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CONFORMAL IMBEDDINGS OF THE COMPLEX PROJECTIVE PLANE AND SELF-DUAL CONNECTIONS

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Abstract.

The purpose of this paper is twofold. First we present a construction of a non-trivial set of conformal imbeddings of the complex projective plane into the quaternion projective plane. Secondly we construct the moduli space of 1-instantons on the complex projective plane in a very explicit manner.

0. Introduction.

In the last decade there has been an intensive study of the moduli spaces of self-dual connections on 4-manifolds, lately this has lead to the discovery of the Donaldson-invariants, distinguishing different differentiable structures on a 4-manifold.

In this paper we show that in some cases the notion of self-dual connections and the notion of conformal maps are related. First we present a construction of a non-trivial set of conformal imbeddings of the complex projective plane into the quaternion projective plane. In fact we prove that the space:

$$\{c \in \mathrm{Gl}_3(\mathrm{H}) \mid \mathrm{Co}(c^*c) = 1\}/U(1)$$

parametrizes a family of conformal imbeddings. Secondly we construct the moduli space of 1-instantions on the complex projective plane in a very explicit manner.

This paper is divided into 4 sections.

The first section contains an introduction to the notation used in the rest of the paper and some preliminaries on the quaternion Hopf-bundle.

The second section contains a sketch, of the construction of the instanton moduli space, of a 4-manifold. We do not go into any details, but refer the reader to one of the papers [D1], [FU] or [L].

In the third section we define the notion of CH-maps (definition 3.1), a CH-map is a certain conformal imbedding of a complex manifold X of real dimension 4 into the quaternion projective space. Then we show that there is

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a 1-1 correspondence between these maps and self-dual connections on a Sp(1)-bundle over this manifold (proposition 3.2), in fact we show there is a equivariant map from the space of CH-maps on X into the moduli space of self-dual connections on X (proposition 3.16). The results presented in this section are new.

In the fourth section the main results are stated. We prove there is a non-trivial family of conformal imbeddings of the complex projective plane into the quaternion projective plane (theorem 4.6). Secondly we prove that if the moduli space of 1-instantons on the complex projective is connected, then it is a cone on CP^2 (theorem 4.9). This result is related to the ideas presented in the paper [D2]. We have not been able to prove, by differentiable-geometric means that the moduli space is connected. This can be done, using algebraic geometry [B].

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1. The quaternion Hopf-bundle.

This section contains preliminaries used in the folowing sections. The quaternion Hopf-bundle and the canonical connection on this is introduced.

Let in the following H denote the quaternions, the non-commutative 4-dimensional real algebra, generated by 1, i, j and k. i, j and k satisfy the relations:

$$i^2 = j^2 = k^2 = -1 \quad \text{and} \quad ij = k, jk = i, ki = j$$

We let $\mathrm{H}^\times$ denote the invertible quaternions, $\mathrm{H}^\times = \mathrm{H}\backslash\{0\}$. On a quaternion $x_0 = ix_i + jx_2 + kx_3$ we have the operations:

$$\begin{aligned} &\mathrm{Re}\{x_0 + ix_1 + jx_2 + kx_3\} = x_0 \\ &\mathrm{Im}\{x_0 + ix_1 + jx_2 + kx_3\} = ix_1 + jx_2 + kx_3 \\ &\mathrm{Co}\{x_0 + ix_1 + jx_2 + kx_3\} = x_0 + ix_1 \\ &\mathrm{Oc}\{x_0 + ix_1 + jx_2 + kx_3\} = x_2 - ix_3 \end{aligned} \tag{1.1}$$

we also have the conjugation maps:

$$\overline{(x_0 + ix_1 + jx_2 + kx_3)} = x_0 - ix_1 - jx_2 - kx_3 \tag{1.2}$$

If n is a positive integer, let $n + 1$ columns af elements from H be denoted H^{n+1}. If $(q_0, q_1, \ldots, q_{n+1})^t \in \mathrm{H}^{n+1}$, then we extend the conjugation map to:

$$((q_0, q_1, \ldots, q_{n+1})^t)^* = (\bar{q}_0, \bar{q}_1, \ldots, \bar{q}_{n+1}).$$

$\mathrm{H}^\times$ acts from the right on $\mathrm{H}^{n+1}\backslash\{0\}$ by usual scalar multiplication, the quotient of this action is the quaternion projective space HP^n, This principal $\mathsf{H}^\times$-fibration is the quaternion Hopf-bundle:

(1.3) $$\pi\colon \mathsf{H}^{n+1}\backslash\{0\} \to \mathsf{HP}^n$$

or for short $\mathsf{H}\gamma_n^1$. We will make the usual identification:

(1.4) $$v\colon T(\mathsf{H}^{n+1}\backslash\{0\}) \to \mathsf{H}^{n+1}\backslash\{0\} \times \mathsf{H}^{n+1}$$

$$v_q\left(\frac{d}{dt}\bigg|_{t=0}(q+vt)\right) = (q, v)$$

where $q \in \mathsf{H}^{n+1}\backslash\{0\}$ $v \in \mathsf{H}^{n+1}$.

Multiplication from the right by the scalar $\lambda \in \mathsf{H}$ defines a map preserving the fiber in the bundle $T(\mathsf{H}^{n+1}\backslash\{0\}) \to \mathsf{H}^{n+1}\backslash\{0\}$, we denote this map by FR_λ (fiber-right):

(1.5) $$\mathrm{FR}_\lambda\colon T_q(\mathsf{H}^{n+1}\backslash\{0\}) \to T_q(\mathsf{H}^{n+1}\backslash\{0\})$$

$$\mathrm{FR}_\lambda\colon (q, v) \mapsto (q, v\lambda)$$

Thus $T(\mathsf{H}^{n+1}\backslash\{0\}) \to \mathsf{H}^{n+1}\backslash\{0\}$ is in a natural way a quaternion vector bundle. In particular $T(\mathsf{H}^{n+1}\backslash\{0\}) \to \mathsf{H}^{n+1}\backslash\{0\}$ is a complex vector bundle, if we restrict the scalars from H to C. This action, FR_λ, should be distinguished from the action coming from the principal bundle structure. If $g \in \mathsf{H}^\times$, g acts on $\mathsf{H}^{n+1}\backslash\{0\}$ by right translation, we denote the map R_g:

(1.6) $$R_g\colon T_q(\mathsf{H}^{n+1}\backslash\{0\}) \to T_{qg}(\mathsf{H}^{n+1}\backslash\{0\})$$

$$R_g\colon (q, v) \mapsto (qg, vg)$$

Observe:

$$\{\lambda \in \mathsf{H} \mid \mathrm{FR}_\lambda \circ R_g = R_g \circ \mathrm{FR}_\lambda \text{ for all } g \in \mathsf{H}^\times\} = \mathsf{R}$$

so the complex structure does not descend to $T\mathsf{HP}^n$.

Now choose some fixed quaternion inner product on H^{n+1} $\langle\!\langle , \rangle\!\rangle$, conjugated H-linear in the first factor, the associated norm is denoted $\|.\|$. This gives rise to a fiber-wise quaternion inner product, conjugated in the first factor, on the quaternion vector bundle $T(\mathsf{H}^{n+1}\backslash\{0\}) \to \mathsf{H}^{n+1}\backslash\{0\}$.
Let $(q, v), (q, w) \in T(\mathsf{H}^{n+1}\backslash\{0\})$ then we define:

(1.7) $$\langle\!\langle (q, v), (q, w) \rangle\!\rangle_q = \|q\|^{-2} \langle\!\langle v, w \rangle\!\rangle$$

We will now define a connection, in the quaternion Hopf-bundle using this inner product. This connection will be called the canonical connection. The kernel for the projection map π is the vertical vectors:

$$V_q = \{(q,v) \in T(\mathsf{H}^{n+1}\backslash\{0\}) \mid \pi_q(q,v) = 0\} \tag{1.8}$$
$$= \{(q,v) \in T(\mathsf{H}^{n+1}\backslash\{0\}) \mid q = v\lambda \text{ for some } \lambda \in \mathsf{H}\}$$

The horizontal distribution for the canonical connection is then defined to be:

$$H_q = \{(q,v) \in T(\mathsf{H}^{n+1}\backslash\{0\}) \mid \langle\!\langle (q,w),(q,v)\rangle\!\rangle = 0 \text{ for all } (q,w) \in V_q\} \tag{1.9}$$

We see:

$$H_q \oplus V_q = T_q(\mathsf{H}^{n+1}\backslash\{0\})$$

and if $g \in \mathsf{H}^\times$:

$$R_g H_q = H_{qg}$$

We have the isomorphism of real vector spaces:

$$\pi_q \mid H_q \colon H_q \to T_q\mathsf{HP}^n.$$

Denote the inverse of this for π_q^{-1}. Observe $\pi_q^{-1} \circ \pi_q$ is H-projection on the horizontal subspace H_q.

If $q\colon \mathsf{H}^{n+1} \to \mathsf{H}^{n+1}$ denotes the identity map, the connection 1-form ω^0 for the canonical connection is:

$$\omega_q^0 = \|q\|^{-2} \langle\!\langle q, dq\rangle\!\rangle \tag{1.10}$$

The curvature 2-form Ω^0 is given by:

$$\Omega_q^0 = \|q\|^{-2}\{\langle\!\langle dq, dq\rangle\!\rangle - \langle\!\langle dq, q\rangle\!\rangle \|q\|^{-2} \langle\!\langle q, dq\rangle\!\rangle\} \tag{1.11}$$

or

$$\Omega_q^0(v,w) = \operatorname{Im}\{\|q\|^{-2} \langle\!\langle \pi_q^{-1} \circ \pi_q(v), \pi_q^{-1} \circ \pi_q(w)\rangle\!\rangle\} \tag{1.12}$$

for $v, w \in T_q(\mathsf{H}^{n+1}\backslash\{0\})$. It is obvious that if we restrict Ω^0 to H_q then we have:

$$\Omega_q^0 \mid H_q = \|q\|^{-2} \langle\!\langle dq, dq\rangle\!\rangle \tag{1.13}$$

From this we see that the Lie algebra of the holonomy group is Im H ([KN p. 84]), since HP^n is simply connected the holonomy is:

$$\mathrm{Sp}(1) = \{g \in \mathsf{H}^\times \mid g^* g = 1\}.$$

Now we can define a Riemannian metric g on HP^n. The manifold $\mathsf{H}^{n+1}\backslash\{0\}$ has a natural Riemannian metric, namely:

$$\operatorname{Re}\{\|q\|^{-2} \langle\!\langle q, v), (q, w\rangle\!\rangle\} \tag{1.14}$$

for $(q,v)(q,w) \in T_q\mathsf{H}^{n+1}\backslash\{0\}$. It descends to HP^n since it is invariant by $R_g, g \in \mathsf{H}^\times$, we define:

$$g_{\pi(q)}(v, w) = \mathrm{Re}\{\|q\|^{-2} \langle\!\langle \pi_q^{-1}(v), \pi_q^{-1}(w) \rangle\!\rangle\} \tag{1.15}$$

for $q \in \mathsf{H}^{n+1} \backslash \{0\}$ and $v, w \in T_q\mathsf{H}\mathsf{P}^n$.

We sumarize:

LEMMA 1.16. *A choice of quaternion inner product* $\langle\!\langle , \rangle\!\rangle$ *on* H^{n+1}, *induces a natural connection* ω^0 (1.10) *on the quaternion Hopf-bundle* (1.3). *The holonomy of this connection is* Sp(1).

It also induces a natural Riemannian metric on $\mathsf{H}\mathsf{P}^n$, *denoted* g.

REMARK 1.17. Observe that all of the above still holds if we replace the quaternions by the field of complex numbers. The Riemannian metric obtained in this way on CP^n, g (using the standard Hermitian inner product on C^{n+1}) is called the Study-Fubini metric on CP^n.

In the forthcoming sections, the following map will be important:

$$\sigma: \mathsf{C}^{2n+2} \backslash \{0\} \to \mathsf{H}^{n+1} \backslash \{0\} \tag{1.18}$$

$$\sigma: (z^0, z_1, \ldots, z_{2n}, z_{2n+1}) \mapsto (z_0 + jz_1, \ldots, z_{2n} + jz_{2n+1})$$

This map is clearly $\mathsf{C}^\times$-equivariant, and holomorphic, when $\mathsf{H}^{n+1} \backslash \{0\}$ is given the complex structure FR_i. It defines a bundle map, between the complex Hopf-bundle $\mathsf{C}\gamma^1_{2n+1}$ and the quaternion Hopf-bundle $\mathsf{H}\gamma^1_n$:

$$\begin{array}{cccc} \sigma: & \mathsf{C}^{2n+2} \backslash \{0\} & \to & \mathsf{H}^{n+1} \backslash \{0\} \\ & \downarrow \pi & & \downarrow \pi \\ \sigma: & \mathsf{CP}^{2n+1} & \to & \mathsf{HP}^n \end{array} \tag{1.19}$$

We see that σ on the base is a fibration with CP^1 as fiber.

2. The moduli space of self-dual connections.

This section contains a sketch, of the construction of the moduli space of self-dual connections on a 4-manifold; for a detalied discussion see [D1], [L] or [FU].

We restrict our attention to the case where X is a simply connected, compact, oriented Riemannian 4-manifold. In the following we study principal Sp(1)-bundles over X. Topologically these are classified by their 2nd Chern-class. We call minus the 2nd Chern number, *the instanton number* of such a Sp(1)-bundle.

If we fix a principal Sp(1)-bundle with instanton number $k \in \mathsf{Z}$, $\pi: Q \to X$, we have the infinite dimensional affine vector space of connections on this, it is denoted $\mathscr{C}(Q; X)$ or when there is no ambiguity $\mathscr{C}_k(X)$. The group of gauge-transformations, bundle maps covering the identity, is denoted $\mathscr{G}(Q; X)$ or $\mathscr{G}_k(X)$. This group acts on space $\mathscr{C}_k(X)$ by pull-back of forms. The quotient is denoted $\mathscr{B}(Q; X)$ or $\mathscr{B}_k(X)$. In general this space may have singularities, coming from reducible connections. (We do not consider the question of completion of these spaces in appropriate Sobolev norms, this should be done to make the spaces in consideration into smooth manifolds).

To each connection $\omega \in \mathscr{C}_k(X)$, we associate the curvature 2-form Ω^ω. This is a 2-form with values in the adjoint bundle.

Now recall that the Riemannian metric and orientation of X, induces a Hodge star-operator $*$.

$$*\colon \wedge^* T^*X \to \wedge^* T^*X$$

It is an involution on 2-forms. We denote the positive (negative)-eigenspace of $*$ restricted to 2-forms for (anti-)self-dual forms on X.

We may now define *the moduli space of (anti-)self-dual connections* on $\pi\colon Q \to X$, $\mathscr{M}^{\binom{+}{-}}(Q; X)$ or $\mathscr{M}_k^{\binom{+}{-}}(X)$. It is a finite dimensional submanifold of $\mathscr{B}_k(X)$ (with some well understood singularities), cut out by the (anti-)self-duality equations:

$$\mathscr{M}_k^{\binom{+}{-}}(X) = \{[\omega] \in \mathscr{B}_k(X) \mid *\,\Omega^\omega = {}_{\binom{+}{-}}\Omega^\omega\} \tag{2.1}$$

(where brackets means equivalence classes of gauge equivalent connections). We have, for pure topological reasons:

$$\begin{aligned} \mathscr{M}_k^+(X) &= \emptyset \quad \text{for} \quad k < 0 \\ \mathscr{M}_k^-(X) &= \emptyset \quad \text{for} \quad k > 0 \end{aligned} \tag{2.2}$$

For $k > 0$ $(k < 0)$ $\mathscr{M}_k^+(X)(\mathscr{M}_k^-(X))$ might be empty, for general 4-manifolds X, but imposing some assumptions on X's intersection form, $\mathscr{M}_k^+((\mathscr{M}_k^-(X))$ is non-empty for a generic metric on X ([T]).

We now restrict our category of manifolds further, namely assume the intersection form of X^4 is positive definite. In this case the dimension of the moduli space can be calculated using the Atiyah-Singer index theorem on an appropriate elliptic complex [AHS]. The dimension turns out to be:

$$\dim \mathscr{M}_k^+(X) = 8k - 3 \tag{2.3}$$

Hence the 1-instantion moduli space, $\mathscr{M}_1^+(X)$ is a smooth 5-manifold, away from the singularities arising from reducible connections.

We summarize some of the features of the moduli spaces, the results are originally due to Donaldson in the paper [D1] (see also [FU] and [L]):

2.4 i) $\mathscr{M}_1^+$ is for a generic Riemannian metric on X a smooth 5-manifold, away from the singularities arising from reducible connections.

2.4 ii) The singularities of $\mathscr{M}_1^+$ are in 1–2 correspondence with the finite set of cohomology classes:

$$\{s \in H^2(X;\mathbf{Z}) \mid \langle s \cup s; [X] \rangle = 1\}$$

and a neighbourhood of a singularity in $\mathcal{M}_1^+$ is a cone on $\mathbf{CP}^2$:

$$\frac{\mathbf{CP}^2 \times [0;1[}{\mathbf{CP}^2 \times \{0\}}$$

2.4 iii) $\mathcal{M}_1^+$ has a collar diffeomorphic to $X \times]0;\varepsilon[$, for some small $\varepsilon > 0$. (Think of the collar as the connections with curvature very concentrated around a single point in X).

We will make this statement more precise since we need it later on. (For more details on the following see [L p. 69–70] or [FU p. 149–]). The collar denoted, $\mathcal{M}_1^+(\varepsilon)$ consists of an open subset of the moduli space and there is a diffeomorphism:

$$p\colon \mathcal{M}_1^+(\varepsilon) \to X \times]0;\varepsilon[$$

$$p\colon [\omega] \mapsto (x(\omega), r(\omega))$$

where $x(\omega)$ denotes the center of ω (see [L] and [FU] for details) and $r(\omega)$ denotes the radius of the connection ω. The radius of ω is defined as follows:

$$r(\omega) = \min\{s \in \mathbf{R}_+ \mid \exists x \in X\colon I^\omega(x,s) = \tfrac{1}{2}\}$$

and

$$I^\omega(x,s) = \frac{1}{8\pi^2}\int_X \beta(d_g(x,z)/s)\,\|\Omega^\omega\|^2(z)\,\mathrm{vol}$$

where β denotes a smooth bump function on $\mathbf{R}$:

$$\beta(u) = \begin{cases} 0 & \text{for} \quad |u| > 1 \\ 1 & \text{for} \quad |u| < \frac{7}{8} \end{cases}$$

$d_g(x,z)$ denotes the distance between x and z in the Riemannian metric on X and $\frac{1}{8\pi^2}\|\Omega^\omega\|^2(z)\,\mathrm{vol}$ denotes the 2nd Chern form of the connection ω (see (3.10)). We should think of $r(\omega)$ as a smooth version of:

$$\rho(\omega) = \min\{s \in \mathbf{R}_+ \mid \exists x \in X\colon \frac{1}{8\pi^2}\int_{B_s(x)} \|\Omega^\omega\|^2\,\mathrm{vol} = 1/2\},$$

i.e. the smallest ball containing $\frac{1}{2}$ of the action of the instanton ω. In fact $\mathcal{M}_1^+(\varepsilon)$ is defined using the radius function:

$$\mathcal{M}_1^+(\varepsilon) = \{[\omega] \in \mathcal{M}_1^+ \mid r(\omega) < \varepsilon\}$$

2.4 iv) $\mathcal{M}_1^+ \backslash \mathcal{M}_1^+(\varepsilon)$ is a compact set.

3. CH-maps and self-dual connections.

In this section we study the relation between certain conformal maps on a complex manifold, denoted CH-maps and self-dual connections on this manifold. We show that there is a 1-1 correspondence. We also introduce a gauge invariant quantity associated to a CH-map, the density function. We end this section defining a group action on the space of CH-maps.

Recall a smooth map $f: X \to Y$ between 2 Riemannian manifold (X, g_X) and (Y, g_Y) is conformal if $f^*(g_Y) = \lambda_f g_X$ for some real positive function λ_f on X. We call λ_f for the conformal weight function for f.

We begin this section with a definition:

DEFINITION 3.1. *If X is a complex manifold of real dimension 4, with a hermitian metric, then a holomorphic map $f: X \to \mathsf{C}P^{2n+1}$, is called a* CH-*map of rank n on X, if the composition $\sigma \circ f: X \to \mathsf{H}P^n$ is a conformal map; here σ is defined in* (1.19) *and H^{n+1} has a fixed quaternion inner product. If f is a* CH-*map, $\bar{f}$ denotes the composition $\sigma \circ f$.*

Furthermore if $P \to X$ is a principal $\mathsf{C}^\times$-bundle, then the space of CH-*maps of rank n on X, compatible with P, $\mathrm{CH}^n(P; X)$, is defined to be the space of* CH-*maps of rank $n f: X \to \mathsf{C}P^{2n+1}$ such that $f^*(\mathsf{C}\gamma^1_{2n+1})$ is isomorphic to P as topological bundles.*

The following proposition makes it possible to find CH-maps on a complex manifold:

THEOREM 3.2. *Let X an complex manifold of real dimension* 4. *Assume h is an hermitian metric on X. Let ω^0 be the canonical connection in the quaternion Hopf-bundle* (1.3). *Assume $f: X \to \mathsf{C}P^{2n+1}$ is a holomorphic map and $\bar{f}$ has differential not identically* 0 *in any point: $\bar{f}_{*,x} \neq 0$ for all $x \in X$.*

Then f is a CH-*map of rank n on X if and only if $\bar{f}^\circ(\omega^0)$ is a self-dual connection on the bundle $\bar{f}^*(\mathsf{H}\gamma^1_n)$, with respect to the hermitian metric on X and the orientation coming from the complex structure.*

Moreover the connection $\bar{f}^*(\omega^0)$ has a reduction to a self-dual Sp(1)-*connection.*

PROOF. First some notation. Fix a point p in the total space of the pulled back bundle $f^*(\mathsf{C}\gamma^1_{2n+1})$, hence $p \in Ef^*(\mathsf{C}\gamma^1_{2n+1})$, let $x = \pi(p)$ and $\bar{p} = \sigma(p)$. Observe the following: The connection ω^0 in $\pi_{\mathsf{H}}: \mathsf{H}^{n+1}\backslash\{0\} \to \mathsf{H}P^n$ defines a R-linear map:

$$\pi^{-1}_{\mathsf{H},\bar{f}(\bar{p})}: T_{\bar{f}(x)}\mathsf{H}P^n \to T_{\bar{f}(\bar{p})}(\mathsf{H}^{n+1}\backslash\{0\})$$

The connection $\bar{f}^*(\omega^0)$ in $\bar{f}^*(\mathsf{H}\gamma^1_n)$ defines a R-linear map:

$$\bar{\pi}^{-1}_{\bar{p}}: T_xX \to T_{\bar{p}}E\bar{f}^*(\mathsf{H}\gamma^1_n)$$

The pull-back of the canonical connection in he bundle $\mathbf{C}\gamma^1_{2n+1}$ to the bundle $f^*(\mathbf{C}\gamma^1_{2n+1})$ defines a C-linear map:

$$\pi_p^{-1}: T_xX \to T_pEf^*(\mathbf{C}\gamma^1_{2n+1})$$

here multiplication by i is defined by the complex structure on T_xX and the complex structure on $T_pEf^*(\mathbf{C}\gamma^1_{2n+1})$. We have the 2 formulas:

$$(\bar{f}_{\bar{p}} \circ \bar{\pi}_{\bar{p}}^{-1})(v) = (\pi^{-1}_{\mathsf{H},\bar{f}(\bar{p})} \circ \bar{f}_x)(v) \tag{3.3}$$

$$(\bar{f}_{\bar{p}} \circ \bar{\pi}_{\bar{p}}^{-1})(v) = ((\pi^{-1}_{\mathsf{H},\bar{f}(\bar{p})} \circ \pi_{\mathsf{H},\bar{f}(\bar{p})}) \circ \sigma_{f(p)} \circ f_p \circ \pi_p^{-1})(v) \tag{3.4}$$

for $v \in T_xX$. Note in the last formula all the maps on the right commutes with the complex structures.

Now assume $\bar{f}$ is conformal. Since $\bar{f}_{\bar{p}} \circ \bar{\pi}_{\bar{p}}^{-1}$ commutes with the complex structures, $\bar{f}_{\bar{p}} \circ \bar{\pi}_{\bar{p}}^{-1}(T_xX)$ is stable by multiplication from the right by i. Let $\Omega^0_{\bar{f}(\bar{p})}$ be the curvature form defined in (1.12) evaluated in the point $\bar{f}(\bar{p})$. We now use the algebraic lemma below, from this we conclude that the 2-form:

$$\Omega^0_{\bar{f}(\bar{p})} \mid \bar{f}_{\bar{p}} \circ \bar{\pi}_{\bar{p}} \circ \bar{\pi}_{\bar{p}}^{-1}(T_xX)$$

is self-dual with respect to the inner product:

$$\mathrm{Re}\{\|\bar{f}(\bar{p})\|^{-2} \langle\langle(\bar{f}(\bar{p}), v), (\bar{f}(\bar{p}), w)\rangle\rangle\} \mid \bar{f}_{\bar{p}} \circ \bar{\pi}_{\bar{p}}^{-1}(T_xX)$$

Since $\bar{f}_{\bar{p}}$ is conformal and orientation preserving, it preserves self-duality, hence $\bar{f}^*(\Omega^0)_{\bar{p}} \circ \bar{\pi}_{\bar{p}}^{-1}$ is self-dual with respect to the hermitian metric h on X.

Let us assume $\Omega^0_{\bar{f}(\bar{p})} \circ \bar{f}_{\bar{p}} \circ \bar{\pi}_{\bar{p}}^{-1}$ is self-dual with respect to the hermitian metric on X. Define a real 2-form θ_X, on the real vector space T_xX:

$$\theta_X(v, e) = \mathrm{Co}\{\Omega^0_{\bar{f}(\bar{p})} \circ \bar{\pi}_{\bar{p}}^{-1}(v, w)\}$$

for $v, w \in T_xX$. This is a real form, since $\Omega^0_{\bar{f}(\bar{p})}$ only takes values in Im H. We see that:

$$\begin{aligned} \theta_X(v, (w)I) &= \mathrm{Co}\{\mathrm{Im}\{\|\bar{f}(\bar{p})\|^{-2} \langle\langle \bar{f}_{\bar{p}} \circ \bar{\pi}_{\bar{p}}^{-1}(v, (w)I)\rangle\rangle\}\} \\ &= i\text{-part of } \{\|\bar{f}(\bar{p})\|^{-2} \langle\langle \bar{f}_{\bar{p}} \circ \bar{\pi}_{\bar{p}}^{-1}(v, (w)I)\rangle\rangle\} \\ &= i\text{-part of } \{\|\bar{f}(\bar{p})\|^{-2} \langle\langle \bar{f}_{\bar{p}} \circ \bar{\pi}_{\bar{p}}^{-1}(v, w)\rangle\rangle i\} \\ &= \mathrm{Re}\{\|\bar{f}(\bar{p})\|^{-2} \langle\langle \bar{f}_{\bar{p}} \circ \bar{\pi}_{\bar{p}}^{-1}(v, w)\rangle\rangle\} \\ &= \bar{f}^*(g_{\mathsf{H}})(v, w) \end{aligned} \tag{3.5}$$

where I denotes the complex structure on T_xX. On the other hand extend θ_X to a C-bilinear form on the complex vector space $T_xX \otimes_{\mathbf{R}} \mathbf{C}$, then we claim that θ_x is of type $(1, 1)$.

To see this, let $v, w \in T_xX \otimes_{\mathbf{R}} \mathbf{C}^{(1,0)}$, hence $(v)I = vi$ and $(w)I = wi$, we have:

$$\theta_X((v)I,(w)I) = \theta_X(vi, wi) = i^2\theta_X(v,w) = -\theta_X(v,w)$$

but:

$$\theta_X((v)I,(w)I) = \mathrm{Co}\{\mathrm{Im}\{\|\bar{f}(\bar{p})\|^{-2} \langle\!\langle f_{\bar{p}}\circ \bar{\pi}_{\bar{p}}^{-1}((v)I,(w)I)\rangle\!\rangle\}\} = \theta_X(v,w)$$

where we used that $\bar{f}_{\bar{p}}\circ \bar{\pi}_{\bar{p}}^{-1}$ commutes with the complex structure. We conclude that $\theta_X(v,w) = 0$ for $v, w \in T_xX \otimes_{\mathsf{R}} \mathsf{C}^{(1,0)}$ or $v, w \in X \otimes_{\mathsf{R}} \mathsf{C}^{(0,1)}$ so it is of type (1, 1).

Now a self-dual 2-form of type (1, 1) is a multiple of the fundamental 2-form, associated to the hermitian structure ([A p. 46–48]). We have for $v, w \in T_xX$:

$$\begin{aligned} \bar{f}^*(g)_X(v,w) &= \theta_X(v,(w)I) \\ &= \lambda(x)h_X((v)I,(w)I) \\ &= \lambda(x)h_X(v,w) \end{aligned}$$

for some $\lambda(x)$, by assumption $\bar{f}_x \neq 0$ hence $\lambda(x) > 0$ and we are done.

For the last statement, the holonomy of the connection $\bar{f}^*(\omega^0)$ is a subgroup of the holonomy of ω^0 ([KN p. 81]), hence it is a subgroup of Sp(1). Now in general a connection has a reduction to the holonomy bundle ([KN p. 84]). In particular $\bar{f}^*(\omega^0)$ has a Sp(1)-reduction.

In the theorem above we used the following algebraic lemma:

LEMMA 3.6. *Let W be a right vector space over the quaternions, assume $\langle\!\langle,\rangle\!\rangle$ is a quaternion inner product on W, conjugated in the first factor. Assume V^4 is a 4-dimensional real subspace of W, also assume that V^4 is stable by multiplication from the right, by a quaternion $\theta \in \mathsf{H}^\times$: $\theta^2 = -1$.*

Then the 2-form on W, $\Omega = \mathrm{Im}\langle\!\langle,\rangle\!\rangle$, when restricted to V^4, is self-dual with respect to the R*-inner product* $\mathrm{Re}\langle\!\langle,\rangle\!\rangle$ *and the orientation defined by the complex structure θ.*

PROOF. Since $\theta^2 = -1$, θ induces a R-linear involution:

$$\mathrm{Ad}(\theta)\colon \mathsf{H}\to\mathsf{H};\ g\mapsto \bar{\theta}g\theta$$

We denote the $+1$, and -1 eigenspaces by V_+ and V_-:

$$V_+ = \{g\in\mathsf{H}\mid \theta g = g\theta\};\ V_- = \{g\in\mathsf{H}\mid \theta g = -g\theta\}$$

Observe $V_+\theta \subset V_+$ and $V_-\theta \subset V_-$, hence V_+ is a subfield of H of real dimension 2. We define:

$$\begin{aligned} P_+&\colon \mathsf{H}\to\mathsf{H};\ P_+ = \tfrac{1}{2}(1 + \mathrm{Ad}(\bar{\theta})) \\ P_-&\colon \mathsf{H}\to\mathsf{H};\ P_- = \tfrac{1}{2}(1 - \mathrm{Ad}(\bar{\theta})) \end{aligned}$$

Then:

$$P_+P_- = P_-P_+ = 0$$

$$P_+^2 = P_+P_-^2 = P_-$$

$$\overline{P_+g} = P_+\bar{g}$$

Now define a new inner product on W with values on the field V_+:

$$\langle v, w \rangle = P_+(\langle\!\langle v, w \rangle\!\rangle) \quad \text{for} \quad v, w \in W$$

We can now find an ortonormal basis for V^4 over the field V_+ with respect to $\langle , \rangle$; denote this basis e_1 and e_3. It follows that $e_1, e_1\theta = e_2, e_3, e_3\theta = e_4$ is a positive ortonormal basis for V^4 over R with the inner product Re$\langle , \rangle$. If $\{e_i\}_{i=1..,4}$ is any positive ortonormal basis for V^4, then $\Omega \mid V^4$ is self-dual ($*\Omega = \Omega$) if and only if the set of equations are fullfilled:

$$\Omega(e_1, e_2) - \Omega(e_3, e_4) = 0$$

$$\Omega(e_1, e_3) - \Omega(e_4, e_2) = 0$$

$$\Omega(e_1, e_4) - \Omega(e_2, e_3) = 0$$

but it is not hard to see:

$$\langle\!\langle e_1, e_2 \rangle\!\rangle - \langle\!\langle e_3, e_4 \rangle\!\rangle = 0$$

$$\langle\!\langle e_1, e_3 \rangle\!\rangle - \langle\!\langle e_4, e_2 \rangle\!\rangle = 0$$

$$\langle\!\langle e_1, e_4 \rangle\!\rangle - \langle\!\langle e_2, e_3 \rangle\!\rangle = 0$$

since $\langle\!\langle e_1, e_3 \rangle\!\rangle \in V_-$.

REMARK 3.7. i) If $f \in \mathrm{CH}^n(P; X)$ then the self-dual connection $\bar{f}^*(\omega^0)$ defines a holonomy Sp(1)-bundle, this is independent of the choice of $f \in \mathrm{CH}^n(P; X)$, since Sp(1) is the maximal compact subgroup of $\mathrm{H}^\times$.

ii) In the above proposition we can replace the notion of holomorphic maps and complex structures with the notion of almost holomorphic maps and almost complex structures.

iii) Observe the above proposition can also be formulated for anti-holomorphic maps.

iv) If $f \in \mathrm{CH}^n(P; X)$ and $c_1(P)$ denotes the 1st Chern class (a $U(1)$-reduction) of the $\mathrm{C}^\times$-principal bundle P, then the 2nd Chern class of (a Sp(1)-reduction) the bundle $\bar{f}^*(\mathrm{H}\gamma_n^1)$ is $-c_1(P)^2$.

The next proposition we do not need in the following, we include it since it is related to theorem 3.2.

PROPOSITION 3.8. *Let* $f \in \mathrm{CH}^n(P; X)$ *then the* Oc-*part* (*see* 3.9) *of the curvature form* $\bar{f}^*(\Omega^0)$, *of the connection* $\bar{f}^*(\omega^0)$, *is of type* (2, 0).

PROOF. In analogy with the proof of proposition 3.2 we define the 2-form:

$$\phi_x(v, w) = \mathrm{Oc}\{\Omega^0_{\bar{f}(\bar{p})} \circ \bar{f}_{\bar{p}} \circ \bar{\pi}_{\bar{p}}^{-1}(v, w)\} \tag{3.9}$$

for $v, w \in T_x X$. Now we extend ϕ_X to a bilinear form on the complex vector space $T_x X \otimes_{\mathbb{R}} \mathbb{C}$. We claim that ϕ_x is of type (2, 0).

To see this, let $v \in T_x X \otimes_{\mathbb{R}} \mathbb{C}^{(0,1)}$, hence $(v)I = vi$ and we let $w \in T_x X \otimes_{\mathbb{R}} \mathbb{C}$ be any vector. We have:

$$\phi_X((v)I, w) = \phi_X(vi, w) = i\phi_X(v, w)$$

but

$$\phi_X((v)I, w) = \mathrm{Oc}\{\mathrm{Im}\{\|\bar{f}(\bar{p})\|^{-2} \langle\langle \bar{f}_{\bar{p}} \circ \bar{\pi}_{\bar{p}}^{-1}((v)I, w)\rangle\rangle\}\} = -i\phi_X(v, w)$$

We conclude that $\phi_X(v, w) = 0$ for $v \in T_x X \otimes_{\mathbb{R}} \mathbb{C}^{(0,1)}$ and $w \in T_x X \otimes_{\mathbb{R}} \mathbb{C}$ hence it is of type (2, 0).

In the last part of this section we will associate a positive function on X, to a CH-map. It is called the density function, we prove this is a gauge-invariant for the connection associated to the CH-map. We also introduce a group action on the space of CH-maps.

Now let $f \in \mathrm{CH}^n(P; X)$, then the curvature form $\bar{f}^*(\Omega^0)$ for the connection $\bar{f}^*(\omega^0)$ is a 2-form on $E\bar{f}^*(\mathrm{H}\gamma_n^1)$ with values in $\mathrm{Im}\,\mathrm{H}$. On H we have the usual norm $\|.\|$, hence it makes sense to calculate the norm square of the values of the form $\bar{f}^*(\Omega^0) \wedge \bar{f}^*(\Omega^0)$, this new form is denoted:

$$\|\bar{f}^*(\Omega^0)\|^2$$

This is a $\mathrm{H}^\times$-invariant horizontal 4-form on $E\bar{f}^*(\mathrm{H}\gamma_n^1)$, thus it descends to a 4-form on X. This form multiplied with the number $(8\pi^2)^{-1}$ is in fact the 4-form on X representing the 2nd Chern class of $\bar{f}^*(\mathrm{H}\gamma_n^1)$ in de Rham cohomology.

On the other hand there is a distinguished 4-form on X, namely the volume form, comming from the Hermitian metric (since a complex manifold has a natural orientation). We denote this: $\mathrm{vol} \in \Omega^4(X; \mathbb{R})$. Thus any smooth 4-form on X can be written $f \cdot \mathrm{vol}$ for some function $f \in C^\infty(X)$. In particular there exist a positive function $\delta \in C^\infty(X)$ such that:

$$\|\bar{f}^*(\Omega^0)\|^2 = \delta \cdot \mathrm{vol} \tag{3.10}$$

We now define:

DEFINITION 3.11. If $f \in \mathrm{CH}^n(P; X)$ then the unique positive function $\delta_f : X \to \mathbb{R}_+$ such that

$$\|\bar{f}^*(\Omega^0)\|^2 = \delta_f \cdot \mathrm{vol}$$

is called the density function for the CH-map f.

LEMMA 3.12. *Consider two* CH-*maps*, $f_1, f_2 \in \mathrm{CH}^n(P; X)$. *If there exists a bundle map* $g: \bar{f}_1^*(\mathrm{H}\gamma_n^1) \to \bar{f}_2^*(\mathrm{H}\gamma_n^1)$ *covering the identity on the base* X, *and* $\bar{f}_1^*(\omega^0) = g^*(\bar{f}_2^*(\omega^0))$, *then the density functions for* f_1 *and* f_2 *are equal.*

PROOF. If $\bar{f}_1^*(\omega^0) = g^*(\bar{f}_2^*(\omega^0))$ then also $\bar{f}_1^*(\Omega^0) = g^*(\bar{f}_2(\Omega^0))$ it follows that:

$$\|\bar{f}_1^*(\Omega^0)\|^2 = \|g^*(\bar{f}_2^*(\Omega^0))\|^2 = \|\bar{f}_2^*(\Omega^0)\|^2$$

and we are done.

Let $P \to X$ be as in proposition 3.2. Let G be a compact Lie group acting on the bundle $P \to X$ by bundle maps, we assume G induces holomorphic isometries on the base space. Then the space $\mathrm{CH}^n(P; X)$ is in a natural way a $\mathrm{Sp}(n+1) \times G$-space, we define an action by the rule:

$$(s, g)\cdot f = \sigma^{-1} \circ s \circ \sigma \circ f \circ g^{-1} \tag{3.13}$$

for $(s, g) \in \mathrm{Sp}(n+1) \times G$ and $f \in \mathrm{CH}^n(P; X)$.

If δ_f is the density function for f then:

$$\delta_{(s,g)\cdot f} = \delta_f \circ g^{-1} \tag{3.14}$$

The action of G on $P \to X$ defines an action of G on the principal $\mathrm{H}^\times$-bundle $P \times_{\mathrm{C}^\times} \mathrm{H}^\times \to X$, and since G is compact we can choose a Sp(1)-reduction $Q \to X$ of $P \times_{\mathrm{C}^\times} \mathrm{H}^\times \to X$ such that G also acts on this reduced bundle. In this way we get a natural action on the moduli space of (anti-)self-dual connections on the bundle $Q \to X$, by pull-back of connection forms (see also [K]).

If $(s, g) \in \mathrm{Sp}(n+1) \times G$ and $[\omega] \in \mathcal{M}^+(Q; X)$ then the action is defined by the rule:

$$(s, g)\cdot[\omega] = [(g^{-1})^*(\omega)] \tag{3.15}$$

We can summarize the above in:

PROPOSITION 3.16. *If* $P \to X$ *and* $Q \to X$ *and* G *are as above then the map:*

$$\psi: \mathrm{CH}^n(P; X) \to \mathcal{M}^+(Q; X)$$

$$\psi: f \mapsto [\bar{f}^*(\omega^0)]$$

is well-defined and $\mathrm{Sp}(n+1) \times G$-*equivariant and if* $\psi(f_1) = \psi(f_2)$ *then* $\delta_{f_1} = \delta_{f_2}$.

Moreover if X *is simply connected and compact, then the instanton number of* $Q \to X$ *is:*

$$\langle c_1(P) \cup c_1(P); [X]\rangle \in \mathbf{Z}$$

PROOF. The statement in this proposition is contained in proposition 3.2, lemma 3.10 and remark 3.9.

4. The $\mathbf{CP}^2$ case.

In this section we specialize to the case where X is the complex projective plane. We construct a non-trivial family of conformal imbeddings of the complex projective plane into the quaternion projective plane (theorem 4.6). Moreover we prove that if the moduli space of 1-instantons on the complex projective plane is connected then it is a cone on $\mathbf{CP}^2$ (theorem 4.9).

Consider the complex projective plane, $\mathbf{CP}^2$ with the Study-Fubini metric $g_{\mathbf{C}}$. This is a compact, simply connected, Kaehler manifold of real dimension 4. Give $\mathbf{H}^3$ the natural quaternion inner product:

$$\langle\!\langle v, w \rangle\!\rangle = v^* \cdot w$$

PROPOSITION 4.1. *Let $\bar{g}_t$ be the family of* C*-linear maps:*

$$\bar{g}_t: \mathbf{C}^3 \to \mathbf{H}^3$$

$$\bar{g}_t = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \dfrac{jt}{\sqrt{1+t^2}} \\ 0 & 0 & \end{bmatrix} \quad \text{for } t \in [0; 1[$$

Then the family $\bar{f}_t = ((\bar{g}_t)^*)^{-1}$ defines a family of CH*-maps of rank* 2 *on* $\mathbf{CP}^2$.

If $z = [z_0 : z_1 : z_3] \in \mathbf{CP}^2$, then the conformal weight function λ_t is:

$$\lambda_t(z) = \frac{|z|^4(1-t^2)}{(|z|^2 - t^2|z_0|^2)^2}$$

an the density function δ_t is:

$$\delta_t(z) = \frac{|z|^4(1-t^2)^2(|z|^4 + 4t^2|z_0|^4)}{(|z|^2 - t^2|z_0|^2)^4}$$

PROOF. We must prove that the linear map: $\bar{f}_t: \mathbf{CP}^2 \to \mathbf{HP}^2$ is conformal for $t \in [0; 1[$. First we discuss when a map $A: \mathbf{CP}^n \to \mathbf{HP}^n$ induced from a C-linear map $A: \mathbf{C}^{n+1} \to \mathbf{H}^{n+1}$ is conformal. Recall that the Study-Fubini metric on $\mathbf{CP}^n$ is induced from the usual hermitian inner product $\langle , \rangle$ on $\mathbf{C}^{n+1}$:

$$g_{\mathbf{C}\pi(z)}(v, w) = \mathrm{Re}\{|z|^{-2}\langle \pi_z^{-1}(v), \pi_z^{-1}(w)\rangle\}$$

where $z \in \mathbf{C}^{n+1}\backslash\{0\}$ and $v, w \in T_{\pi(z)}\mathbf{CP}^n$. We have the same formula for the quaternion projective space, now we use the quaternion inner product $\langle\!\langle , \rangle\!\rangle$.

$$g_{\mathbf{H}\pi(q)}(v, w) = \mathrm{Re}\{\|q\|^{-2}\langle\!\langle \pi_q^{-1}(v), \pi_q^{-1}(w)\rangle\!\rangle\}$$

where $q \in \mathbf{H}^{n+1}\backslash\{0\}$ and $v, w \in T_{\pi(q)}\mathbf{HP}^n$.

From this it follows that A is conformal if and only if for any $z \in \mathbf{C}^{n+1}\backslash\{0\}$ there exists a positive real number $\lambda(z)$ such that for all $v, w \in \mathbf{C}^{n+1}$:

(4.2) $\operatorname{Re}\{|z|^{-2}\langle\pi_z^{-1}\pi_z(v), \pi_z^{-1}\pi_z(w)\rangle\} = \operatorname{Re}\{\|Az\|^{-2}\langle\langle\pi_{az}^{-1}\pi_{az}(v), \pi_{az}^{-1}\pi_{az}(w)\rangle\rangle$

or since:

$$\pi_z^{-1}\pi_z(v) = (1 - |z|^{-2}zz^*)(v)$$

and:

$$\pi_{az}^{-1}\pi_{az}(v) = (1 - \|Az\|^{-2}Azz^*A^*)(v)$$

(4.2) is equivalent to:

$$\operatorname{Co}\{|z|^{-2}(1 - |z|^{-2}zz^*)\} = \lambda(z)\operatorname{Co}\{\|Az\|^{-2}(A^*A - \|Az\|^{-2}A^*Azz^*A^*A)\}$$

or:

(4.3) $|z|^{-2}(1 - |z|^{-2}zz^*) = \lambda(z)\operatorname{Co}\{\|Az\|^{-2}(A^*A - \|Az\|^{-2}A^*Azz^*A^*A)\}$

For the maps $\bar{f}_t$ this equation can be checked (it takes some calculations), the conformal weight function is:

$$\lambda_t(z) = \frac{|z|^4(1 - t^2)}{(|z|^2 - t^2|z_0|^2)^2}$$

and the density function is:

$$\delta_t(z) = \frac{|z|^4(1 - t^2)^2(|z|^4 + 4t^2|z_0|^4)}{(|z|^2 - t^2|z_0|^2)^4}$$

The following lemma is a preparation to the next theorem:

LEMMA 4.4. *Let C be the space:*

$$C = \{c \in \mathrm{Gl}_3(\mathsf{H}) \mid \operatorname{Co}\{c^*c\} = 1\}$$

Then we have:

i) *C is the total space for a left principal* Sp(3)*-bundle, with a left* $U(3)$*-action. The action on C is defined by:*

$$s \cdot c = s \circ c \quad \textit{for} \quad c \in C \quad \textit{and} \quad s \in \mathrm{Sp}(3)$$

$$g \cdot c = c \circ g^* \quad \textit{for} \quad c \in C \quad \textit{and} \quad g \in U(3)$$

ii) *The base space of this principal* Sp(3)*-bundle,* $B = \mathrm{Sp}(3)\backslash C$, *is diffeomorphic to the 6-ball:*

$$B = \left\{b = \begin{bmatrix} 0 & b_3 & b_2 \\ -b_3 & 0 & b_1 \\ -b_2 & -b_1 & 0 \end{bmatrix} \;\middle|\; b_i \in \mathsf{C}\; |b_1|^2 + |b_2|^2 + |b_3|^2 < 1\right\}$$

and the projection map $\pi\colon C \to B$ *is given by* $\pi(c) = \operatorname{Oc}\{c^*c\}$.

iii) *If* $b \in B$ *then there exist a* $g \in \mathrm{SU}(3)$ *such that:*

$$\bar{g}bg^* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & t \\ 0 & -t & 0 \end{bmatrix} \quad \textit{where } t = \sqrt{|b_1|^2 + |b_2|^2 + |b_3|^2}$$

PROOF. i) Observe that the Sp(3)-action on C is free and it commutes with the $U(3)$-action.

iii) First some notation:

$$E_{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \quad E_{31} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad E_{12} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We want to solve the equation in $g \in \mathrm{SU}(3)$:

$$\bar{g}E_{23}g^* = b_1E_{23} - b_2E_{31} + b_3E_{12}$$

where we without loss of generality assume $|b|^2 = |b_1|^2 + |b_2|^2 + |b_3|^2 = 1$. If $g \in \mathrm{SU}(3)$ and the 3 columns in $\bar{g}$ are the 3 unit vectors g_1, g_2 and g_3, then:

$$\bar{g}E_{23}g^* = \det(e_1, g_2, g_3)E_{23} - \det(e_2, g_2, g_3)E_{31} + \det(e_3, g_2, g_3)E_{12}$$

where e_i is the ith canonical basis vector of $\mathbf{C}^3$ and $\det(e_i, g_2, g_3)$ is the determinant of the matrix with columns e_i, g_2 and g_3. If we denote $g_2 \times g_3 = (\det(e_1, g_2, g_3), -\det(e_2, g_2, g_3), \det(e_3, g_2, g_3))$ then we should solve:

(4.5) $$(b_1, b_2, b_3) = g_2 \times g_3$$

In general, if $u \in \mathrm{SU}(3)$ and the columns of u are u_1, u_2 and u_3 then $\bar{u}_1 = u_2 \times u_3$, hence an element of the form $(\bar{b}, g_2, g_3) \in \mathrm{SU}(3)$ with $\bar{b}$ in the first column and any 2 vectors in the 2 last columns (making the matrix a SU(3)-matrix) will solve equation (4.5).

ii) if $c \in C$ then by iii) there exist a $g \in \mathrm{SU}(3)$ such that:

$$c^*c = 1 + jb = 1 + \bar{g}tE_{23}g^* \quad \text{where} \quad |b| = t$$

or:

$$(cg)^*(cg) = 1 + tE_{23} \quad \text{where} \quad |b| = t$$

Hence the eigenvalues of $1 + tE_{23}$ must be strictly positive, but since this is equivalent to $0 \leqq |b| < 1$, we conclude that π is well defined. We prove π is surjective.

If $b \in B$ then again using iii) we see that $1 + jb$ is a positive definite matrix, thus it has a square root c (determined up to an element in Sp(3)):

$$c^*c = 1 + jb$$

hence π is surjective.

The fiber of π is Sp(3) since:

$$\pi(c) = \pi(c') \Leftrightarrow c^*c = c'^*c'$$

but then:

$$(c'c^{-1})^*(c'c^{-1}) = (c^*)^{-1}c'^*c'c^{-1} = 1$$

We can now prove that there exist a non-trivial set of conformal imbeddings of CP^2 into HP^2:

THEOREM 4.6. *Let C be the space:*

$$C = \{c \in \mathrm{Gl}_3(\mathrm{H}) \mid \mathrm{Co}\{c^*c\} = 1\}$$

Then there is a $\mathrm{Sp}(3) \times U(3)$*-equivariant map:*

$$\phi: C \to \mathrm{CH}^2(\mathrm{C}^3\backslash\{0\}; \mathrm{CP}^2)$$

$$\phi: c \mapsto \sigma^{-1} \circ (c^*)^{-1}$$

where $\mathrm{CH}^2(\mathrm{C}^3\backslash\{0\}; \mathrm{CP}^2)$ *has the action of* $\mathrm{Sp}(3) \times U(3)$ *defined in* (3.15). *On its image* ϕ *has fiber* $U(1)$, *that is the subgroup* $\{1\} \times U(1) \subset \mathrm{Sp}(3) \times U(3)$ *acts freely and transitively on each fiber.*

PROOF. First we prove ϕ is equivariant. If $(s, g) \in \mathrm{Sp}(3) \times U(3)$ and $c \in C$ then:

$$\phi((s,g)\cdot c) = \phi(scg^*) = \sigma^{-1}(s(c^*)^{-1}g^*) = (s,g)\cdot\phi(c)$$

Now we prove $\phi(c) \in \mathrm{CH}^2(\mathrm{C}^3\backslash\{0\}; \mathrm{CP}^2)$ for any $c \in C$. If $\bar{g}_t$ is the map defined in proposition 4.1 then we see:

$$\bar{g}_t^*\bar{g}_t = 1 + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & jt \\ 0 & -jt & 0 \end{bmatrix} \quad \text{where} \quad 0 \leqq t < 1$$

and by the proposition $\phi(\bar{g}_t) \in CH^2(\mathrm{C}^3\backslash\{0\}; \mathrm{CP}^2)$. Now if $c \in C$ then by lemma 4.4 ii) and iii) there exists a $g \in \mathrm{SU}(3)$ such that:

$$(cg^*)^*(cg^*) = 1 + tE_{23} \quad \text{where} \quad 0 \leqq t < 1$$

thus $\phi(cg^*) \in \mathrm{CH}^2(\mathrm{C}^3\backslash\{0\}; \mathrm{CP}^2)$, but ϕ is equivariant:

$$\phi(c) = (1, g^*)\phi(cg^*)$$

and $\mathrm{CH}^2(\mathrm{C}^3\backslash\{0\}; \mathrm{CP}^2)$ is a $\mathrm{Sp}(3) \times U(3)$ space.

We will study the equivariant map $\alpha := \psi \circ \phi$:

$$\alpha = \psi \circ \phi : C \to \mathcal{M}_1^+(\mathrm{CP}^2)$$

Since Sp(3) $\times$ $U(3)$ ($U(1)$ imbedded in the diagonal) acts trivially on $\mathcal{M}_1^+(\mathrm{CP}^2)$ ($U(1)$ act by non-trivial gauge transformations), we form the quotient map $\bar{\alpha}$:

$$\begin{array}{cc} & C \to \mathcal{M}_1^+(\mathrm{CP}^2) \\ (4.8) & \downarrow \quad \nearrow \\ & U(1)\backslash B \end{array}$$

Here we used lemma 4.4 to make the identification Sp(3) $\times$ $U(1)|C \simeq U(1)\backslash B$. Note that:

$$U(1)\backslash B \simeq \mathrm{CP}^2 \times [0;1[/\mathrm{CP}^2 \times \{0\}$$

hence $(U(1)\backslash B)\backslash\{0\}$ is a smooth 5-manifold. In the moduli space $\mathcal{M}_1^+(\mathrm{CP}^2)$ there is only 1 singularity, corresponding to the fact that (see 2.4 ii)):

$$\#\{s \in H^2(\mathrm{CP}^2;\mathrm{Z}) \mid \langle s \cup s; [X]\rangle = 1\} = 2$$

The gauge class containing the reducible self-dual connections is denoted $[a]$. Away from $[a]$ $\mathcal{M}_1^+(\mathrm{CP}^2)$ is a smooth 5-dimensional manifold. It is not hard to see that $\bar{\alpha}(0) = [a]$, so $\bar{\alpha}$ restricts to a smooth map:

$$\bar{\alpha}: (U(1)\backslash B))\backslash\{0\} \to \mathcal{M}_1^+(\mathrm{CP}^2)\backslash\{[a]\}$$

We now state and prove the theorem.

THEOREM 4.9. *Let* CP^2 *be the complex projective plane with the Study-Fubini metric. Denote by* $\mathcal{M}_1^+(\mathrm{CP}^2)$ *the moduli space of* 1*-instantons on* CP2.

If $\mathcal{M}_1^+(\mathrm{CP}^2)$ *does not have a 5-dimensional smooth, compact component, then* $\mathcal{M}_1^+(\mathrm{CP}^2)$ *is a cone on* CP^2.

A bijection from the cone to $\mathcal{M}_1^+(\mathrm{CP}^2)$ *is given by the map* $\bar{\alpha}$:

$$\bar{\alpha}: U(1)\backslash B \to \mathcal{M}_1^+(\mathrm{CP}^2)$$

(*B is defined in lemma* 4.4 ii)).

Furthermore $\bar{\alpha}$ *is* PU(3)*-equivariant.*

PROOF. From proposition 4.1 and proposition 3.16 it follows that $\bar{\alpha}$ is injective. The inverse function theorem then tells us that Im $\bar{\alpha}$ is open in $\mathcal{M}_1^+(\mathrm{CP}^2)$. We *claim* that Im $\bar{\alpha}$ is *closed* in $\mathcal{M}_1^+(\mathrm{CP}^2)$, and Im $\bar{\alpha}$ *contains the collar* and *the reducible connection class* [a]. The theorem now follows since Im $\bar{\alpha}$ has to be a component of $\mathcal{M}_1^+(\mathrm{CP}^2)$ containing both the collar and the reduction $[a]$. Hence by 2.4 iv) the complement to Im $\bar{\alpha}$ has to be a smooth 5-dimensional compact manifold, but the assumption is that no such component exists.

We prove the claim above. From 2.4 iii) we know there exist a small $\varepsilon > 0$ such that:

$$p: \mathcal{M}_1^+(\varepsilon) \to \mathrm{CP}^2 \times]0; \varepsilon[$$

is a diffeomorphism. Now consider the family $\bar{g}_t$ defined in proposition 4.1. By the lemma below

$$r(\alpha(\bar{g}_t)) \to 0 \quad \text{for} \quad t \to 1-$$

thus there exists a $t_1 \in [0; 1[$ close to 1, such that $\alpha(\bar{g}_{t_1}) \in \mathcal{M}_1^+(\varepsilon)$, so if we put $\varepsilon_1 = r(\bar{g}_{t_1}))$ then by continuity:

$$p(\alpha(\{\bar{g}_t \mid t \in [t_1; 1[\ \})) = \{x_0\} \times]0; \varepsilon_1]$$

where $x_0 = [1:0:0]$. It now follows from the $U(3)$-equivariance of α and p, that $p \circ \alpha$ has to be surjective on $\mathrm{CP}^2 \times]0; \varepsilon_1]$, or α is surjective on the clousure of $\mathcal{M}_1^+(\varepsilon_1)$ in $\mathcal{M}_1^+$, denoted $(\mathcal{M}_1^+(\varepsilon_1))^{cl}$.

Let $\varepsilon_0 < \varepsilon_1$ then $\alpha^{-1}(\mathcal{M}_1^+ \backslash \mathcal{M}_1^+(\varepsilon_0))$ are elements in C inducing instantons with radius equal to ε_0 or bigger radius than ε_0. This has to be a compact subset of C (it is the inverse image in C of a subset of $U(1)\backslash B$ of the form $(\mathrm{CP}^2 \times [0; 1-\delta])/(\mathrm{CP}^2 \times \{0\})$).

We conclude that:

$$C = \alpha^{-1}(\mathcal{M}_1^+ \backslash \mathcal{M}_1^+(\varepsilon_0)) \cup \alpha^{-1}((\mathcal{M}_1^+(\varepsilon_1)^{cl})$$

hence:

$$\alpha(C) = \alpha(\alpha^{-1}(\mathcal{M}_1^+ \backslash \mathcal{M}_1^+(\varepsilon_0))) \cup \mathcal{M}_1^+(\varepsilon_1)^{cl}$$

that is $\alpha(C)$is the union of two closed subsets of $\mathcal{M}_1^+$, so it has to be closed. This finishes the proof.

In the theorem above we used the following lemma:

LEMMA 4.10. *Let $\bar{g}_t$ be the family of linear maps defined in proposition* 4.1. *Let* α: $C \to \mathcal{M}_1^+(\mathrm{CP}^2)$ *be the map defined in* 4.7.

Then if $r(\alpha(\bar{g}_t))$ denotes the radius of the instanton $\alpha(\bar{g}_t)$, defined by the CH-*map* $\phi(\bar{g}_t)$ *we have:*

$$r(\alpha(\bar{g}_t)) \to 0 \quad \textit{for} \quad t \to 1-$$

PROOF. If $x_0 = [1:0:0]$ and $z = [z_0:z_1:z_2] \in \mathrm{CP}^2$, then we have the formula for $r(\alpha(\bar{g}_t))$:

$$\min\left\{ s \in \mathrm{R}_+ \,\middle|\, \frac{1}{8\pi^2} \int_{\mathrm{CP}^2} \beta(d_g(x_0, z)/s))\delta_t(z)\,\mathrm{vol} = \tfrac{1}{2} \right\}$$

where:

$$\delta_t(z) = \frac{|z|^4 (1-t^3)^2 (|z|^4 + 4t^2 |z_0|^4)}{(|z|^2 - t^2 |z_0|^2)^4}$$

From this it is not hard to prove the lemma.

We have not been able to prove that $\mathcal{M}_1^+(\mathrm{CP}^2)$ does not have a 5-dimensional smooth, compact component. This can be done using the monad construction in twistor theory. In [B] Buchdal proves that the moduli space is a cone on CP^2.

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