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L^2-INTEGRABILITY OF SECOND ORDER DERIVATIVES FOR POISSON'S EQUATION IN NONSMOOTH DOMAINS

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Abstract.

We define a certain class of domains with corners directed outwards only, thus being natural extensions of convex domains. We show that such a domain Ω can be approximated with smooth domains Ω_m of the same type. Using a technique based on integration by parts we derive an *a priori* estimate

$$\|u\|_{H^2(\Omega_m)} \leqq C(\Omega)\,\|\Delta u\|_{L^2(\Omega_m)} \quad for \quad u \in H^2(\Omega_m) \cap H_0^1(\Omega_m)$$

where $C(\Omega)$ is independent of m. This enables us to obtain a solution u in $H^2(\Omega)$ of the Dirichlet problem

$$\begin{cases} \Delta u = f \in L^2(\Omega) \\ \gamma u \;= 0. \end{cases}$$

Here γ is the trace operator on the boundary of Ω.

1. Introduction.

The purpose of this paper is to prove that the Dirichlet problem for Poisson's equation has a solution $u \in H^2(\Omega)$ for certain non-smooth domains Ω. The work in this area can provisionally be divided into two groups, see [9] and [5] and the references therein. In the first group attention is focused on global smoothness conditions on the boundary; in the other group the singularities are localized, and one considers a finite number of singularities on the boundary, such as edges, polyhedral angles, conical points, etc. It is desirable to treat not only a finite number of singularities, but to give a global smoothness condition in the spirit of the first group of works, allowing for not necessarily localized singularities of the type mentioned in the other group. This paper gives such a suitable definition of domains and proves that the second order derivatives, as mentioned above, are in L^2 for these domains. The domains we consider here are natural extensions of convex domains. The main resut is the following.

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THEOREM 1.1. *Suppose Ω is a strongly Lipschitz domain in R^n of finite width. If Ω satisfies an outer ball condition of uniform radius, then the unique solution $u \in H_0^1(\Omega)$ of the Dirichlet problem*

$$\begin{cases} \Delta u = f \in L^2(\Omega) \\ \gamma u = 0 \text{ on } \partial\Omega, \end{cases}$$

has all its second order derivatives in $L^2(\Omega)$, i.e. $u \in H^2(\Omega)$.

Here the Sobolev space $H^m(\Omega)$ is, as usual, the space of functions in $L^2(\Omega)$ with distributional derivatives of order up to m, in $L^2(\Omega)$. This space is equipped with the Hilbert-space norm $\|\cdot\|_{H^m(\Omega)}$ given by

$$\|u\|_{H^m(\Omega)} = \{ \sum_{0 \leq |\alpha| \leq m} \|D^\alpha u\|^2_{L^2(\Omega)} \}^{\frac{1}{2}}.$$

By $H_0^m(\Omega)$ we denote the completion of $C_0^\infty(\Omega)$ with respect to the above norm. For more facts concerning Sobolev spaces, see [1] and [5]. Here the "outer ball condition" means the following.

DEFINITION 1.2. An open set Ω in R^n is said to satisfy an *outer ball condition* if for each $p \in \partial\Omega$ there exists an open ball B in R^n such that $B \subset \Omega^c$ and $p \in \partial B$. Such a ball "tangent to $\partial\Omega$" will also be called an outer *supporting* ball at $p \in \partial\Omega$. The ball conditions is said to be of *uniform radius* $R > 0$ if the radius of the ball B can be taken to be R for each $p \in \partial\Omega$.

Finite width has the following natural meaning.

DEFINITION 1.3. A set in R^n is said to be of *finite width* if it lies between two parallel hyperplanes.

An open subset Ω of R^n is said to be Lipschitz if its boundary is locally given as a Lipschitz function. That is, for every $x \in \partial\Omega$ there is a rectangular neighborhood V of x in R^n and a, with the usual coordinate system, isometric coordinate system $\{y_1, \ldots, y_n\}$ such that $V = \{(y_1, \ldots, y_n)\colon -a_j < y_j < a_j,\ 1 \leqq j \leqq n\}$ and fulfilling the following properties. For every $y' = (y_1, \ldots, y_{n-1}) \in V'$, $|\varphi(y')| \leqq a_n/2$, $\Omega \cap V = \{y = (y', y_n) \in V\colon y_n < \varphi(y')\}$, and $\partial\Omega \cap V = \{y = (y', y_n) \in V\colon y_n = \varphi(y')\}$. Here V' is the projection of V onto the first $n-1$ coordinates. "Strongly Lipschitz" is just a requirement of uniformity of the Lipschitz properties and reduces to an ordinary Lipschitz-condition in the case of a bounded domain. We will just indicate this extension here and concentrate on the bounded case. We refer to [2] for a complete treatment. (See 5.3 below for the definition.) Theorem 1.1 includes the known cases for bounded domains being either C^2 or convex. As can be seen from the considerations below the main requirement for obtaining a solution in H^2 seems to be that all singularities on the boundary are directed out from the domain. In this paper we prove that this is indeed the case under the additional condition that the boundary is Lipschitz, i.e. the singularities are corners or edges. Our condition for having the corners directed out from the domain is formulated by an outer ball condition.

The proof of Theorem 1.1 rests on a uniform estimate of the second order derivatives of solutions u_m of the same problem in smooth approximating domains, and the standard weak convergence procedure inferred from this uniform estimate. The estimate is obtained from the following formula derived, basically, by the use of Green's formula twice. The formula dates back to Caccioppoli [3] amongst others. We have,

$$(1) \qquad \sum_{i,j=1}^{n} \int_{\Omega_m} \left(\frac{\partial^2 u_m}{\partial x_i \partial x_j} \right)^2 dx = \int_{\Omega_m} (\Delta u_m)^2 \, dx + \int_{\partial \Omega_m} \operatorname{tr} \mathfrak{B}_m \left(\frac{\partial u_m}{\partial \nu} \right)^2 d\sigma$$

where $\operatorname{tr} \mathfrak{B}_m$ is the trace of the second fundamental quadratic form on $\partial\Omega_m$, i.e. the mean curvature. Hence, the uniform estimate comes down to an estimate of the boundary integral in the previous formula. The here indicated method of proof was first exploited by Kadlec for the Dirichlet problem of a convex domain [7]. Extensions of this result to other boundary conditions are due to Grisvard and Iooss [6]. For more elaborate estimates of (1) in a slightly different direction, see [9] and the very nice results on $C^{1,\alpha}$ domains, i.e. [8], [10] and [11].

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2. Frame results.

In this section we present the technique on which the subsequent sections relies. Since the H^2-estimate depends heavily on the mean curvature of the boundary, we give a brief review of one of its definitions.

Letting Ω be a bounded open subset of R^n with a C^2 boundary, we can define its second fundamental quadratic form denoted $\mathfrak{B}$. Let $p \in \partial\Omega$. Then at p, $\mathfrak{B}(p)$, or just $\mathfrak{B}$, is the bilinear form

$$\begin{aligned} \mathfrak{B}(\zeta, \eta) &= - \sum_{i,k=1}^{n-1} \left\langle \frac{\partial N}{\partial s_i}, \Theta_k \right\rangle \zeta_i \eta_k \\ &= - \left\langle \frac{\partial N}{\partial \zeta}, \eta \right\rangle, \end{aligned}$$

where ζ, η are tangent vectors to $\partial\Omega$ at p with components $\{\zeta_1, \ldots, \zeta_{n-1}\}$, $\{\eta_1, \ldots, \eta_{n-1}\}$ in the basis $\{\Theta_1, \ldots, \Theta_{n-1}\}$ furnished by the tangent vectors at p of $n-1$ curves $\Lambda_1, \ldots, \Lambda_{n-1}$ passing through p and being orthogonal there. Also,

$s_1, \ldots, s_{n-1}$ are the arclengths along $\Lambda_1, \ldots, \Lambda_{n-1}$, N is the outer unit normal, and $\langle \cdot, \cdot \rangle$ denotes the scalar product. The trace of this form is

$$\operatorname{tr} \mathfrak{B} = -\sum_{i=1}^{n-1} \left\langle \frac{\partial N}{\partial s_i}, \Theta_i \right\rangle.$$

Let v be any vectorfield on $\partial\Omega$. We denote by v_n the component of v in the direction of the unit outer normal n, whereas v_T denotes the projection of v on the tangent hyperplane to $\partial\Omega$. In other words,

$$v_n = \langle v, n \rangle \qquad \text{and} \qquad v_T = v - v_n n.$$

In the same way ∇_T is the projection of the gradient operator on the tangent hyperplane,

$$\nabla_T u = \nabla u - \frac{\partial u}{\partial n} n.$$

THEOREM 2.1. (Theorem 3.1.1.1, [5]). *Let Ω be a bounded open subset of R^n with a C^2 boundary and assume $v \in H^1(\Omega)^n$. Then*

$$\begin{aligned}\int_\Omega |\operatorname{div} v|^2\, dx - \sum_{i,j=1}^{n} \int_\Omega \frac{\partial v_i}{\partial x_j} \frac{\partial v_j}{\partial x_i}\, dx = \\ -2\langle (\gamma_v)_T, \nabla_T \langle \gamma_v, n \rangle \rangle \\ - \int_{\partial\Omega} \{\mathfrak{B}((\gamma v)_T, (\gamma v)_T) + \operatorname{tr} \mathfrak{B}) \langle (\gamma v), n \rangle^2\}\, d\sigma,\end{aligned}$$

where γ is the restriction operator to the boundary $\partial\Omega$.

We can now formulate the preliminary version of the main theorem under the hypothesis of the existence of suitable approximating domains.

THEOREM 2.2. *Let Ω be an open bounded set of width w in R^n with a Lipschitz boundary. Suppose there is a sequence of bounded open sets $\{\Omega_m\}_{m \geqq 1}$ such that $\Omega_m \uparrow \Omega$ and $\partial\Omega_m$ is C^∞. Further, assume that there is a $\delta > 0$ and a $\mu \in C^\infty(\bar{\Omega})^n$ such that $\langle \mu, n_m \rangle \geqq \delta$ on $\delta\Omega_m$. Here n_m is the outer unit normal on $\partial\Omega_m$. Then, if there is a constant c such that $\operatorname{tr} \mathfrak{B}_m \leqq c$ on the boundary $\partial\Omega_m$ for every m, then there exists for each $f \in L^2(\Omega)$ a unique $u \in H_0^1(\Omega)$ being the solution of $\Delta u = f$ in Ω. Furthermore, $u \in H^2(\Omega)$ and*

$$\|u\|_{H^2(\Omega)} \leqq C \|f\|_{L^2(\Omega)}$$

with $C = C(n, c, w, K)$, where $K = 2\delta^{-1} \|\mu\|_{C^1(\bar{\Omega})}$.

PROOF. Since $u \in H_0^1(\Omega_m)$, a standard Poincaré inequality, (see [1, p. 158]), together with an integration by parts give

$$\|u\|_{L^2(\Omega_m)} \leqq \frac{w^2}{2} \|\Delta u\|_{L^2(\Omega_m)} \quad \text{and} \quad \sum_{i=1}^{n} \int_{\Omega_m} \left| \frac{\partial u}{\partial x_i} \right|^2 dx \leqq \frac{w^2}{2} \|\Delta u\|^2_{L^2(\Omega_m)}.$$

The standard estimate,

$$\int_{\partial\Omega_m} |\gamma u|^2 \, d\sigma \leqq K \left\{ \lambda^{1/2} \int_{\Omega_m} |\nabla u|^2 \, dx + \lambda^{-1/2} \int_{\Omega_m} |u|^2 \, dx \right\}, \tag{2}$$

holds for all $u \in H^1(\Omega_m)$ and $\lambda \in (0, 1)$. See e.g. [5, Chap. 1].

Claim: There is a constant C such that for all $m \geqq 1$

$$\|u\|_{H^2(\Omega_m)} \leqq C \|\Delta u\|_{L^2(\Omega_m)}$$

for every $u \in H^2(\Omega_m) \cap H_0^1(\Omega_m)$ and $C = C(c, w, K)$. For the proof of this we apply Theorem 2.1 to $v = \nabla u$ observing that since $\gamma u = 0$ on $\partial\Omega_m$ we also have $(\gamma v)_T = \gamma \nabla_T u = 0$ on $\partial\Omega_m$. Consequently, if there is a constant c independent of m such that $\operatorname{tr} \mathfrak{B}_m \leqq c$ for all points on the boundary we have an upper bound

$$\sum_{i,j=1}^{n} \int_{\Omega_m} \left| \frac{\partial^2 u}{\partial x_i dx_j} \right|^2 dx \leqq \int_{\Omega_m} |\Delta u|^2 \, dx + c \int_{\partial\Omega_m} |\gamma(\nabla u)|^2 \, d\sigma.$$

Using (2) with $\lambda = (2cK)^{-2}$ we get an estimate of the L^2 norm of the second order derivatives in terms of the L^2 norm of the Laplacian. Thus, $\|u\|_{H^2(\Omega_m)} \leq C \|\Delta u\|_{L^2(\Omega_m)}$ where $C = \left\{ 2(1 + c^2K^2w^2) + \frac{w^2}{2} + \frac{w^4}{4} \right\}$.

Having proved the claim, we turn to the actual proof of the theorem. Let f be given and let $u_m \in H^2(\Omega_m)$ be the solution of the same problem as u but in Ω_m. We know that such a solution exists since the boundary of Ω_m is C^∞. From *Claim* follows by a standard weak convergence argument, the existence of a solution u with the desired properties. See [5, Chap. 3]. The estimate of the theorem follows from a straightforward calculation using that

$$\|u\|_X^2 = \langle u, u \rangle_X = \langle u - u_m, u \rangle_X + \langle u_m, u \rangle_X \leqq \varepsilon + \|u_m\|_X \|u\|_X, \quad \forall \varepsilon > 0$$

if m big enough, where X is H^1 and L^2, respectively.

3. Approximating domains.

In this section the objective is to give an approximation procedure for a particularly simple domain.

DEFINITION 3.1. Let r and θ be polar coordinates for R^n, $0 \leqq r < \infty$, $\theta \in S^{n-1}$ (the unit sphere in R^n). A domain D in R^n is a *starlike domain* (with respect to the origin) if there exists a function $\beta: S^{n-1} \to R$, β continuous and strictly positive, so that $D = \{r\theta: 0 \leqq r < \beta(\theta), \theta \in S^{n-1}\}$. If D is a starlike domain in R^n and there is a constant M such that $|\beta(\theta_2) - \beta(\theta_1)| \leqq M|\theta_2 - \theta_1|$ for all $\theta_1, \theta_2 \in S^{n-1}$, we say that D is a *starlike Lipschitz domain with a Lipschitz constant bounded by M*. Here $|\cdot|$ denotes the Euclidean distance in R^n.

The following notational conventions will be convenient.

NOTATIONS 3.2. We denote by U the upper hemisphere of S^{n-1}, i.e. $U = \{\theta \in S^{n-1}: \theta_n > 0\}$, and we let $\phi: B'_r(0) \to U \subset S^{n-1}$ be the C^∞ diffeomorphism onto an open subset of U defined by $\phi(\xi) = (\xi, \sqrt{1 - |\xi|^2})$. Here $B'_r(0)$ is the open ball of radius r in R^{n-1} centered at the origin. Let $\theta_0 = (0, \ldots, 0, 1) \in S^{n-1}$, the north-pole. Further, we put

$$\rho_0 = \min_{\theta \in S^{n-1}} \beta(\theta) \qquad \rho_1 = \max_{\theta \in S^{n-1}} \beta(\theta)$$

The following lemma contains some easily proved consequences of the fact that β is Lipschitz.

LEMMA 3.3. *Let D be a starlike Lipschitz domain with a Lipschitz constant bounded by M. Then*

a) *D is $C^{0,1}$.*

b) *For almost every $\theta \in S^{n-1}$ there exists an outer unit normal to ∂D at $\beta(\theta)\theta$. For such a $\theta \in U$, the upper hemisphere, the outer unit normal $N(x)$ to ∂D at $x = \beta(\theta)\theta$ is given by $N(x) = \dfrac{n(x)}{|n(x)|}$ where*

$$n(x) = \frac{1}{|x|}\left\{1 + \frac{1}{|x|^2}\left\langle \nabla(\beta \circ \phi)\left(\phi^{-1}\left(\frac{x}{|x|}\right)\right), x'\right\rangle\right\} x - \frac{1}{|x|}\left(\nabla(\beta \circ \phi)\left(\phi^{-1}\left(\frac{x}{|x|}\right)\right), 0\right)$$

and

$$|n(x)|^2 = 1 - \frac{1}{|x|^4}\left\langle \nabla(\beta \circ \phi)\left(\phi^{-1}\left(\frac{x}{|x|}\right)\right), x'\right\rangle^2 + \frac{1}{|x|^2}\left|\nabla(\beta \circ \phi)\left(\phi^{-1}\left(\frac{x}{|x|}\right)\right)\right|^2.$$

c) *If there exists a normal at the point $x \in \partial D$, then*

$$\alpha_0 \equiv \frac{1}{\sqrt{1 + \frac{(n-1)M^2}{\rho_0^2}}} \leqq \left\langle N(x), \frac{x}{|x|} \right\rangle.$$

We next construct the approximating smooth domains needed to be able to compute the second fundamental quadratic form (i.e. the curvature). First, however, we introduce some more notation.

NOTATIONS 3.4. Let $W = \{\phi(x'): x' \in B_r'(0)\} \subset U$ where U and ϕ are as in Notations 3.2 and $r \in (0, 1)$ is a real number which will be specified later.

Letting, for each $\theta \in S^{n-1}$, R_θ be a ON-transformation that rotates the north-pole θ_0 onto θ we define $\phi_\theta: B_r'(0) \to W_\theta \subset S^{n-1}$ by $\phi_\theta(\xi) = (R_\theta \circ \phi)(\xi)$ where $W_\theta = \phi_\theta(B_r'(0))$. Thus ϕ_θ is a difffeomorphism of $B_r'(0)$ onto W_θ. Due to the compactness of S^{n-1}, we can cover S^{n-1} by a finite set $\{W_\nu\}$ of these W_θ. Denote by $\{q_\nu\}$ a partition of unity on S^{n-1} subordinate to $\{W_\nu\}$ so that $q_\nu \in C_0^\infty(W_\nu)$, $0 \leqq q_\nu \leqq 1$ and $\sum_\nu q_\nu = 1$. Since the support of q_ν is compactly contained in $\{W_\nu\}$, $\phi_\nu^{-1}(\text{supp}(q_\nu))$ is a compact set contained in $B_{r_\nu}'(0)$ for some $0 < r_\nu < r$. Set $l_\nu = r - r_\nu$ and $l = \min_\nu l_\nu$. Define

$$G_\nu: B_r'(0) \to R$$

by

$$G_\nu(\xi) = ((q_\nu \beta) \circ \phi_\nu)(\xi).$$

Since $\text{supp}(G_\nu) \subset B_{r_\nu}'(0) \subset B_r'(0)$ we extend G_ν to zero outside $B_r'(0)$. Extending $G_\nu \circ \phi_\nu^{-1}$ to all of S^{n-1} by letting it be zero outside W_ν yields, for $x \in S^{n-1}$,

$$\beta(x) = \sum_\nu q_\nu(x)\beta(x) = \sum_\nu (G_\nu \circ \phi_\nu^{-1})(x).$$

Further, let $\eta \in C_0^\infty(R^{n-1})$ with

$$\text{supp}(\eta) \subset \{\xi \in R^{n-1}: |\xi| < 1\},$$

$\eta \geqq 0$, $\int_{R^{n-1}} \eta(\xi)\, d\xi = 1$, and $\eta_\varepsilon(\xi) = \varepsilon^{-(n-1)}\eta\left(\dfrac{\xi}{\varepsilon}\right)$.

As an approximation of β we take for $x \in S^{n-1}$ and ε small enough

$$\tilde{\beta}_\varepsilon(x) = \sum_\nu (G_\nu * \eta_\varepsilon) \circ \phi_\nu^{-1}(x),$$

where we extend $(G_\nu * \eta_\varepsilon) \circ \phi_\nu^{-1}$ to all of S^{n-1} by letting it be zero outside W_ν. The $*$ denotes convolution. Using the property $\int_{R^{n-1}} \eta_\varepsilon(\xi)\, d\xi = 1$, the following lemma is easily proved; we note that the partition of unity $\{q_\nu\}$ only depends on W_ν, i.e. on r.

LEMMA 3.5. *There is a constant* c_0 *such that* $\|\beta - \tilde{\beta}_\varepsilon\|_{L^\infty(S^{n-1})} \leqq c_0\varepsilon$ *and* $c_0 = c_0(n, \rho_1, M, r)$.

We now put, for $x \in S^{n-1}$,

$$\beta_\varepsilon(x) = -c_0\varepsilon + \tilde{\beta}_\varepsilon(x)$$
$$\beta^\varepsilon(x) = c_0\varepsilon + \tilde{\beta}_\varepsilon(x),$$

and take as approximating domains

$$D_\varepsilon = \{r\theta\colon 0 \leqq r < \beta_\varepsilon(\theta)\}$$
$$D^\varepsilon = \{r\theta\colon 0 \leqq r < \beta^\varepsilon(\theta)\}.$$

The following properties are direct consequences of the approximation procedure.

LEMMA 3.6. *There are constants* $c_j = c_j(n, \rho_1, M, r)$ *for* $j = 1, 2$ *such that for all small enough* $\varepsilon > 0$:
a) $\beta_\varepsilon \in C^\infty(S^{n-1})$
b) $0 < \frac{1}{2}\rho_0 \leqq \beta_\varepsilon(\theta) \leqq \beta(\theta) \leqq \beta^\varepsilon(\theta) \leqq \frac{3}{2}\rho_1$ *for each* $\theta \in S^{n-1}$
c) $D_\varepsilon \subset D \subset D^\varepsilon$
d) $\operatorname{dist}(\partial D_\varepsilon, \partial D^\varepsilon) \leqq 2c_0\varepsilon$
e) ∂D_ε *and* ∂D^ε *are* C^∞
f) $\max_\nu \|G_\nu * \eta_\varepsilon\|_{\nu} {}_{C_\nu^1(R^{n-1})} \leqq c_1$
g) $\max_{1 \leqq k \leqq n} \|(\phi_\nu)_k\|_{C^2(B_r^\nu(0))} < (1 - r^2)^{-\frac{3}{2}}$
h) $\max_{1 \leqq k \leqq n-1}(\max_{0 \leqq |\alpha| \leqq 2} |D^\alpha(\phi_\nu^{-1} \circ \phi)_k(0)|^2) \leqq 1$ *for each* ν
i) $|\beta_\varepsilon(y) - \beta_\varepsilon(x)| \leqq c_2|y - x|$ *for* $x, y \in S^{n-1}$.

4. Estimate of tr $\mathfrak{B}_m$.

Since tr $\mathfrak{B}_\varepsilon$ is a geometrical invariant it suffices to give an estimate at just one point.

LEMMA 4.1. *Suppose* $x \in \partial D_\varepsilon$. *Let* $\varepsilon > 0$ *be fixed and let* $\dfrac{x}{|x|} = \theta_0$, *the north-pole of* S^{n-1}. *Then we have*

$$\operatorname{tr}\mathfrak{B}_\varepsilon(x) = \frac{1}{|x|^2}\left(1 + \frac{1}{|x|^2}|\nabla(\beta_\varepsilon \circ \phi)(0)|^2\right)^{-\frac{1}{2}} \sum_{i=1}^{n-1}\Bigg[\sum_{\{v:\theta_0\in W_v\}}$$

$$\Bigg\{\sum_{m,l=1}^{n-1} v_l(\varepsilon,i,v)v_m(\varepsilon,i,v)\frac{\partial^2(G_v * \eta_\varepsilon)}{\partial\vartheta_l\partial\vartheta_m}((\phi_v^{-1}\circ\phi)(0))$$

$$+ \sum_{m,j,k=1}^{n-1}(\Theta_i^\varepsilon(x))_j(\Theta_i^\varepsilon(x))_k\frac{\partial^2(\phi_v^{-1}\circ\phi)_m}{\partial\xi_j\partial\xi_k}(0)\frac{\partial(G_v * \eta_\varepsilon)}{\partial\vartheta_m}((\phi_v^{-1}\circ\phi)(0))\Bigg\}$$

$$- |x|\left(1 + \left\langle\frac{x}{|x|}, \Theta_i^\varepsilon(x)\right\rangle^2\right)\Bigg],$$

where we have denoted by $\{\vartheta_1,\ldots,\vartheta_{n-1}\}$ *the coordinates for* $G_v * \eta_\varepsilon$ *in* R^{n-1} *and* $v_m(\varepsilon,i,v) = \langle(\Theta_i^\varepsilon)'(x), \nabla(\phi^{-1v}\circ\phi)_m(0)\rangle$. *Here* $\mathfrak{B}_\varepsilon(x)$ *is the second fundamental quadratic form on* ∂D_ε *at the point* $x\in\partial D_\varepsilon$.

PROOF. We first derive an expression for $\operatorname{tr}\mathfrak{B}_\varepsilon(x)$ on ∂D_ε only involving β_ε, without any explicit reference to the boundary ∂D. Suppose now x is any point in ∂D_ε such that $\frac{x}{|x|}\in U$, the upper hemisphere. The expression for the outer unit normal $N_\varepsilon(x)$ as in Lemma 3.3 shows that the normal is defined in a neighbourhood of ∂D_ε. Take $\{x_1,\ldots,x_{n-1},x_n\}$ as coordinates in R^n and suppose that in these coordinates $p = x$ for $p\in\partial D_\varepsilon$. Then $\Lambda_i^\varepsilon(0) = x$ for $i = 1,\ldots,n-1$, and if $(N_\varepsilon)_j$ denotes the jth component of the outer unit normal, we have, with the notations taken from the introduction of Section 2, that

$$\frac{d}{ds_i}((N_\varepsilon)_j\circ\Lambda_i^\varepsilon)(0) = \sum_{k=1}^{n}\frac{\partial(N_\varepsilon)_j}{\partial x_k}(x)\frac{d}{ds_i}(\Lambda_i^\varepsilon)_k(0) = \sum_{k=1}^{n}\frac{\partial(N_\varepsilon)_j}{\partial x_k}(x)(\Theta_i^\varepsilon)_k.$$

This shows that

$$\operatorname{tr}\mathfrak{B}_\varepsilon(x) = -\sum_{i=1}^{n-1}\sum_{j,k=1}^{n}(\Theta_i^\varepsilon(x))_j(\Theta_i^\varepsilon((x))_k\frac{\partial(N_\varepsilon)_j}{\partial x_k}(x).$$

Now

$$\frac{\partial(N_\varepsilon)_j}{\partial x_k}(x) = \frac{\partial}{\partial x_k}\left(\frac{(n_\varepsilon)_j(x)}{|n_\varepsilon(x)|}\right)$$

$$= |n_\varepsilon(x)|^{-1}\frac{\partial(n_\varepsilon)_j}{\partial x_k}(x) - \frac{(n_\varepsilon)_j(x)}{|n_\varepsilon(x)|^2}\frac{\partial}{\partial x_k}(|n_\varepsilon(x)|).$$

However, $\Theta_i^\varepsilon(x)$ is in the tangent space of ∂D_ε at x so that $\langle\Theta_i^\varepsilon(x), n_\varepsilon(x)\rangle = 0$ for $1 \leq i \leq n-1$. Hence,

$$\operatorname{tr}\mathfrak{B}_\varepsilon(x) = -|n_\varepsilon(x)|^{-1}\sum_{i=1}^{n-1}\sum_{j,k=1}^{n}(\Theta_i^\varepsilon(x))_j(\Theta_i^\varepsilon(x))_k\frac{\partial(n_\varepsilon)_j}{\partial x_k}(x).$$

It is tedious but straightforward to compute $\dfrac{\partial(n_\varepsilon)_j}{\partial x_k}(x)$, to find that

$$\operatorname{tr}\mathfrak{B}_\varepsilon(x) = |n_\varepsilon(x)|^{-1}\sum_{i=1}^{n-1}$$

$$\Bigg[\Bigg[\Bigg\{\sum_{j,k=1}^{n-1}|x|^{-2}\Bigg[(\Theta_i^\varepsilon(x))_j(\Theta_i^\varepsilon(x))_k - 2\left\langle\frac{x}{|x|},\Theta_i^\varepsilon(x)\right\rangle(\Theta_i^\varepsilon(x))_j\frac{x_k}{|x|}$$

$$+\left\langle\frac{x}{|x|},\Theta_i^\varepsilon(x)\right\rangle^2\frac{x_j}{|x|}\frac{x_k}{|x|}\Bigg]\frac{\partial^2(\beta_\varepsilon\circ\phi)}{\partial\xi_j\partial\xi_k}\left(\phi^{-1}\left(\frac{x}{|x|}\right)\right)\Bigg\}$$

$$+|x|^{-2}\left(\left\langle\frac{x}{|x|},\Theta_i^\varepsilon(x)\right\rangle^2 - 1\right)\left\langle\frac{x'}{|x|},\nabla(\beta_\varepsilon\circ\phi)\left(\phi^{-1}\left(\frac{x}{|x|}\right)\right)\right\rangle$$

$$-|x|^{-1}\left(1+\left\langle\frac{x}{|x|},\Theta_i^\varepsilon(x)\right\rangle^2\right)\Bigg]\Bigg].$$

Using the expression for $\beta_\varepsilon\circ\phi$ in the above formula, yields, after letting $\dfrac{x}{|x|} = \theta_0$, the result of the lemma.

In the following, D will denote a starlike Lipschitz domain with a Lipschitz constant bounded by M satisfying a uniform outer ball condition of radius R. We aim for a uniform estimate in ε from above of $\operatorname{tr}\mathfrak{B}_\varepsilon$.

For almost every $\theta\in S^{n-1}$ there exists a normal to ∂D at the point $x = \beta(\theta)\theta\in\partial D$. Thus there must be an outer ball tangent to ∂D at x with radius R and center on the line defined by the normal. Denote by m_θ the center of this tangential ball. Let ω'_θ be the angle between m_θ and θ, or, rather, the corresponding radius vectors. Further, let l be a line through the origin that is tangent to the ball tangent at $\beta(\theta)\theta$ with center m_θ. The angle between l and m_θ is the same for all such lines l. We denote this angle by ω_θ. Any point on the tangential ball where such a line intersects we denote by q_θ. Also, for $p\in S^{n-1}$ we let $\omega_\theta(p)$ be the angle between p and m_θ, $\omega'_\theta(p)$ the angle between p and θ. Let $\alpha_1 = 1 + \dfrac{R^2(\rho_0+R)}{2(\rho_1+R)^3}\alpha_0^2$ where α_0 is the constant of Lemma 3.3. Then for almost every $\theta\in S^{n-1}$ we have $\cos\omega'_\theta \geqq \alpha_1\cos\omega_\theta > 0$. To see this, note that the boundary ∂D is Lipschitz so that it is clear that for some constant $c > 1$ we have $c\omega'_\theta < \omega_\theta$ for a.e. $\theta\in S^{n-1}$. This implies the estimate for some constant. A more precise computation gives the constant α_1. Let $\gamma\in R$ be defined by $\gamma = (1+\alpha_1)/2$. Then $1 < \gamma < \alpha_1$ and for almost every $\theta\in S^{n-1}$ there is a set $O_\theta\subset S^{n-1}$ defined by

$$O_\theta = \left\{ p \in S^{n-1} : \left\langle p, \frac{m_\theta}{|m_\theta|} \right\rangle > \gamma \cos \omega_\theta \right\},$$

where m_θ and ω_θ are given above. Since $\left\langle \theta, \frac{m_\theta}{|m_\theta|} \right\rangle = \cos \omega'_\theta \geqq \alpha_1 \cos \omega_\theta >$ $\gamma \cos \omega_\theta, \theta \in O_\theta$ and O_θ is clearly open. Thus O_θ is a neighborhood of θ in S^{n-1}. Let $a \in R$, $a > 0$, and define $U_\theta^a = \{p \in S^{n-1} : |p - \theta| < a\}$.

Lemma 4.2. *There exists a constant $a > 0$ such that for almost every $\theta \in S^{n-1}$*

$$U_\theta^a \subset O_\theta$$

and $a = a(n, \rho_0, \rho_1, M, R)$.

Proof. As in the statement for the constant α_1 above we have for some constant c that $c\omega'_\theta < \omega_\theta$ for a.e. $\theta \in S^{n-1}$. A similar estimate then naturally holds for points in a neighbourhood of these θ. Moreover, it can be seen that the size of the neighborhood can be taken the same for the θ. That is $c\omega_\theta(p) < \omega_\theta$ for a.e. $\theta \in S^{n-1}$ and p in a neighborhood of θ whose size can be taken the same for each θ. This is essentially the result of the lemma. It is not difficult to make the estimates more detailed to justify the above assertments.

We now use this constant a to fix the radius r of the ball $B'_r(0)$ being the domain of the coordinate map ϕ as in Notation 3.4. Let so $x, y \in W = \{\phi(x') : x' \in B'_r(0)\}$. Then $|x - y| \leqq 2\sqrt{2}r$ as is clear from geometrical considerations. Choose r so that $2\sqrt{2}r \leqq a$ where a is the constant obtained in Lemma 4.2. With this choice the range of the coordinate maps, W_ν, will be contained in $U^a_{\phi_\nu(\vartheta)}$ for each $\vartheta \in B'_r(0)$ and this will enable us to give an uniform estimate in ε of the term involving second order derivatives in $\operatorname{tr} \mathfrak{B}_\varepsilon$. Recalling Notation 3.4 we have $l = \min_\nu l_\nu$, $l_\nu = r - r_\nu > 0$ where r_ν is the radius of the ball $B'_{r_\nu}(0)$ containing the support of G_ν.

Lemma 4.3. *Let $v \in R^{n-1}$. There exists a constant $K_0 = K_0(n, \rho_0, \rho_1, M, R)$ such that for each ν we have that for almost every $\vartheta \in B'_{r_\nu + \frac{1}{3}l_\nu}(0)$*

$$(\beta \circ \phi_\nu)(\vartheta + tv) + (\beta \circ \phi_\nu)(\vartheta - tv) - 2(\beta \circ \phi_\nu)(\vartheta) \leqq K_0 |v|^2 t^2$$

for all $t \in R$ with $|t| < \frac{1}{3}\frac{1}{|v|}$.

Proof. Using the symbols above we have that $W_\nu \subset U^a_{\phi_\nu(\vartheta)}$ for each $\vartheta \in B'_r(0)$ since if $p \in W_\nu$ we have $|p - \phi_\nu(\vartheta)| < 2\sqrt{2}r < a$. Also, while diffeomorphisms preserve the zero sets, Lemma 4.2 ensures that $U^a_{\phi_\nu(\vartheta)} \subset O_{\phi_\nu(\vartheta)}$ for almost every $\vartheta \in B'_r(0)$. Let ν be fixed but arbitrary. For $|t| < \frac{1}{3}\frac{1}{|v|}$ and $\vartheta \in B'_{r_\nu + \frac{1}{3}l_\nu}(0)$ we have

$|\vartheta + tv| \leqq r_v + (l_v/3) + (l/3\,|v|)|v| \leqq r_v + (2l_v/3)$ so that $\vartheta + tv \in B'_r(0)$ and $\phi_v(\vartheta + tv) \in W_v \subset U^a_{\phi_v(\vartheta)} \subset O_{\phi_v(\vartheta)}$.

Consider the line through $\phi_v(\vartheta + tv)$ from the origin. The supporting outer ball at $\beta(\phi_v(\vartheta))\phi_v(\vartheta)$ will be intersected by this line since $\phi_v(\vartheta + tv) \in O_{\phi_v(\vartheta)}$. So there is a $s \in R$ depending on t such that $s = s(t) =$ *distance from the origin to the point on the supporting outer ball.* Noting that $s(t)\phi_v(\vartheta + tv)$ lies on the supporting outer ball gives $|s(t)\phi_v(\vartheta + tv) - m_{\phi_v(\vartheta)}|^2 = R^2$ so that

$$s(t) = \langle \phi_v(\vartheta + tv), m_{\phi_v(\vartheta)} \rangle - \{\langle \phi_v(\vartheta + tv), m_{\phi_v(\vartheta)} \rangle^2 + R^2 - |m_{\phi_v(\vartheta)}|^2\}^{\frac{1}{2}}$$

where $m_{\phi_v(\vartheta)}$ is the center of the supporting outer ball at $\beta(\phi_v(\vartheta))\phi_v(\vartheta)$. The second derivative is given by

$$\begin{aligned} s''(t) = &- (R^2 - |m_{\phi_v(\vartheta)}|^2)(\varepsilon^{\vartheta}_{v,v}(t))^{-\frac{3}{2}} \\ &\times \left\{ \sum_{k=1}^{n} \sum_{l=1}^{n-1} v_l (m_{\phi_v(\vartheta)})_k \frac{\partial(\phi_v)_k}{\partial \vartheta_l}(\vartheta + tv) \right\}^2 \\ &+ [1 - \langle \phi_v(\vartheta + tv), m_{\phi_v(\vartheta)} \rangle (e^{\vartheta}_{v,v}(t))^{-\frac{1}{2}}] \\ &\times \left\{ \sum_{k=1}^{n} \sum_{l,m=1}^{n-1} v_l v_m (m_{\phi_v(\vartheta)})_k \frac{\partial^2(\phi_v)_k}{\partial \vartheta_l \partial \vartheta_m}(\vartheta + tv) \right\}. \end{aligned}$$

where $e^{\vartheta}_{v,v}(t) \equiv \langle \phi_v(\vartheta + tv), m_{\phi_v(\vartheta)} \rangle^2 + R^2 - |m_{\phi_v(\vartheta)}|^2$. Now $\phi_v(\vartheta + tv) \in O_{\phi_v(\vartheta)}$ so that by the definition of $O_{\phi_v(\vartheta)}$ we have

$$(\gamma\,|m_{v_v(\vartheta)}| \cos \omega_{\phi_v(\vartheta)})^2 \leqq \langle \phi_v(\vartheta + tv), m_{\phi_v(\vartheta)} \rangle^2.$$

It is a consequence of this and the fact $(R^2 - |m_{\phi_v(\vartheta)}|^2) = -(|m_{\phi_v(\vartheta)}| \cos \omega_{\phi_v(\vartheta)})^2$, that the estimate $e^{\vartheta}_{v,v}(t) \geqq (\gamma^2 - 1)\rho_0^2$ holds true for all v, almost every $\vartheta \in B'_{r_v + \frac{1}{3}l_v}(0)$ and $t \in R$ with $|t| < \frac{1}{3}\frac{l}{|v|}$. This shows that $s \in C^\infty\left(\left\{t: |t| < \frac{1}{3}\frac{l}{|v|}\right\}\right)$. From this last estimate, the estimate for $\max_{1 \leqq k \leqq n} \|(\phi_v)_k\|_{C^2(B'_r(0))}$ in Lemma 3.6 and the choice of the constant a, we obtain $|s''(t)| \leqq K_0\,|v|^2$ if $|t| < \frac{1}{3}\frac{l}{|v|}$, for some constant K_0 only depending on n, ρ_0, ρ_1, M and R. By the above expansion

$$(\beta \circ \phi_v)(\vartheta + tv) \leqq s(t) \leqq (\beta \circ \phi_v)(\vartheta) + s'(0)t + \tfrac{1}{2}K_0\,|v|^2\,t^2,$$

since $s(0) = (\beta \circ \phi_v)(\vartheta)$ in view of the fact that the outer ball is tangent to $\partial\Omega$ at $\beta(\phi_v(\vartheta))\phi_v(\vartheta)$. The lemma follows by summing up for t and $-t$.

LEMMA 4.4. *Let $v \in R^{n-1}$. There is a constant $K_1 = K_1(n, \rho_0, \rho_1, M, R)$ such that for each v we have for almost every $\vartheta \in B'_{r_v + \frac{1}{3}l_v}(0)$ and for all $t \in R$ with $|t| < \frac{1}{3}\frac{l}{|v|}$ that*

$$G_v(\vartheta + tv) + G_v(\vartheta - tv) - 2G_v(\vartheta) \leqq K_1\,|v|^2\,t^2.$$

PROOF. By definition $G_\nu(\vartheta) = (q_\nu \circ \phi_\nu)(\vartheta)(\beta \circ \phi_\nu)(\vartheta)$. Rewriting the symmetric difference we get

$$
\begin{aligned}
&G_\nu(\vartheta + tv) + G_\nu(\vartheta - tv) - 2G_\nu(\vartheta) \\
&\quad = (q_\nu \circ \phi_\nu)(\vartheta + tv)\{(\beta \circ \phi_\nu)(\vartheta + tv) + (\beta \circ \phi_\nu)(\vartheta - tv) - 2(\beta \circ \phi_\nu)(\vartheta)\} \\
&\quad + (\beta \circ \phi_\nu)(\vartheta - tv)\{(q_\nu \circ \phi_\nu)(\vartheta + tv) + (q_\nu \circ \phi_\nu)(\vartheta - tv) - 2(q_\nu \circ \phi_\nu)(\vartheta)\} \\
&\quad - 2\{(q_\nu \circ \phi_\nu)(\vartheta + tv) - (q_\nu \circ e\phi_\nu)(\vartheta)\}\{(\beta \circ \phi_\nu)(\vartheta - tv) - (\beta \circ \phi_\nu)(\vartheta)\}.
\end{aligned}
$$

For fixed ϑ and ν let $f(t) = (q_\nu \circ \phi_\nu)(\vartheta + tv)$. *Since* $q_\nu \in C_0^\infty(W_\nu))$, we have $f \in C_0^\infty(R)$, where $q_\nu \circ \phi_\nu$ is extended to zero outside $B_r'(0)$. Now $f''(t) = \langle v, H(q_\nu \circ \phi_\nu)(\vartheta + tv)v\rangle$ with $H(q_\nu \circ \phi_\nu)(\vartheta + \mathrm{s}v)$ as the Hessian of $q_\nu \circ \phi_\nu$. Thus, using the estimate for the second order derivatives of $q_\nu \circ \phi_\nu$ given in Lemma 3.6, it is easily sen that the first two lines in the rewrite of the symmetric difference of $G_\nu(\vartheta + tv)$ above, can be estimated by $K_1 |v|^2 t^2$ for some constant K_1. In the same way, each of the factors of the third line can be estimated by $K_1 |v| t$ for some constant K_1. The lemma follows.

LEMMA 4.5. *If $\varepsilon > 0$ is small enough and $\theta_0 \in W_\nu$, then*

$$\sum_{l,m=1}^{n-1} v_l v_m \frac{\partial^2 (G_\nu * \eta_\varepsilon)}{\partial \vartheta_m \partial \vartheta_l}((\phi_\nu^{-1} \circ \phi)(0)) \leqq K_1 |v|^2$$

for all $v \in R^{n-1}$. Here $K_1 = K_1(n, \rho_0, \rho_1, M, R)$ is the constant obtained in Lemma 4.4.

PROOF. Define $\psi_{\nu,v}^\varepsilon(t) = (G_\nu * \eta_\varepsilon)((\phi_\nu^{-1} \circ \phi)(0) + tv)$. We now claim that for sufficiently small $\varepsilon > 0$ we have that $\psi_{\nu,v}^\varepsilon \in C_0^\infty(R)$ and $(\psi_{\nu,v}^\varepsilon)''(0) \leqq K_1 |v|^2$. The claim is clearly true for $v = 0$ so assume that $v \neq 0$. That $\psi_{\nu,v}^\varepsilon \in C_0^\infty(R)$ follows from Lemma 3.6. To prove the second part of the claim we note that for $|t| < \frac{1}{3}\dfrac{l}{|v|}$ we have $\psi_{\nu,v}^\varepsilon(t) = \int_{B'_{r_\nu + l_\nu}(0)} G_\nu(\vartheta + tv)\eta_\varepsilon((\phi_\nu^{-1} \circ \phi)(0) - \vartheta)\, d\vartheta$ since $\operatorname{supp}(G_\nu) \subset B_{r_\nu}(0)$. Consequently, according to the preceding lemma, $\psi_{\nu,v}^\varepsilon(t) + \psi_{\nu,v}^\varepsilon(-t) - 2\psi_{\nu,v}^\varepsilon(0) \leqq K_1 |v|^2 t^2$. Using this estimate we derive $(\psi_{\nu,v}^\varepsilon)''(0)t^2 \leqq K_1 |v|^2 t^2 + \dfrac{2}{3!}(\max|(\psi_{\nu,v}^\varepsilon)'''|)|t|^3$ and thus $(\psi_{\nu,v}^\varepsilon)''(0) \leqq K_1 |v|^2 + \frac{2}{3}[\max|(\psi_{\nu,v}^\varepsilon)'''|]\, |t|$. Now let $t \searrow 0$ for fixed $\varepsilon > 0$ to obtain $(\psi_{\nu,v}^\varepsilon)''(0) \leqq K_1 |v|^2$. The claim is established. To obtain the result of the lemma just compute the second derivative of $\psi_{\nu,v}^\varepsilon$.

LEMMA 4.6. *There exists a constant $c > 0$ such that for each sufficiently small $\varepsilon > 0$*

$$\operatorname{tr}(\mathfrak{B}_\varepsilon) \leqq c$$

for all points on the boundary ∂D_ε and $c = c(n, \rho_0, \rho_1, M, R)$.

PROOF. For fixed but arbitrary $\varepsilon > 0$ and $1 \leqq i \leqq n-1$ let v be the vector with components $v_m = \langle (\Theta_i^\varepsilon)'(x), \quad \nabla(\phi_v^{-1} \circ \phi)_m(0) \rangle$ for $1 \leqq m \leqq n-1$. Then $|v| \leqq c\,|\Theta_i^\varepsilon(x)|$ for some constant c. Now take $x \in \partial D_\varepsilon$ with $\frac{x}{|x|} = \theta_0$, the north-pole, and use Lemma 4.5 to infer that from the formula of Lemma 4.1 it follows that $\operatorname{tr} \mathfrak{B}_\varepsilon(x) \leqq c$. Finally, since $\operatorname{tr} \mathfrak{B}_\varepsilon$ is a geometrical invariant we get the same estimate for all $p \in \partial D_\varepsilon$.

5. Solvability in H^2 of the Dirichlet problem.

We can now apply the preliminary version, Theorem 2.2, to the starlike domain D and the domains D_ε. There only remains to check the uniformity of the boundary trace estimate. This can be done in several ways. The following lemma is easily proved, see e.g. [2].

LEMMA 5.1. *There exists $\delta > 0$ and $\mu \in C^\infty(R^n)^n$ such that for all sufficiently small $\varepsilon > 0$*

$$\langle \mu, N_\varepsilon \rangle \geqq \delta, \qquad \text{a.e. } on\ \partial D_\varepsilon,$$

where N_ε is the outer unit normal on ∂D_ε and $\sigma = \sigma(n, \rho_0, \rho_1, M, R)$.

The main result in the case of a starlike domain now follows directly from Theorem 2.2, Lemma 4.6 and the foregoing lemma.

THEOREM 5.2. *Let D be a starlike Lipschitz domain with a Lipschitz constant bounded by M satisfying a uniform outer ball condition of radius R. Then there exists for each $f \in L^2(D)$ a unique $u \in H_0^1(D)$ being the solution of $\Delta u = f$ in D. Furthermore, the solution $u \in H^2(D)$ and $\|u\|_{H^2(D)} \leqq C\,\|f\|_{L^2(D)}$ with $C = C(n, \rho_0, \rho_1, M, R)$.*

Using a partition of unity together with some requirement of uniformity, one can achieve results similar to the above theorem for more general domains, for example strongly Lipschitz ones. The crucial point in maintaining the uniformity for these, in general, unbounded domains is, apart from a bound of Lipschitz constants, that the size of the rectangular neighborhoods can be taken to be the same.

DEFINITION 5.3. Let Ω be open in R^n, bounded or unbounded. We say that Ω is *strongly Lipschitz* if it is Lipschitz in the sense given in the introduction, and there are positive real numbers a' and a_n such that for all $x \in \partial\Omega$ we have $a_j(x) = a'$ for $1 \leqq j \leqq n-1$ and $a_n(x) = a_n$ for the rectangular neighborhood of x. We also require that an upper bound of the Lipschitz constant can be taken the same for all functions φ_x and that x is the origin in the new coordinates.

As easily seen bounded Lipschitz domains are special instances of the above definition. For the proof of Theorem 1.1 we refer to [2]. To avoid less cheerful details we content ourselves here with the result for a starlike Lipschitz domain, Theorem 5.2.

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