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A HOMOTOPY THEORETICAL DERIVATION OF Q MAP$(K, -)_+$

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1. Introduction.

1.1. In his calculus of functors [6] Goodwillie defines the derivative functor of a homotopy functor from the category of (un)based spaces to the category of based spaces. Throughout this note space will mean a topological space of the homotopy type of a CW-complex. Goodwillie also shows that for a certain class of functors called analytic functors, the derivative functor to a large extend determines the functor itself.

The main application of this theory has been to relate Waldhausens functor A to topological cyclic homology TC; [3]. A crucial point is Goodwillies calculation of the derivative functor of $Q\,\mathrm{Map}(K, -)_+$, [5; 2.4]. This paper gives a new proof of this result by homotopy theoretical means. In section 1 we recall Goodwillies results and explain notation, in section 2 we extend the notion of configuration spaces to apply as models for certain spaces of sections and finally in section 3 we use these models to prove Goodwillies theorem.

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1.2. Let F be a homotopy functor from unbased to based spaces. The derivative of F at X in the direction $x \in X$ is the spectrum $\partial_{(X,x)}F$ associated with the infinite loopspace

$$\underset{\underset{j}{\rightarrow}}{\mathrm{holim}}\, \Omega^j\, \mathrm{hfib}(F(X \vee_x S^j) \to F(X)).$$

Here hfib denotes the homotopy theoretical fiber. We note the formal similarity with standard calculus.

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1.3. Connected with an analytic functor is a natural number ρ called the modulus. We also say that an analytic functor of modulus ρ is ρ-analytic. For example the identity functor and Waldhausens functor A are both 1-analytic while $Q\,\mathrm{Map}(K, -)_+$ is dim K-analytic. We now make precise to what extent an analytic functor is determined by its derivative functor.

Let $\phi\colon F \to G$ be a natural transformation of ρ-analytic functors. Suppose there exists a number n such that

$$\partial_{(X,x)}\phi\colon \partial_{(X,x)}F \to \partial_{(X,x)}G$$

is an equivalence for all n-fold suspensions X. Then the induced transformation on reduced functors

$$\tilde{\phi}\colon \tilde{F}(X) \xrightarrow{\simeq} \tilde{G}(X)$$

is an equivalence provided X is ρ-connected.

We remark that in [6; 5.2] Goodwillie demands the stronger hypothesis that $\partial_{(X,x)}\phi$ is an equivalence for all based spaces (X, x). However the proof of [6; 5.2] works equally well in the category of spaces of the homotopy type of an n-fold suspension. Our hypothesis then shows that $\tilde{\phi}$ is an equivalence when X is the n-fold suspension of a ρ-connected space. Finally we can apply (the proof of) [6; 4.12] which states, that a natural transformation of ρ-analytic functors is an equivalence on ρ-connected spaces if it is one on suspensions of ρ-connected spaces, to prove that $\tilde{\phi}$ is an equivalence on ρ-connected spaces.

1.4. For N a parallizable compact d-dimensional manifold with boundary and $x \in X$ we define a space

$$E_x = \{(z,f) \in N \times \mathrm{Map}(N, X) \mid f(z) = x\}.$$

Projection onto the first coordinate defines a map $E_x \to N$, and it is an easy consequence of the tubular neighborhood theorem that this map makes E_x into a bundle over N. The fiber over $n \in N$ is the space of based maps $\mathrm{Map}(N, n; X, x)$. In this bundle we may fiberwise add a basepoint and suspend j times. The bundle obtained this way has total space

$$\Sigma^j_N(E_x \amalg N) = (S^j \times E_x) \bigcup_{* \times E_x} N$$

and the fiber over $n \in N$ *is* $\Sigma^j\mathrm{Map}(N, n; X, x)_+$.

THEOREM Goodwillie). *The derivative $\partial_{(X,x)}Q\,\mathrm{Map}(N, X)_+$ is naturally equivalent to the spectrum associated with the infinite loopspace*

$$\operatorname*{holim}_{\substack{\to \\ j}} \Omega^j\Gamma(\Sigma^j_N(\mathrm{E}_x \amalg N) \to N).$$

We will prove the theorem only in the case when X is a dim N-fold suspension. However in view of 1.3 this is not a serious limitation since it suffices for the known applications including [3].

1.5. Here we define the natural transformation in theorem 1.4. To begin with one notes that the infinite loopspace 1.1 which defines $\partial_{(X,x)}Q\,\mathrm{Map}(N,X)_+$ is naturally equivalent to the infinite loopspace

$$\underset{\substack{\rightarrow \\ j}}{\mathrm{holim}}\, \Omega^j\left(\frac{\mathrm{Map}(N, X \vee_x S^j)}{\mathrm{Map}(N,X)}\right).$$

The natural transformation 1.4 is therefore given by maps

$$\phi_j\colon \frac{\mathrm{Map}(N, X \vee_x S^j)}{\mathrm{Map}(N,X)} \to \Gamma(\Sigma^j_N(E_x \amalg N) \to N),$$

which we will now define. Let p_i be the maps

$$X \overset{p_1}{\leftarrow} X \underset{x}{\vee} S^j \overset{p_2}{\rightarrow} S^j$$

that collaps S^j and X respectively. We may compose the mapping

$$f_j\colon N \times \mathrm{Map}(N, X \underset{x}{\vee} S^j) \to S^j \times N \times \mathrm{Map}(N,X)$$

given by $f_j = (p_2 \circ \mathrm{ev}) \times pr_1 \times p_{1*}$, with the projection

$$S^j \times N \times \mathrm{Map}(N,X) \to (S^j \times N \times \mathrm{Map}(N,X)) \underset{* \times N \times \mathrm{Map}(N,X)}{\bigcup} N.$$

and then adjoin to obtain a map

$$\tilde{f}_j\colon \mathrm{Map}(N, X \underset{x}{\vee} S^j) \to \Gamma((S^j \times N \times \mathrm{Map}(N,X)) \cup N \to N).$$

We observe that $\tilde{f}_j$ factors through the inclusion

$$\Gamma((S^j \times E_x) \underset{* \times E_x}{\bigcup} N \to N) \hookrightarrow \Gamma((S^j \times N \times \mathrm{Map}(N,X)) \cup N \to N)$$

such that we have a map

$$\tilde{\phi}_j\colon \mathrm{Map}(N, X \underset{x}{\vee} S^j) \to \Gamma((S^j \times E_x) \underset{* \times E_x}{\bigcup} N \to N).$$

Finally we note that $\tilde{\phi}_j$ maps $\mathrm{Map}(N,X) \subset \mathrm{Map}(N, X \vee_x S^j)$ to the basepoint such that the required map ϕ_j is obtained. In section 3 we will prove that ϕ_j is approximately $2j$-connected, when X is a dim N-fold suspension.

1.6. In the form we have stated theorem 1.4 we demand that N is a parallizable compact manifold with boundary. However if K is a finite CW-complex we can always embed K in such a manifold N without changing the homotopy type. Such an embedding always exists with $\dim N > 2 \cdot \dim(K)$.

2. The combinatorial model.

2.1. In [1], [2] and [8] configuration space of particles in a manifold with labels in a CW complex F are considered as models for certain mapping spaces. This section extends the notion of configuration space to allow the space F to vary as the fiber of a bundle over the manifold. We obtain a model for the space of sections in a certain related bundle.

Let N be a smooth compact manifold, $N_0 \subset N$ a compact submanifold, $\pi: E \to N$ a bundle over N with a preferred section o: N $\hookrightarrow E$, and let F denote the typical fiber. The symmetric group Σ_k, k $\geqq 1$, acts on the space

$$\tilde{C}^k(\pi) = \{(e_1, \ldots, e_k) \in E \times \ldots \times E \,|\, \pi e_i \neq \pi e_j, \text{ if } i \neq j\}.$$

We let $C^k(\pi)$ denote the orbitspace and set $C^0(\pi) = *$. The configuration space of particles in N modulo N_0 with local labels in F is then defined as

$$C(N, N_0; \pi) = \left(\coprod_{k=0}^{\infty} C^k(\pi) \right) \Big/ \approx .$$

Here $(e_1, \ldots, e_k) \approx (e_1, \ldots, e_{k-1})$, if $\pi e_k \in N_0$ or $\varepsilon_k \in \operatorname{im} o$. When $E = N \times F$ is a trivial bundle we write $C(N, N_0; F)$ instead of $C(N, N_0; \pi)$. This is the case treated in [1], [2].

For $\xi = (e_1, \ldots, e_k) \in C(N, N_0; \pi)$ let $z_i = \pi e_i$. One may think of ξ as particles $z_i \in N$ where each particle have a coefficient e_i in the fiber above it, and write $\xi = \Sigma z_i e_i$. The relation $\approx$ then implies that particles in N_0 and particles with coefficient in $\operatorname{im} o$ are annihilated.

The number of particles in configurations induces a filtration

$$* = C_0(N, N_0; \pi) \subset C_1(N, N_0; \pi) \subset \ldots \subset C(N, N_0; \pi);$$

$$C_n(N, N_0; \pi) = \left(\coprod_{k=0}^{n} C^k(\pi) \right) \Big/ \approx .$$

We let $D_n(N, N_0; \pi) = C_n(N, N_0; \pi)/C_{n-1}(N, N_0; \pi)$ denote the filtration quotients.

2.2. Here we shall reveal some properties of the construction $C(N, N_0; \pi)$. Two properties are immediate from the definition, namely the product formula

$$C(N, N_0; \pi) = C(N_1, N_1 \cap N_0; \pi) \times C(N_2, N_2 \cap N_0; \pi)$$

when $N_1 \cup N_2 = N$ and $N_1 \cap N_2 \subset N_0$, and excision

$$C(N, N_0; \pi) = C(N - U, N_0 - U; \pi)$$

when $U \subset N_0$ and U is open in N.

We define a map $(N, N_0; \pi) \to (N', N'_0; \pi')$ to be a map of bundles which preserves the preferred section and induces a map of pairs $(N, N_0) \to (N', N'_0)$. Clearly with this notion $C(N, N_0; \pi)$ becomes a functor.

LEMMA. *Let $H \subset N$ be a compact codimension zero submanifold and suppose that either $(H, N \cap N_0)$ or F is connected. Then the cofibration*

$$(H, H \cap N_0) \to (N, N_0) \xrightarrow{q} (N, H \cup N_0)$$

induces a quasi fibration

$$C(H, H \cap N_0; \pi) \to C(N, N_0; \pi) \xrightarrow{Q} C(N, H \cup N_0; \pi).$$

PROOF. We shall use the work by Dold and Thom on quasi fibrations, [4]. To reduce notation we use

$$\mathscr{F} \to \mathscr{E} \xrightarrow{Q} \mathscr{B}$$

as short for the sequence we are to prove is a quasi fibration.

From 2.1 we have a filtration of $\mathscr{B}$ with $\mathscr{B}_k = C_k(N, H \cup N_0; \pi)$. There is an induced filtration on $\mathscr{E}$, $\mathscr{E}_k = Q^{-1}(\mathscr{B}_k)$, and [4; 2.15] states that it suffices to prove that $Q|_{\mathscr{E}_k}$ is a quasi fibration for all k. We proceed to show this.

First we observe that $\mathscr{E}_k - \mathscr{E}_{k-1}$ consists of configurations $\xi = \Sigma z_i e_i$ where exactly k particles are in $N - (H \cup N_0)$ (with coefficients off $\operatorname{im} o$) while the remaining particles belong to $H \cup N_0$. Thus since the particles in $N - (H \cup N_0)$ may be distinguished from those in $H \cup N_0$ we have homeomorphisms

$$\mathscr{E}_k - \mathscr{E}_{k-1} = (\mathscr{B}_k - \mathscr{B}_{k-1}) \times \mathscr{F}.$$

From this we will show by induction on k that $Q|_{\mathscr{E}_k}$ is a quasi fibration. We shall apply [4; 2.2] in the induction step.

If $\partial H = \emptyset$ the lemma follows easily from the product and excision formulae listed above, so we may assume $\partial H \neq \emptyset$. Let $f\colon \partial H \times (-\varepsilon, \varepsilon) \to N$ be a tubular neighborhood of ∂H such that $f(\partial H \times 0) = \partial H$ and $f(\partial H \times (-\varepsilon, 0]) \subset H$. We set

$$U = H \cup f\left(\partial H \times \left[0, \frac{\varepsilon}{2}\right)\right),$$

and define $U_k \subset \mathscr{B}_k$ to be those configurations $\xi = \Sigma z_i e_i$, where at least one $z_i \in U$. We may identify $\mathscr{B}_{k-1} \subset U_k$ with the configurations where at least one particle belong to H; then U_k is an open neighborhood of $\mathscr{B}_{k-1}$ in $\mathscr{B}_k$.

We wish to use [4; 2.10] to show that $Q|_{Q^{-1}(U_k)}$ is a quasi fibration. By a somewhat lenghty but straightforward argument one sees that there is a deformation retract $r_k\colon U_k \to \mathscr{B}_{k-1}$. The idea is this: one pushes the particles in U towards H (where they disappear) until at most $k-1$ particles are left outside of H. One may also see that there is a map $\bar{r}_k\colon Q^{-1}(U_k) \to \mathscr{B}_{k-1}$ lying over r_k, and finally it is not too hard to show that the composition

$$\mathscr{F} \overset{\cong}{\leftarrow} Q^{-1}(b) \overset{\bar{r}_k}{\to} Q^{-1}(r_k(b)) \overset{\cong}{\to} \mathscr{F}$$

is a homotopy equivalence, provided $(H, H \cap N_0)$ or F is connected. Thus the hypothesis of [4; 2.10] is satisfied and consequently $Q|_{Q^{-1}(U_k)}$ is a quasi fibration.

We can now apply [4; 2.2] and obtain that $Q|_{\mathscr{B}_k}$ is a quasi fibration, that is we have proved the induction step.

2.3. Let $W = N \cup (\partial N \times [0,1))$ and let $p\colon W \to N$ be the map which collapses the collar. The fiberwise one point compactification $\dot{T}W$ of the tangent bundle has a preferred section at infinity. The fiberwise smash product $t_\pi = \dot{T}W \wedge_W p^*E$ again has a preferred section o. If N is parallelizable, t_π is equivalent to the $\dim N$-fold fiberwise suspension of p^*E over W. This is the case in 1.4.

We let $\Gamma(W - N_0, W - N; \pi)$ denote the space of sections of t_π that are defined outside N_0 and equals o outside N. There is a natural map

$$\gamma\colon C(N, N_0; \pi) \to \Gamma(W - N_0, W - N; \pi),$$

where $\gamma(\xi)$ is the section of t_π that maps $z \in W - N_0$ to the image of ξ under the composition

$$C(N, N_0; \pi) \to C(N, N_0 \cup (N - \operatorname{int} D(z)); \pi) \cong C(D(z), \partial D(z); \pi)$$

$$\cong C(D(z), \partial D(z); D(z) \times F) \underset{\cong}{\overset{\exp_z}{\longleftarrow}} C(C_z W, \partial D_z W; D_z W \times F)$$

$$\overset{R}{\to} C_1(D_z W, \partial D_z W; D_z W \times F) = (D_z W/\partial D_z W) \wedge F \overset{i_z}{\to} t_\pi.$$

The first map is the natural quotient, the second excision while the third is induced by the local trivialization. The map R is a retract of the inclusion constructed as follows. In a configuration $\xi = \Sigma z_i e_i$ of particles in $D_z W$ one may push all particles in ξ towards the boundary (where they disappear) at a speed proportional to their distance to z. One continues until at most one particle is left; this is $R(\xi)$. Finally ι_z is the fiber inclusion.

Proposition. *If either (N, N_0) or F is connected then γ is a homotopy equivalence.*

Proof. When $\pi\colon E \to N$ is a trivial bundle this is proved in [1] and [8]. If

π: $E \to N$ is non-trivial we may cover N by compact codimension zero submanifolds $N_1, \dots, N_k$ such that $\pi|_{N_i}$ is trivial, and proceed by induction on k.

We set $N' = N_1 \cup \dots \cup N_{k-1}$ and obtain by naturality a commutative diagram

$$\begin{array}{ccc} C(N', N' \cap N_0; \pi) & \xrightarrow{\gamma} & \Gamma(W - (N' \cap N_0), W - N'; \pi) \\ \downarrow & & \downarrow \\ C(N, N_0; \pi) & \xrightarrow{\gamma} & \Gamma(W - N_0, W - N; \pi) \\ \varrho \downarrow & & \varrho \downarrow \\ C(N, N' \cup N_0; \pi) & \xrightarrow{\gamma} & \Gamma(W - (N' \cup N_0), W - N; \pi) \end{array}$$

The first column is a quasi fibration by lemma 2.2 and it is standard that the second column is a fibration. Now the excision formula gives

$$\begin{aligned} C(N, N' \cup N_0; \pi) &\cong C(N - \operatorname{int} N', N_0 - (N_0 \cap \operatorname{int} N'); \pi) \\ &= C(N_k - (N_k \cap \operatorname{int} N'), (N_0 \cap N_k) - (N_0 \cap N_k \cap \operatorname{int} N'); \pi). \end{aligned}$$

and the top and bottom maps in the diagram above are therefore equivalences by induction. Hence so is the middle map.

Our main interest is the case when N is parallizable.

COROLLARY. *Suppose N is a parallizable compact d-dimensional manifold, possibly with boundary. Then*

$$C(N, \partial N; \pi) \xrightarrow{\gamma} \Gamma(\Sigma_N^d E \to N)$$

is an equivalence provided $(N, \partial N)$ or F is connected.

3. Proof of Goodwillies theorem.

3.1. We recall the notation in 1.4; N is a paralizable compact d-dimensional manifold possibly with boundary and X is a space that is a d-fold suspension. We let (Y, x) be a pointed space such that $(X, x) = \Sigma^d(Y, x)$. From 2.3 we have equivalences

$$\gamma\colon C(N, \partial N; Y \vee_x S^j) \xrightarrow{\simeq} \operatorname{Map}(N, X \vee_x S^{j+d})$$

$$\gamma\colon C(N, \partial N; \Sigma_N^j(E_x \amalg N) \to N) \xrightarrow{\simeq} \Gamma(\Sigma_N^{j+d}(E_x \amalg N) \to N).$$

The former induces an equivalence of $C(N, \partial N; Y) \subset C(N, \partial N; Y \vee_x S^j)$ with the subspace $\operatorname{Map}(N, X) \subset \operatorname{Map}(N, X \vee_x S^{j+d})$.

We want to realize the map ϕ_j of 1.5 on the configuration space models. But since we only need to show that ϕ_j is an equivalence in the (stable) range $\leqq 2j$, we are allowed to replace the models by stably equivalent ones.

3.2. The space $\tilde{C}^p(\mathrm{id}_N)$ is of the Σ_p-homotopytype of a free Σ_p-CW-complex. Indeed if $\Delta^p \subset N^p$ is the thick diagonal of p-tuples $(z_1, \ldots, z_p)$ with at least two coordination equal, and $\tilde{\Delta}^p$ a tubular neighborhood stable under the Σ_p-action, then

$$\tilde{C}^p(\mathrm{id}_N) = N^p - \Delta^p \simeq N^p - \tilde{\Delta}^p.$$

The following lemma shows that we may replace $C(N, \partial N; \Sigma_N^j(E_x \amalg N) \to N)$ by the configuration space of only one particle.

LEMMA. *Let $F \to E \xrightarrow{\pi} N$ be a bundle and suppose that F is j-connected. Then the inclusion*

$$C_p(N, N_0; \pi) \hookrightarrow C(N, N_0; \pi)$$

is $(p + 1)(j + 1) - 1$ connected, provided $j \geqq 1$.

PROOF. First we assume that π is a trivial bundle. Let $\mathscr{D} \subset \tilde{C}^p(\mathrm{id}_B)$ be the subspace of p-tuples $(z_1, \ldots, z_p)$ with at least one $z_i \in N_0$. Then clearly $(\tilde{C}^p(\mathrm{id}_N), \mathscr{D})$ is Σp-equivalent to a free Σ_p-CW pair. We denote the cofiber by $\tilde{C}^p(N, N_0)$ and can then write the cofiber of the inclusion $C_{p-1}(N, N_0; \mathrm{F}) \hookrightarrow C_p(N, N_0; F)$ as

$$D_p(N, N_0; F) = F^{\wedge p} \wedge_{\Sigma_p} \tilde{C}^p(N, N_0).$$

The skeleton filtration of $\tilde{C}^p(N, N_0)$ induces a filtration of $D_p(N, N_0; F)$;

$$F_k D_p(N, N_0; F) = F^{\wedge p} \wedge_{\Sigma_p} \tilde{C}^p(N, N_0)^{(k)}$$

and we have cofibrations

$$F_{k-1} D_p(N, N_0; F) \hookrightarrow F_k D_p(N, N_0; F) \to F^{\wedge p} \wedge_{\Sigma_p} (\vee \Sigma_{p+} \wedge S^k).$$

In particular

$$\begin{aligned} F_0 D_p(N, N_0; F) &= F^{\wedge p} \wedge_{\Sigma_p} \tilde{C}^p(N, N_0)^{(0)} = F^{\wedge p} \wedge_{\Sigma_p} (\vee \Sigma_{p+} \wedge S^0) \\ &= \vee (F^{\wedge p} \wedge S^0) = \vee F^{\wedge p}. \end{aligned}$$

This is a $p(j + 1) - 1$ connected space and an easy Mayer-Vietoris argument now shows that $D_p(N, N_0; \mathrm{F})$ is homology $p(j + 1) - 1$ connected. It follows that the inclusion $C_{p-1}(N, N_0; \mathrm{F}) \hookrightarrow C_p(N, N_0; F)$ is homology $p(j + 1) - 1$ connected.

We prove that $\pi_1 C_p(N, N_0; F)) = 1$ by induction on p. A series of applications of the van Kampen theorem to the cofibrations above shows that $\pi_1 D_p(N, N_0; F) = 1$. But by induction and the van Kampen theorem

$$\pi_1 D_p(N, N_0; F) = \pi_1 C_p(N, N_0; F) * 1 = \pi_1 C_p(N, N_0; F).$$

Thus the Whitehead theorem establishes the lemma in the case where π is a trivial bundle.

To prove the general case we can proceed as in the proof of proposition 2.3 and cover N by compact codimension zero submanifolds $\{N_i\}$ such that the restriction of π to each N_i is trivial.

3.3. We introduce a new filtration

$$C^{(0)}(N, \partial N; Y \vee S^j) \subset C^{(1)}(N, \partial N; Y \vee S^j) \subset \ldots \subset C(N, \partial N; Y \vee S^j),$$

with $C^{(k)}(N, \partial N; Y \vee S^j)$ being the configurations which have at most k particles in $S^j - x$; $C^{(0)}(N, \partial N; Y \vee S^j) = C(N, \partial N; Y)$. Denote the filtration quotients $D^{(k)}$.

We also introduce spaces $\mathscr{E}^{(p)}$ given by

$$\mathscr{E}^{(p)} = \left(\coprod_{k=p}^{\infty} \tilde{C}^k(\mathrm{id}_N) \times_{\Sigma_{k-p}} Y^{k-p}\right)\Big/ \sim$$

with $(z_1, \ldots, z_k; y_{p+1}, \ldots, y_k) \sim (z_1, \ldots, z_{k-1}; y_{p+1}, \ldots, y_{k-1})$ when $y_k = x$. Here the Σ_{k-p}-action on $\tilde{C}^k(\mathrm{id}_N)$ is on the last $k - p$ coordinates.

We can define maps

$$\pi^{(p)}: \mathscr{E}^{(p)} \to \tilde{C}^p(\mathrm{id}_N)$$

Indeed in $\tilde{C}^k(\mathrm{id}_N)$ we may project onto the first p coordinates and the relation $\sim$ does not concern these coordinates. It is shown in [5] that $\pi^{(p)}$ is actually a bundle.

There is an action by Σ_p on $\tilde{C}^k(\mathrm{id}_N)$ which permutes the first p coordinates. This action induces an action on $\mathscr{E}^{(p)}$ which is obviously free. In fact $\mathscr{E}^{(p)}$ is of the Σ_p-homotopy type of a free Σ_p-CW-complex. To see this we observe that the fiber in the bundle $\pi^{(p)}$ is equivalent to a CW-complex while the basespace according to 3.2 is Σ_p-equivalent a Σ_p-CW-complex; the claim follows.

LEMMA. *The inclusion*

$$C^{(p)}(N, \partial N; Y \vee S^j) \hookrightarrow C(N, \partial N; Y \vee S^j)$$

is $(p + 1) \cdot j$-connected, provided $j \geqq 2$.

PROOF. It suffices to prove that the filtration quotients $D^{(p)}$ are $p \cdot j$-connected. Since we may distinguish particles in S^j from those not in S^j

$$D^{(p)} = \left(\bigvee_{k=p}^{\infty} (\tilde{C}^k(\mathrm{id}_N) \times_{\Sigma_p \times \Sigma_{k-p}} (\underbrace{S^j \times \ldots \times S^j}_{p} \times \underbrace{Y \times \ldots \times Y}_{k-p}))_+\right)\Big/ \approx$$

where

$$(z_1,.,z_k;u_1,.,u_p,y_{p+1},.,y_k) \approx * \text{ if } u_p = x, \text{ or } z_p \in \partial N$$

$$(z_1,.,z_k;u_1,.,u_p,y_{p+1},.,y_k) \approx (z_1,.,z_{k-1};u_1,.,u_p,y_{p+1},.,y_{k-1}),$$

if $y_k = x$, or $z_k \in \partial N$.

In $\mathscr{E}^{(p)}$ consider the subspace $\mathscr{A}^{(p)}$ of p-tuples $(z_1,.,z_p,z_{p+1},.,z_k,y_{p+1},.,y_k)$ where $z_i \in \partial N$ for some $1 \leqq i \leqq p$. Then $(\mathscr{E}^{(p)},\mathscr{A}^{(p)})$ is Σ_p-equivalent to a free Σ_p-CW-pair. Let $\bar{\mathscr{E}}^{(p)}$ be the cofiber.

Now we observe that

$$D^{(p)} = \underbrace{(S^j \wedge \ldots \wedge S^j)}_{p} \wedge_{\Sigma_p} \bar{\mathscr{E}}^{(p)}$$

and can argue as in 3.2 that $D^{(p)}$ is $p \cdot j$-connected.

3.4. Recall that $E_x \subset N \times \mathrm{Map}(N,X)$ is the subspace of pairs (z,f) such that $f(z) = x$. We identify the bundle $\pi^{(1)}\colon \mathscr{E}^{(1)} \to N$.

LEMMA. *There is a natural equivalence of bundles $\gamma\colon \mathscr{E}^{(1)} \xrightarrow{\simeq} E_x$.*

PROOF. The inclusions $\tilde{C}^k(\mathrm{id}_N) \hookrightarrow N \times \tilde{C}^{k-1}(\mathrm{id}_N)$ are compatible such that their collection

$$\coprod_{k=1}^{\infty} \tilde{C}^k(\mathrm{id}_N) \times_{\Sigma_{k-1}} Y^{k-1} \to \coprod_{k=1}^{\infty} N \times \tilde{C}^{k-1}(\mathrm{id}_N) \times_{\Sigma_{k-1}} Y^k$$

induces an inclusion $\mathscr{E}^{(1)} \hookrightarrow N \times C(N,\partial N;Y)$. This fits into a commutative diagram

$$\begin{array}{ccc} \mathscr{E}^{(1)} & \xrightarrow{\bar{\gamma}} & E_x \\ \downarrow & & \downarrow \\ N \times C(N,\partial N;Y) & \xrightarrow[\simeq]{\mathrm{id}_N \times \gamma} & N \times \mathrm{Map}(N,X) \end{array}$$

of bundles over N. The action of $\bar{\gamma}$ on the geometrical fibers is given by

$$C(N - \{n\},\emptyset;Y) \xrightarrow{\gamma} \mathrm{Map}(N,\{n\};X)$$

which is a natural equivalence by proposition 2.2.

3.5. In the bundles $\mathscr{E}^{(1)} \to N$ and $E_x \to N$ we may fiberwise add a basepoint and suspend j times. Then γ induces an equivalence $\psi_j\colon \Sigma^j_N(\mathscr{E}^{(1)} \amalg N) \to \Sigma^j_N(E_x \amalg N)$.

Now $C_p(N,N_0;\pi)$ is a functor in the sense of 2.2 and it is easy to see that we have equivalences

$$\psi_{j*}\colon C_p(N,\partial N;\Sigma^j_N(\mathscr{E}^{(1)} \amalg N) \to N) \to C_p(N,\partial N;\Sigma^j_N(E_x \amalg N) \to N).$$

We also observe the immediate but striking fact that

$$C_1(N, \partial N; \Sigma_N^j(\mathscr{E}^{(1)} \amalg N) \to N) = D^{(1)} = \frac{C^{(1)}(N, \partial N; Y \vee S^j)}{C(N, \partial N; Y)}.$$

Consequently we can place ψ_{j*} in a diagram

$$\begin{array}{ccc}
\dfrac{\mathrm{Map}(N, X \vee_x S^j)}{\mathrm{Map}(N, X)} & \xrightarrow{\phi_j} & \Gamma(\Sigma_N^j \amalg N) \to N) \\
{\scriptstyle \simeq_{2j}} \uparrow & & \uparrow {\scriptstyle \simeq_{2j}} \\
\dfrac{C^{(1)}(N, \partial N; Y \vee S^j)}{C(N, \partial N; Y)} & \underset{\simeq}{\xrightarrow{\psi_{j*}}} & C_1(N, \partial N; \Sigma_N^j(\mathrm{E}_x \amalg N) \to N)
\end{array}$$

To prove theorem 1.4 in the case where X is dim N-fold suspension it only remains to check that the diagram commutes. However this is almost a tautologi.

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