for all $y \in Yx_n$,

$$\mathrm{Re}(h(\gamma\zeta_n(y)c(y)^{-1})) > \tfrac{1}{4}.$$

Thus, $\gamma \in \tilde{Y}_1c(x_n)$ implies that

$$\mathrm{Re}(L^{s(x_n)}(F)(\xi_n)(\gamma)) \geqq \tfrac{1}{4}\lambda_{s(x_n)}(Yx_n)$$
$$= \tfrac{1}{4}\lambda_{r(x_n)}(Y).$$

Hence,

$$\|L^{s(x_n)}(F)\xi_n\|^2 = \int_{\mathfrak{G}} |L^{s(x_n)}(F)(\xi_n)(\gamma)|^2 \, d\lambda_{s(x_n)}(\dot\gamma)$$
$$\geqq \int_{Yx_n} \tfrac{1}{16}\lambda_{r(x_n)}(Y)^2 \, d\lambda_{s(x_n)}(\dot\gamma)$$
$$= \tfrac{1}{16}\lambda_{r(x_n)}(Y)^3.$$

Since $\|\xi_n\|^2 = \lambda_{r(x_n)}(Y)$ and $\xi_n \in \mathscr{E}_{s(x_n),2}$, $\|L^{s(x_n)}(F)P_2^{s(x_n)}\| \geqq \frac{1}{4}\lambda_{r(x_n)}(Y)$. Since $r(x_n) \to z$, it follows, as in the argument at the end of the proof of Theorem 2.3 in [7], that $\lambda_{r(x_n)}(Y)$ is bounded away from zero. This concludes the proof of Lemma 4.5.

PROOF OF THEOREM 4.3. Define $q:(-\infty,\infty) \to [0,\infty)$ by

$$q(t) = \begin{cases} 0 & \text{if } t \leqq a, \\ 2(t-a) & \text{if } a \leqq t \leqq 2a, \\ t & \text{if } 2a \leqq t. \end{cases}$$

Since $F * F$ is positive, so is $L^{s(x_n)}(F * F)P_2^{s(x_n)}$. Thus, by Lemma 4.5, $L^{s(x_n)}(F * F)P_2^{s(x_n)}$ has an eigenvalue at least as large as $2a$, and so $q(L^{s(x_n)}(F * F)P_2^{s(x_n)})$ is positive with norm at least $2a$. Since $L^{s(x_n)}(F * F)P_i^{s(x_n)} = P_i^{s(x_n)}L^{s(x_n)}(F * F)P_i^{s(x_n)}$ for $i = 1, 2$, we have

$$q(L^{s(x_n)}(F * F)P_i^{s(x_n)}) = q(L^{s(x_n)}(F * F))P_i^{s(x_n)} = L^{s(x_n)}(q(F * F))P_i^{s(x_n)},$$

because q can be approximated uniformly by polynomials p with $p(0) = 0$. A similar argument shows that

$$q(L^{s(x_n)}(F * F)) = q(L^{s(x_n)}(F * F)(P_1^{s(x_n)} + P_2^{s(x_n)}))$$
$$= q(L^{s(x_n)}(F * F))P_1^{s(x_n)} + q(L^{s(x_n)}(F * F))P_2^{s(x_n)}.$$

In view of Lemma 4.4, it follows that $u \mapsto \mathrm{Tr}(L^u(q(F * F))P_1^u)$ is continuous at z. In particular, $\mathrm{Tr}(L^{s(x_n)}(q(F * F))P_1^{s(x_n)})$ converges to $\mathrm{Tr}(L^z(q(F * F))P_1^z) = \mathrm{Tr}(L^z(q(F * F))$. (The last equality follows from the fact that $g(r(\gamma)) = g^{(1)}(\gamma)$ provided $\gamma \in \mathfrak{E}_z$.) On the other hand, it follows from Lemma 4.5 that

$$\mathrm{Tr}(q(L^{s(x_n)}(F * F)P_2^{s(x_n)})) = \mathrm{Tr}(L^{s(x_n)}(q(F * F))\, P_2^{s(x_n)}) \geqq 2a > 0.$$

Since

$$\mathrm{Tr}(L^{s(x_n)}(q(F * F))) = \mathrm{Tr}(L^{s(x_n)}(q(F * F))\, P_1^{s(x_n)}) + \mathrm{Tr}(L^{s(x_n)}(q(F * F))\, P_2^{s(x_n)}),$$

it follows that $\mathrm{Tr}(L^{s(x_n)}(q(F * F)))$ does not converge to $\mathrm{Tr}(L^z(q(F * F)))$. But $q(F * F)$ is a positive element of the Pedersen ideal [8; p. 134]; thus, $C^*(\mathfrak{G}; \mathfrak{E}, \lambda)$ cannot have continuous trace. This completes the proof of Theorem 4.3, as well as the the proof of our Main Theorem.

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