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A CHARACTERIZATION OF BALANCED RATIONAL NORMAL SURFACE SCROLLS IN TERMS OF THEIR OSCULATING SPACES II

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In this note we prove a result conjectured by the second and third author [3, Conjecture (i), p. 216); the proof was found independently by the first author and by the second and third authors. The results needed in the proof are contained in [3]; it only remains to apply a theorem of Van de Ven [6, Theorem III, p. 406] to the adjunction divisor. We would like to take this opportunity to point out adjunction theory as a very powerful tool in projective geometry!

Let $X \subset \mathbf{P}^N$ be a smooth, complex, projective surface. Recall [3, p. 216] that the mth osculating space to X at a point x is the linear subspace $\mathrm{Osc}_X^m(x)$ determined by the partial derivatives of order $\leqq m$ of the coordinate functions, with respect to a system of local parameters for X at x and evaluated at x. More precisely, let $\mathscr{P}_X^m(1)$ denote the sheaf of principal parts of order m of $\mathcal{O}_X(1) := \mathcal{O}_{\mathbf{P}^N}(1)|_X$. Then $\mathscr{P}_X^m(1)$ is locally free with rank $\binom{m+2}{2}$, and there are homomorphisms

$$a^m : \mathcal{O}_X^{N+1} \to \mathscr{P}_X^m(1)$$

such that $\mathrm{Im}\,(a^m(x))$ defines the mth order osculating space to X at x, i.e., such that

$$\mathrm{Osc}_X^m(x) := \mathbf{P}(\mathrm{Im}(a^m(x))) \subset \mathbf{P}^N.$$

For a general surface, one expects the dimension of $\mathrm{Osc}_X^m(x)$ to be $\binom{m+2}{2} - 1$ for almost all points x of X and all m such that $\binom{m+2}{2} - 1 \leqq N$. Points where this dimension is smaller than expected are "flat" points of the surface – often called points of hyperosculation. If the surface X contains a *line* through the

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point x, then the dimension of $\mathrm{Osc}_X^m(x)$ is at most $2m$. Hence a *ruled surface* has the property that *all* its mth order osculating spaces have dimension at most $2m$. There are, however, non-ruled surfaces with this property (Togliatti [5], Dye [1]).

Suppose the surface X is a *balanced rational normal scroll of degree* $2n$, i.e., X is isomorphic to $\mathrm{P}^1 \times \mathrm{P}^1$ embedded in P^{2n+1} via $\mathrm{pr}_1^* \mathcal{O}_{\mathrm{P}^1}(1) \otimes \mathrm{pr}_2^* \mathcal{O}_{\mathrm{P}^1}(n)$. It is shown in [2, p. 1060] that in this case *all* mth order osculating spaces to X have dimension $2m$, for $m \leqq n$. The theorem we shall prove shows that this property characterizes the balanced rational normal scrolls.

THEOREM. *Let* $X \subset \mathrm{P}^{2n+1}$, $n \geqq 2$, *be a smooth, projective surface, not contained in a hyperplane, such that* $\dim \mathrm{Osc}_X^m(x) = 2m$ *for all* $x \in X$ *and all* $m \leqq n$. *Then* X *is a balanced rational normal scroll of degree* $2n$.

PROOF. In [3, Theorem, p. 221] the second and third author proved that any surface satisfying the hypotheses of the theorem is birationally ruled, but not isomorphic to P^2. Moreover, the theorem was shown to hold if X is geometrically ruled (i.e., has no exceptional curves of the first kind) or if X is rational. Finally, it was shown that the theorem holds provided $n \leqq 4$.

We shall prove the theorem by showing that the assumptions $n \geqq 5$, X is not rational, X is birationally ruled and contains at least one exceptional curve, lead to a contradiction.

Let H denote the class of a hyperplane section of X and let K denote the class of a canonical divisor on X. Since X spans P^{2n+1}, we have

$$\dim H^0(X, \mathcal{O}_X(1)) \geqq 2n + 2 \geqq 12$$

and $\deg X = H^2 \geqq 2n \geqq 10$. By a theorem of Van de Ven [6, Theorem III, p. 406] it follows that $K + H$ is very ample unless there exists an exeptional curve E on X with $E \cdot H = 1$. Let us show that such an E cannot occur. Recall from [3, p. 220] that X possesses a line bundle $\mathcal{Q}$, given by

$$\mathcal{Q} := \mathrm{Coker}\,(a^2 : \mathcal{O}_X^{N+1} \to \mathcal{P}_X^2(1)),$$

with first Chern class

$$c_1(\mathcal{Q}) = \frac{1}{n-1}(2nK + (n+3)H).$$

Since $E \cdot K = -1$, $E \cdot H = 1$ implies

$$c_1(\mathcal{Q}) \cdot E = -\frac{n-3}{n-1}.$$

But this is not an integer since we have assumed $n \geqq 5$. So $K + H$ must be very ample.

We can therefore argue as in [3, p. 221], using $K + H$ instead of H. The theorem of Sommese [4, Theorem (1.5), p. 377] and Van de Ven [6, Theorem II, p. 403] allows us to conclude that $2K + H$ is generated by its global sections, in particular that $(2K + H)^2 \geqq 0$ holds.

Let $\tau = \frac{1}{3}(K^2 - 2c_2(X))$ denote the Hirzebruch index of X. Since X is birationally ruled and not isomorphic to $\mathbf{P}^2$, we have $K^2 = 8(1 - q) - t$ and $c_2(X) = 4(1 - q) + t$, where t denotes the number of exceptional curves on X with respect to a relatively minimal model, and q denotes the genus of the base curve of this minimal model. Therefore $\tau = -t$ holds, hence $\tau \leqq 0$.

Recall [3, (2′), p. 220; p. 221] that the numerical characters of X satisfy the formulas

$$n(2n + 1)K^2 + 2n(n + 5)K \cdot H + 2(n^2 + 2n + 3)H^2 = 0$$

and

$$3n(2n + 1)\tau = -2(n + 3)(nK \cdot H + (n + 1)H^2).$$

From these two formulas we deduce the following equality

$$(2K + H)^2 = \frac{54}{n + 3}\tau - 3\frac{n - 4}{n}H^2.$$

Since $\tau \leqq 0$, $n \geqq 5$, and $H^2 > 0$, we get $(2K + H)^2 < 0$, which is the desired contradiction.

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