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HOMOGENEOUS CONNECTIONS AND MODULI SPACES

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Abstract.

We develop a method, using a homogeneous technique, that enables us to reprove that the moduli space of one-instantons on the four-sphere is hyperbolic five-space. The same method is used to prove the that moduli space of anti-self-dual connections ($p_1 = -3$ and $w_2 \neq 0$) on the complex projective plane is a single point.

0. Introduction.

In the last decade there has been an intensive study, of the moduli spaces of self-dual connections on 4-manifolds, lately this has lead to the discovery of the Donaldson-invariants, distinguishing different differentiable structures on a 4-manifold. The first moduli space it is natural to study is the one-instanton moduli space on the four-sphere. This moduli space is known. M. F. Atiyah, N. Hitchin and I. Singer [AHS] prove this is hyperbolic five-space.

In this paper we will present another proof of this theorem. We will develop a technique, making the theorem accessible by homogeneous methods (actions of compact Lie groups and representation theory of compact Lie groups). This technique also enables us to prove that the moduli space of anti-self-dual connections ($p_1 = -3$ and $w_2 \neq 0$) on the complex projective plane, is one point (this problem was originally raised, by D. Kotschick). In both cases we study a group action on the bundle of anti-self-dual skew 2-tensors on the manifold.

The paper is divided into 5 sections.

The first section introduce the notion of fiber-transitive group actions on a principal bundle. We classify these up to equivalence. In the second section we calculate the homogeneous connections on a fiber-transitive principal bundle. This result is originally due to Wang [W], we present a proof that differs from the original proof. We end this section with a calculation of the irreducible homogeneous gauge classes of connections on a fiber-transitive principal bundle. This result will be important in the proof of our main theorems. The third section

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contains a construction of homogeneous metrics on homogeneous spaces, our main interest are the four-sphere with the standard metric and the complex projective plane with the Study-Fubini metric. Everything in this section is standard. In the fourth section we include a sketch of the construction of the moduli space of self-dual connections on a four-manifold. We prove that the moduli space of irreducible anti-self-dual connections on the complex projective plane with the Study-Fubini metric is a smooth manifold. The fifth section contains our main results, using the homogeneous methods developed in the first three sections, we reprove that the moduli space of one-instantons on the four-sphere is hyperbolic five space. Our other main result is that the moduli space of anti-self-dual connections ($p_1 = -3$ and $w_2 \neq 0$) on the complex projective plane, with the Study-Fubini metric, is a single point.

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1. Fiber-transitive group actions.

This section introduces the notion of fiber-transitive group actions on a principal bundle (definition 1.4). We prove that fiber-transitive principal bundle have a normal-form (lemma 1.5) and we end this section by classifing fiber-transitive principal bundles up to equivalence (proposition 1.9).

Let G be a Lie group and let H be a closed subgroup of G. H acts freely on G from the right and we have a principal H-bundle.

$$\pi: G \to G/H \tag{1.1}$$

More general, suppose K is some Lie group and χ some Lie group homomorphism: $\chi: H \to K$. We form the principal K-bundle:

$$\pi: G \times_\chi K \to G/H \tag{1.2}$$

G acts from the left on this principal K-bundle, we denote this action ϕ:

$$\phi: G \to \mathrm{Aut}(G \times_\chi K; G/H); \quad \phi: g_1 \to \{(g_2; k) \to (g_1 g_2; k)\} \tag{1.3}$$

here (and in the following) we let $\mathrm{Aut}(G \times_\chi K; G/H)$ denote the group of bundle maps of the principal bundle $\pi: G \times_\chi K \to G/H$. The corresponding action on the base space M is denoted $\bar{\phi}$, we see this action is transitive on M.

DEFINITION 1.4. Let G act on the principal K-bundle $\pi: P \to M$ by bundle maps, we denote the G-action $\phi: G \to \mathrm{Aut}(P; M)$. Then ϕ is called *fiber-transitive* if the action is transitive on the base space M.

We include the following lemma, namely any fiber-transitive action can be put on the form 1.2.

LEMMA 1.5. *Let G be a Lie group and suppose ϕ: $G \to \mathrm{Aut}(P; M)$ is an action on the principal K-bundle π: $P \to M$. u_0 is a choice of base point in P, and put $\pi(u_0) = x_0$. Let G_{x_0} denote the isotropy subgroup of G on the base space.*

Then we have: The map χ: $G_{x_0} \to K$ defined by the equation:

$$\phi(h)(u_0) = u_0 \chi(h); \textit{for } h \in G_{x_0}$$

is a group homomorphism.

If we define F to be the map:

$$F\colon G \times_\chi K \to P;\ F(g; k) = \phi(g)(u_0)k$$

then we have:

i) *F is G-K-equivariant, so the diagram below commutes and the maps are G-K-equivariant:*

$$\begin{array}{ccc} G \times_\chi K & \xrightarrow{F} & P \\ \pi\downarrow & & \downarrow\pi \\ G/G_{x_0} & \xrightarrow{F} & M \end{array}$$

also $F(1 \cdot G_{x_0}) = x_0$ and $F(g_1(g_2 \cdot G_{x_0})) = \bar{\phi}(g_1)(F(g_2 \cdot G_{x_0}))$

ii) *F is injective.*

iii) *If G acts fiber-transitive, then F is a G-isomorphism, of principal K-bundles.*

REMARK 1.6. From the lemma above we see that if G acts fiber-transitive, then the principal K-bundle π: $P \to M$, has a reduction to the subgroup $\mathrm{Im}(\chi)$.

We need the following definitions:

1.7. DEFINITION. If $\bar{\phi}$: $G \to \mathrm{Diff}(M)$ is a G-action on M, then a lift of $\bar{\phi}$ to the principal K-bundle π: $P \to M$ is a *G-action* ϕ: $G \to \mathrm{Aut}(P; M)$, such that:

$$\pi(\phi(g)(p)) = \bar{\phi}(g)(\pi(p)) \text{ for all } p \in P \text{ and } g \in G$$

If ϕ_1 and ϕ_2 are two lifts of the G-action $\bar{\phi}$: $G \to \mathrm{Diff}(M)$, then we say that ϕ_1 and ϕ_2 are *equivalent*, if there exists a bundle isomorphism, F: $P \to P$ covering the identity, such that:

$$F(\phi_1(g)(p)) = \phi_2(g)(F(p)) \text{ for all } p \in P \text{ and } g \in G$$

DEFINITION 1.8. Let π: $P \to M$ be a principal K-bundle, assume $\bar{\phi}$ is a transitive G-action on M, then a homomorphism of Lie groups χ: $G_{x_0} \to K$ (G_{x_0} is the isotropy group of the action $\bar{\phi}$ on the base space) is said to be *compatible* with P and $\bar{\phi}$, if there exists a bundle isomorphism F: $G \times_\chi K \to P$ such that on the base space:

i) $F(1 \cdot G_{x_0}) = x_0$

ii) $F(g_1(g_2 \cdot G_{x_0})) = \bar{\phi}(g_1)F(g_2 \cdot G_{x_0})$ for $g_1, g_2 \in G$

The set of homomorphisms compatible with P and $\bar{\phi}$ is denoted $\mathrm{Hom}^{(P,\bar{\phi})}(G_{x_0}, K)$ or when there is no ambiguity $\mathrm{Hom}^P(G_{x_0}, K)$.

With this notation we have:

PROPOSITION 1.9. *Let π: $P \to M$ be a principal K-bundle with a transitive G-action on the base M, let u_0 be a base point in P and $\pi(u_0) = x_0$.*

If G_{x_0} denotes the isotropy group at the point x_0, then the set of equivalence classes of lifts of the G-actions to the principal K-bundle π: $P \to M$ (fixed to be $\bar{\phi}$ on the base), are in bijective correspondence with the set of equivalence classes:

$$\mathrm{Hom}^P(G_{x_0}, K)/\mathrm{Inner}(K)$$

where Inner(K) *denotes inner automorphisms of K.*

PROOF. Assume ϕ_1 and ϕ_2 are equivalent lifts of $\bar{\phi}$, by lemma 1.5 they define two homomorphisms χ_1 and χ_2 compatible with P. We must show that they differ by an inner automorphism of K. By definition there exists a bundle isomorphism, F: $P \to P$ covering the identity, such that:

$$F(\phi_1(g)(p)) = \phi_2(g)(F(p)) \text{ for all } p \in P \text{ and } g \in G$$

Define k_0 to be the element in K that satisfy the equation: $F(u_0) = u_0 k_0$, then if $h \in G_{x_0}$ we have:

$$F(\phi_1(h)(u_0))) = F(u_0 \chi_1(h)) = F(u_0)\chi_1(h)$$

on the other hand:

$$F(\phi_1(h)(u_0)) = \phi_2(h)(F(u_0)) = F(u_0)\, k_0^{-1}\, \chi_2(h)\, k_0$$

hence $\chi_1(h) = k_0^{-1} \chi_2(h)\, k_0$.

Now let us assume that there exists a $k_0 \in K$ such that for all $h \in G_{x_0}$, $\chi_1(h) = k_0^{-1} \chi_2(h) k_0$.

Then consider the map:

$$F\colon G \times_{\chi_1} K \to G \times_{\chi_1} K;\ F(g; k) = (g, k_0 k)$$

This is a well-defined G-equivariant bundle map, covering the identity. Now it is not hard to see that χ_1 and χ_2 define equivalent G-actions (covering $\bar{\phi}$ on the base) on the bundle π: $P \to M$.

2. Homogeneous connections.

In this section we calculate the homogeneous connections on a fiber-transitive principal bundle (proposition 2.8). This result is originally due to Wang [W], we present a proof that differs from the original proof. We end this section with a calculation of the irreducible homogeneous gauge classes of connections on a fiber-transitive principal bundle (corollary 2.18). The result will be important in the proof of our main theorems.

If V is some finite dimensional H-module, let the corresponding representation of H be denoted $\tau: H \to Gl(V)$; we form the associated vector bundle, with fiber V:

$$\pi: P \times_\tau V \to M$$

Sections in this bundle is denoted, $\Gamma(P \times_\tau V; M)$. If ϕ is a G-action on $\pi: P \to M$, it induces an action on the sections $\Gamma(P \times_\tau V; M)$. We denote the G-invariant sections, by the symbol: $\Gamma^\phi(P \times_\tau V; M)$, or where there is no ambiguity $\Gamma^G(P \times_\tau V; M)$.

The following proposition makes it possible to calculate the homogeneous sections:

PROPOSITION 2.1. *Let G be a Lie group and H a closed subgroup of G. Let V be some finite dimensional H-module, then:*

$$\Gamma^G(G \times_\tau V; G/H) \cong V^H$$

where V^H denotes the H-invariant vectors in the H-module V.

PROOF. The proof is standard, see [BtD p. 141–142]. The proposition can be thought of as a special case of the Frobenius reciprocity theorem for finite groups see [R].

The following construction gives rise to a important H-module. Let $\mathfrak{G}$ denote the Lie algebra of G, and $\mathfrak{H}$ the Lie algebra of H. The Lie algebra of H is naturally included in the Lie algebra of G, if we think of the Lie algebras as the tangent spaces at the element $1 \in H \subset G$. G acts on $\mathfrak{G}$ by the adjoint action:

$$\mathrm{Ad}: G \to Gl(\mathfrak{G}) \tag{2.2}$$

If we restrict this action to the subgroup H, then the Lie algebra of H is a H-submodule of $\mathfrak{G}$.

DEFINITION 2.3. If we can choose a complement, $\mathfrak{M}$ to $\mathfrak{H}$ in $\mathfrak{G}$, such that this complement is $\mathrm{Ad}(H)$-invariant:

$$\mathfrak{G} = \mathfrak{H} \oplus \mathfrak{M}$$

Then we say that the homogeneous space G/H is reductive.

If G/H is reductive, we call the representation corresponding to the H-module $\mathfrak{M}$, for the linear *isotropy representation*:

$$\lambda: H \to \mathrm{Gl}(\mathfrak{M}) \tag{2.4}$$

More general, for any non-negative integer p, we form the p^{th} exterior power of the isotropy representation:

$$\lambda^p\colon H \to \mathrm{Gl}(\Lambda^p \mathfrak{M}) \tag{2.5}$$

We have the lemma:

2.6. LEMMA. *There is a natural isomorphism of real vector bundles:*

$$\{G \times_{\lambda^p} \Lambda^p \mathfrak{M} \to G/H\} \cong \{\Lambda^p TM \to M\}$$

This isomorphism is G-equivariant, when G acts on the first component, on the left hand side; and it acts by the p^{th} exterior power of the differential, on the right hand side.

PROOF. Define the G-isomorphism as follows:

$$\psi\colon G \times_{\lambda^p} \Lambda^p \mathfrak{M} \to \Lambda^p\, TM;\ \psi(g; v) = (\pi(g), \pi_*(g \cdot v))$$

This is well-defined, it is clearly linear and a bijection in each fiber, since for any $g \in G$:

$$T_g G = \ker \pi_* \oplus g \cdot \mathfrak{M} \tag{2.7}$$

The affine space of connections on $\pi\colon P \to M$ is denoted $\mathscr{C}(P;\ M)$. If ϕ is a G-action on $\pi\colon P \to M$ the it induces a G-action on $\mathscr{C}(P;\ M)$, by pull-back of 1-forms. The connections invariant by the G-action ϕ, are denoted $\mathscr{C}^{\phi}(P; M)$, or $\mathscr{C}^{G}(P; M)$ and they are called homogeneous connections.

Proposition 2.1 enables us to calculate the homogeneous connections, when the action ϕ is fiber-transitive. The following result is originally due to Wang [W], we include a proof of the first part of the theorem, that differs from the original proof:

THEOREM 2.8 ([W]). *Let G be a Lie group and H a closed subgroup of G. Assume G/H is a reductive homogeneous space, and let $\mathfrak{M}$ denote a* Ad(H)*-invariant complement to $\mathfrak{H}$ in $\mathfrak{G}$. Suppose K is some other Lie group, with Lie algebra $\mathfrak{K}$ and let $\chi\colon H \to K$ be a homomorphism of Lie groups.*

Then the homogeneous connections $\mathscr{C}^G(G \times_\chi K; G/H)$ are in bijective correspondence with the vector in the vector space:

$$\mathrm{Hom}_H(\mathfrak{M}; (\mathrm{Ad} \circ \chi)^*(\mathfrak{K}))$$

Moreover the homogeneous connection corresponding to the homomorphism $\zeta \in \mathrm{Hom}_H(\mathfrak{M}; (\mathrm{Ad} \circ \chi)^*(\mathfrak{K}))$, has curvature form given by the formula:

$$j^*(\Omega^\zeta)_1(v, w) = \tfrac{1}{2}\{([\zeta(v); \zeta(w)] - \zeta([v; w]_m)) - \chi([v; w]_h)\}\ \textit{for}\ v, w \in \mathfrak{M},$$

where j is the inclusion map:

(2.9) $$j\colon G \to G \times_\chi K; \quad j(g) = (g; 1),$$

suffix m and h denote projection on the subalgebras $\mathfrak{M}$ *and* $\mathfrak{H}$; $[;]$ *is the Lie bracket in* $\mathfrak{G}$.

$j^*(\Omega^\zeta)_1$ *is a map of H-modules*:

$$j^*(\Omega^\zeta)_1 \in \mathrm{Hom}_H(\Lambda^2\mathfrak{M}; (\mathrm{Ad}\circ\chi)^*(\mathfrak{K}))$$

PROOF. Since G/H is reductive, there is an natural origon for the homogeneous connections, on $\pi\colon G \times_\chi K \to G/H$, the horizontal distribution for this connection is constructed as follows: In the bundle $\pi\colon G \to G/H$ we have the connection defined by the horizontal distribution:

(2.10) $$H_g = g \cdot \mathfrak{M} \quad \text{for all } g \in G$$

This induces a homogeneous connection in the bundle $\pi\colon G \times_\chi K \to G/H$, by the bundle map j.

Now any other homogeneous connection can be obtained from this, by adding a G-invariant 1-form on G/H, with value in the adjoint bundle $G \times_{\mathrm{ad}\circ\chi} \mathfrak{K} \to G/H$. By proposition 2.1 and lemma 2.6 the G-invariant 1-form on G/H with values in the adjoint bundle $\pi\colon G \times_{\mathrm{ad}\circ\chi} \mathfrak{K} \to G/H$, is parametrized by the finite dimensional real vector space:

$$\mathrm{Hom}_H(\mathfrak{M}; (\mathrm{Ad}\circ\chi)^*(\mathfrak{K})).$$

This finishes the first part of the proof.

We obmit the proof of the curvature formula, since it is a rather long computation and we do not need it later on (see [KNI p. 106]). In general, the curvature form for a connection is a 2-form of adjoint type, on the total space of the bundle, so:

$$j^*(\Omega^\zeta)_1 \in \mathrm{Hom}_H(\Lambda^2\mathfrak{M}; (\mathrm{Ad}\circ\chi)^*(\mathfrak{K}))$$

REMARK 2.11. If $(G, H, \mathfrak{M}, \mathfrak{H})$ is a symmetric space, then $[\mathfrak{M}, \mathfrak{M}]$ is contained in $\mathfrak{H}$, so in this case the formula above for the curvature form reduces to:

$$j^*(\Omega^\zeta)_1(v, w) = \tfrac{1}{2}\{[\zeta(v); \zeta(w)] - \chi([v; w]_h)\} \text{ for } v, w \in \mathfrak{M}$$

There is another important action on the affine space of connections, namely the action of the gauge group, $\mathscr{G}(P; M)$. The gauge group is defined to be:

(2.12) $$\{f\colon P \to P \mid f(pk) = f(p)k \text{ for all } p \in P\ k \in K, \bar{f} \equiv \mathrm{id}\}$$

If f is a gauge transformation and ω is a connection, then also $f^*(\omega)$ is a connection, we can form the quotient of this action:

(2.13) $$\mathscr{B}(P; M) = \mathscr{C}(P; M)/\mathscr{G}(P; M).$$

If we, like above, have a G-action ϕ on the bundle $\pi: P \to M$, then this action induces an action on $\mathscr{B}(P; M)$, this action on $\mathscr{B}(P; M)$ depends only on the G-action $\bar{\phi}$ on the base, and not on the choice of lift of $\bar{\phi}$. The G-action on $\mathscr{B}(P; M)$ is given by:

$$\mathscr{B}(P; M) \times G \to \mathscr{B}(P; M); \quad ([\omega], g) \mapsto [\phi(g)^*(\omega)] \tag{2.14}$$

where brackets mean equivalence class of gauge equivalent connections. The gauge transformations commuting with the action ϕ is denoted $\mathscr{G}^\phi(P; M)$, we also have the space of G-invariant gauge classes, we denote this space $\mathscr{B}^G(P; M)$, it is defined to be:

$$\{[\omega] \in \mathscr{B} \mid \exists \phi \text{ a lift of } \bar{\phi}, \text{ and } \forall g \in G, \exists f \in \mathscr{G}: \phi(g)^*(\omega) = f^*(\omega)\} \tag{2.15}$$

DEFINITION 2.16. A connection ω in the K-principal bundle $\pi: P \to M$, is called *irreducible* if the holonomy of ω is the structure group K.

The subspace of irreducible connections in $\mathscr{C}(P; M)$, is denoted $\tilde{\mathscr{C}}(P; M)$, the gauge group act on this; the quotient is denoted $\tilde{\mathscr{B}}(P; M)$. The ϕ-invariant irreducible connections, called $\tilde{\mathscr{C}}^\phi(P, M)$. The G-invariant part of $\tilde{\mathscr{B}}(P; M)$, is $\tilde{\mathscr{B}}^G(P, M)$.

PROPOSITION 2.17. *Let $\pi: P \to M$ be a principal K-bundle. Suppose the center of K is trivial. Let $\bar{\phi}: G \to \mathrm{Diff}(M)$ be a G-action on M, we let $\mathscr{L}$ denote a set of representatives for each equivalence class of lifts of $\bar{\phi}$ to $\pi: P \to M$.*

Then the space of irreducible G-invariant gauge classes of connections is:

$$\tilde{\mathscr{B}}^G(P; M) \cong \bigcup_{\phi \in \mathscr{L}} \tilde{\mathscr{C}}^\phi(P; M)/\mathscr{G}^\phi(P; M)$$

PROOF. Let $[[\omega]] \in \tilde{\mathscr{C}}^\phi/\mathscr{G}^\phi$, then $[[\omega]]$ is represented by a ϕ-invariant connection ω. Clearly $[\omega]$ is G-invariant. Now assume, the class $[\omega] \in \tilde{\mathscr{B}}^G$ is G-invariant. Pick a fixed representative ω for the class $[\omega]$. By assumption there exist a lift ϕ_1 of $\bar{\phi}$, so that for any $g \in G$, there exist a $f \in \mathscr{G}$, such that $\phi_1(g)^*(\omega) = f^*(\omega)$, hence

$$(\phi_1(g) \circ f^{-1})^*(\omega) = \omega.$$

Since ω is irreducible and the center of K is trivial, the only gauge transformation preserving ω, is the identity so $\phi_1(g) \circ f^{-1}$ is the unique bundle map covering $\overline{\phi(g)}$ and preserving ω.

Define another fiber-transitive action ϕ_2 (covering $\bar{\phi}$) on the bundle $\pi: P \to M$:

$$\phi_2: G \to \mathrm{Aut}(P; M); \quad \phi_2(g) = \phi_1(g) \circ f^{-1}$$

By assumption there exist a unique $\phi_3 \in \mathscr{L}$ and a $F \in \mathscr{G}$ (determined up to an element in $\mathscr{G}^{\phi}$) such that:

$$F \circ \phi_2(g) \circ F^{-1} = \phi_3(g) \text{ for all } g \in G$$

Now consider the map:

$$\tilde{\mathscr{B}}^G \to \bigcup_{\phi \in \mathscr{L}} \tilde{\mathscr{C}}^{\phi}/\mathscr{G}^{\phi}$$

$$[\omega] \to (F \circ \phi_2 \circ F^{-1}; [[F^{-1*}(\omega)]])$$

It is not hard to see, this map is well-defined, and we are done.

COROLLARY 2.18. *Let π: $P \to M$ be a principal K-bundle. Let u_0 be a base point in P and $\pi(u_0) = x_0$. Assume M is reductive, and the center of K is trivial.*

If $\bar{\phi}$: $G \to \mathrm{Diff}(M)$ is a transitive G-action on M, then the space of irreducible G-invariant gauge classes of connections is:

$$\tilde{\mathscr{B}}^G(P; M) \cong \bigcup_{\phi \in \mathscr{L}} \tilde{\mathscr{C}}^G(P; M)/\mathscr{G}^G(P; M)$$

where

$$\mathscr{L} = \mathrm{Hom}^P(G_{x_0}, K)/\mathrm{Inner}(K)$$

and $\mathrm{Inner}(K)$ *denotes inner automorphisms of K*

$$\tilde{\mathscr{C}}^{\phi}(P; M)/\mathscr{G}^{\phi}(P; M)$$

in the subspace of $\mathrm{Hom}_H(\mathfrak{M}; (\mathrm{Ad} \circ \chi)^*(\mathfrak{K})$ *corresponding to irreducible connections (see proposition* 2.8 *for this correspondence).*

PROOF. Combine the proposition above, proposition 1.9, and the proposition 2.8.

REMARK 2.19. By definition a irreducible connection has K as holonomy group. In general the image of the curvature form for a connection restricted to a fixed tangent space $T_{u_0}P$ is the Lie algebra of the holonomy group [KNI p. 81]. From this we can conclude that if a homomorphism $\zeta \in \mathrm{Hom}_H(\mathfrak{M}; (\mathrm{Ad} \circ \chi)^*(\mathfrak{K}))$ corresponds to an irreducible connection then the induced map:

$$j^*(\Omega^{\zeta})_1 \in \mathrm{Hom}_H(\Lambda^2 \mathfrak{M}; (\mathrm{Ad} \circ \chi)^*(\mathfrak{K}))$$

$$v \wedge w \mapsto \tfrac{1}{2}\{([\zeta(v); \zeta(w)] - \zeta([v; w]_m)) - \chi([v; w]_h)\}$$

must be surjective.

3. Riemannian metric on homogeneous spaces.

This section contains a construction of a homogeneous metric on a homogeneous space (lemma 3.1), our main interest is the four-sphere with the standard metric and the complex projective plane with the Study-Fubini metric (remark 3.3). All statements in this section are standard.

We now want to give the homogeneous space M, a riemanninan metric, we have:

LEMMA 3.1. *If H is compact, then the Lie algebra of G, $\mathfrak{G}$, has an $\mathrm{Ad}(H)$-invariant positive definite inner product.*

If $\langle ; \rangle_1$ denotes such an inner product, then there is a unique extension of this, to a riemannian metric on G, such that multiplication from the left by elements of G, induces isometrics of G. The extension is given by the formula:

$$\langle v; w\rangle_g = \langle g^{-1}v; g^{-1}w\rangle_1$$

for any $g \in G$ and $v, w \in T_g G$. This riemannian metric on G, descends to G/H, by the canonical projection:

$$\pi: G \to G/H$$

PROOF. Since H is compact, any inner product on $\mathfrak{G}$, can by averaging be made into an $\mathrm{Ad}(H)$-invariant inner product. The rest of the lemma is trivial, when we observe that the tangent bundle of any Lie group is trivial, in a natural way:

$$TG \simeq G \times \mathfrak{G}$$

DEFINITION 3.2. A riemannian metric on G/H, induced from an $\mathrm{Ad}(H)$-invariant inner product on $\mathfrak{G}$, is called a *homogeneous* metric on M.

REMARK 3.3. i) A homogeneous metric is not unique (even when we mod out positive constants), since the linear isotropy representation might not be irreducible as a real representation.

ii) In the following our main interest will be the two homogeneous spaces:

$$S^4 \simeq \frac{\mathrm{SO}(5)}{1 \oplus \mathrm{SO}(4)}, \quad \mathbf{CP}^2 \simeq \frac{\mathrm{SU}(3)}{S(U(1) \times U(2))}$$

in both cases the linear isotropy representation is irreducible, so the homogeneous metrics are determined up to a positive constant in both cases. The homogeneous metric on S^4 is called the standard metric. The homogeneous metric on $\mathbf{CP}^2$ is called the Study-Fubini metric.

iii) If H is compact and connected, the linear isotropy representation is equivalent to a representation in $SO(\mathfrak{M})$. A choice of representation of H into $SO(\mathfrak{M})$ equivalent to the linear isotropy representation, defines in a natural way a orientation of G/H.

iv) Observe that an $\mathrm{Ad}(H)$-invariant inner product on $\mathfrak{G}$, induces a natural splitting of $\mathfrak{G}$, namely:

$$\mathfrak{M} = \{v \in \mathfrak{G} \mid \langle v, \mathfrak{H} \rangle_1 = 0\}$$

Hence M is reductive, when H is compact.

4. The moduli space of self-dual connections.

This section contains a sketch, of the construction of the moduli space of self-dual connections on a four-manifold; for details see [AHS], [D], [L] or [FU]. For reference we include one of the main theorems in [AHS] (theorem 4.2). It states that under certain conditions, on the riemannian metric on the four-manifold, the moduli space of irreducible self-dual connections is a smooth manifold. We prove that the moduli space of irreducible anti-self-dual connections on the complex projective plane, with the Study-Fubini metric, is a smooth manifold (theorem 4.4).

Let M be compact, oriented riemannian 4-manifold. In this section we will consider principal K-bundles over M, where K is a compact semi-simple Lie group. Let us fix a principal K-bundle $\pi\colon P \to M$.

To each connection $\omega \in \mathscr{C}(P; M)$, we associate the curvature 2-form Ω^ω. This is a 2-form with values in the adjoint bundle. Now recall that the riemannian metric and an orientation of M induces a Hodge star-operator $*$.

$$*\colon \textstyle\bigwedge^* T^*M \to \bigwedge^* T^*M$$

It is an involution on 2-forms. We denote the positive-(negative-) eigenspace of $*$ restricted to 2-forms for (anti-)self-dual forms on M^4.

We may now define the *moduli space of (anti-)self-dual connections* $\mathscr{M}^{(\stackrel{+}{-})}(P; M)$. It is a submanifold of $\mathscr{B}(P; M) = \mathscr{C}(P; M)/\mathscr{G}(P; M)$, cut out by the (anti-)self-duality equation:

$$\mathscr{M}^{(\stackrel{+}{-})}(P; M) = \{[\omega] \in \mathscr{B}(P; M) \mid * \Omega^\omega = {}_{(\stackrel{+}{-})}\Omega^\omega\} \tag{4.1}$$

In general this space may have singularities coming from reducible connections. (we do not consider the question of completion of these spaces in appropriate Sobolev norms, this should be done to make the spaces in consideration into smooth manifolds).

The Atiyah-Singer index theorem makes it possible to calculate the dimension of $\mathscr{M}^+(P; M)$ $(\mathscr{M}^-(P; M))$ by study of an appropriate elliptic complex.

THEOREM 4.2. ([AHS]). *Let M be a compact self-dual riemannian 4-manifold with positive scalar curvature. Let P be a principal K-bundle over M where K is a compact semi-simple Lie group.*

Then, the moduli space of irreducible self-dual connections on P is either empty or a manifold of dimension:

$$2p_1(\underline{\mathfrak{K}}) - \tfrac{1}{2}\dim K(\chi - \tau)$$

$2p_1(\underline{\mathfrak{K}})$ is the first Pontrjagin class of the vector bundle associated to P by the adjoint representation: $\underline{\mathfrak{K}}$, χ is the Euler characteristic of M, and τ is the signature of M.

REMARK 4.3. i) An oriented riemannian 4-manifold is self-dual if the negative Weyl tensor vanishes.

ii) The 4-sphere with the standard metric, and any of the two orientations is self-dual.

iii) The complex projective plane with the Study-Fubini metric and the standard orientation (coming from the complex structure) is self-dual, hence if we change orientation of the complex plane, it becomes anti-self-dual.

iv) For any reductive homogeneous space the Levi-Civita connection associated to a homogeneous metric has non-negative scalar curvature (see [KNII p203]).

We now prove the following proposition; we assume the reader is familiar with the proof of theorem 4.2 in [AHS] (see also [L p. 47–]):

THEOREM 4.4. *Let CP^2 be the complex projective plane, with the Study-Fubini metric, give CP^2 the standard orientation. Let P be a principal K-bundle over CP^2 where K is a compact semi-simple Lie group.*

Then the moduli space of irreducible anti-self-dual connections on P is either empty or a manifold of dimension:

$$-2(\dim K + p_1(\underline{\mathfrak{K}}))$$

where $p_1(\underline{\mathfrak{K}})$ is the first Pontrjagin class of the bundle associated to P by the adjoint representation $\underline{\mathfrak{K}}$. (p_1 is negative for anti-self-dual connections see 4.5).

PROOF. We change the orientation of CP^2 to the opposite orientation (denote CP^2 with this orientation $\overline{\mathrm{CP}^2}$). We want to use the proof of theorem 4.2, but this needs a slight modification since $\overline{\mathrm{CP}^2}$ is *anti*-self-dual. If we examine the proof it is not hard to see that the proof will apply in our case, if the 2nd cohomology group of the fundamental complex on $\overline{\mathrm{CP}^2}$ vanishes. We will prove this is the case.

Let ω be a anti-self-dual connection on $P \to \mathrm{CP}^2$, and let D denote the associated covariant derivative on the vector bundle associated to $P \to \mathrm{CP}^2$ by the adjoint representation, $\underline{\mathfrak{K}}$, then we have the anti-self-dual fundamental complex:

$$0 \to \Omega^0(\underline{\mathfrak{K}}) \xrightarrow{D} \Omega^1(\underline{\mathfrak{K}}) \xrightarrow{D_+} \Omega^2_+(\underline{\mathfrak{K}}) \to 0$$

where $D_+ = P_+ \circ D$ and P_+ is linear projection on the self-dual 2-forms with-values in $\underline{\mathfrak{K}}$.

The 2nd cohomology of the complex above is equal to the kernel of the operator $D_+ D_+^*$, so we want to prove that $\ker(D_+ D_+^*) = 0$. We do this using a standard technique, namely a Weitzenbröck formula, on self-dual 2-forms (see [FU p. 111 + appendix c]):

$$D_+ \circ D_+^* = \nabla^* \circ \nabla - 2W^+(\cdot) + \frac{\kappa}{3}$$

where the operators on the left and right hand side are operators from $\Omega^2_+(\underline{\mathfrak{K}})$ to it self. In the formula $\nabla^* \circ \nabla$ is the trace Laplacian (we only need to know it is a positive operator), W^+ denotes the self-dual part of the Weyl curvature and κ is the scalar curvature.

From this formula we see that $\ker(D_+ \circ D_+^*)) = 0$ if $\frac{\kappa}{3} - 2W^+(\cdot)$ is a positive operator on $\Omega^2_+(\underline{\mathfrak{K}})$. We will show this is the case.

It suffices to prove that in any point $x \in \mathrm{CP}^2$ the map:

$$\frac{\kappa}{3} - 2W^+(\cdot)\colon \Lambda^2_+ T_x^*\mathrm{CP}^2 \to \Lambda^2_+ T_x^*\mathrm{CP}^2$$

has only positive eigenvalues. It is not to hard to see (see [Sal p. 77]) that there exists an ortonormal basis ϕ_1 ϕ_2 and ϕ_3 for $\Lambda^2_+ T_x^*\mathrm{CP}^2$ (the first basis vector is the Kaehler form for the Study-Fubini metric), such that in this basis:

$$W^+ = \frac{\kappa}{24}\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

and:

$$\frac{\kappa}{3} - 2W^+(\cdot) = \frac{\kappa}{24}\begin{bmatrix} 4 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix}$$

If we specialize to SO(3)-bundles and we assume M also is simply connected, then it is known, that SO(3)-bundles over M are topologically classified by their first Pontrjagin class p_1 and there second Stiefel-Whitney class w_2 [DW]. It is not hard to prove, for pure topological reasons:

(4.5) $$\mathcal{M}^+ = \emptyset \text{ for } p_1 < 0$$

$$\mathcal{M}^- = \emptyset \text{ for } p_1 > 0$$

For $p_1 > 0$ ($p_1 < 0$) $\mathscr{M}^+$ ($\mathscr{M}^-$) might be empty, for general 4-manifolds M, but imposing some assumptions on M's intersection form (for example M positive definite) we have $\mathscr{M}^+(\mathscr{M}^-)$ non-empty for a generic metric on M, and it is a manifold of the dimension predicted by the formula in theorem 4.2 ([FU]).

The singularities in the moduli space come from reducible connections (if $P \to M$ is non-trivial and the structure group is SO(3), a reduction has $U(1)$ as structure group). When M is simply connected, compact, oriented, and positive definite the reducible self-dual connections are in 1-2 correspondence with the finite set of cohomology classes [FS]:

$$\{s \in H^2(M, \mathrm{Z}) \mid \langle s \cup s, [\mathrm{M}] \rangle = p_1(P) \text{ and } s = w_2(P) (\mathrm{mod}\, 2)\} \tag{46}$$

hence the singularities in the moduli space are well understood in this case.

5. The proofs of the main theorems.

This section contains our main results, using the homogeneous methods developed in the first three sections, we reprove that the moduli space of one-instantions on the four-sphere is hyperbolic five space (theorem 5.1). The same methods apply to demonstrate that the moduli space of anti-self-dual connections ($p_1 = -3$ and $w_2 \neq 0$) on the complex projective plane, with the Study-Fubini metric is a single point (theorem 5.2). In both cases we study the moduli space on the SO(3)-bundle of anti-self-dual skew 2-tensors on the manifold. We also show that certain principal SO(n + 1)-bundles on certain homogeneous four-spaces have anti-self-dual connections (proposition 5.5)

We prove the following 2 theorems:

THEOREM 5.1. ([AHS]). *The moduli space of anti-self-dual* SO(3)*-connections on the bundle with $p_1 = -4$, and $w_2 = 0$ over S^4, with the standard metric, is diffeomorphic to hyperbolic* 5*-space, that is:*

$$\mathscr{M}^- \simeq \mathrm{SO}_0(5,1)/\mathrm{SO}(5)$$

Note that by using an orientation reversing diffeomorphism of S^4 we get the symmetric result for self-dual connections:

THEOREM 5.1′. *The moduli space of self-dual* SO(3)*-connections on the bundle with $p_1 = 4$ and $w_2 = 0$ over S^4, with the standard metric, is diffeomorphic to hyperbolic* 5*-space that is:*

$$\mathscr{M}^+ \simeq \mathrm{SO}_0(5,1)/\mathrm{SO}(5)$$

This is in fact special for S^4, the moduli space of self-dual SO(3)-connections on bundle with $p_1 = 4$ $w_2 = 0$ over CP^2, is a cone on CP^2 [K], but the moduli space of anti-self-dual SO(3)-connections on bundle with $p_1 = -4$ $w_2 = 0$ over CP^2, is empty (Donaldson (1985) has showed that there is a 1-1 correspondence between anti-self-dual connections on a complex algebraic surface and stable bundles on this surface, using a result of Schwarzenberger (1961), classifing certain stable bundles on $\mathbf{CP}^2$ gives this result).

THEOREM 5.2. *The moduli space of anti-self-dual* SO(3)*-connections on the bundle with* $p_1 = -3$, *and* $w_2 \neq 0$ *over* $\mathbf{CP}^2$, *with the Study-Fubini metric, is a point, and the gauge class is represented by a homogeneous connection.*

First some facts from representation theory:

PROPOSITION 5.4. *The irreducible complex representations of the Lie group* $\mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$, *are classified by two non-negative integers* $\alpha, \beta \in \mathbf{N}_0$, *we denote the representation corresponding to the pair* (α, β) *for* $\chi_{\alpha,\beta}$.

Any irreducible complex representations of SO(4) *is a lift by the spin homomorphism* σ: $\mathrm{Spin}(4) \to \mathrm{SO}(4)$ *of a irreducible representations* $\chi_{\alpha,\beta}$ *of* Spin(4), *where* $\alpha + \beta$ *is even.* $\chi_{\alpha,\beta} (\alpha, \beta \in \mathbf{N}_0)$ *has the following properties:*

i) $\chi_{\alpha,\beta}\colon \mathrm{SU}(2) \times \mathrm{SU}(2) \to U((\alpha + 1)(\beta + 1))$

ii) *If* $\alpha + \beta$ *is even,* $\chi_{\alpha,\beta}$ *is even,* $\chi_{\alpha,\beta}$ *has a real structure. If* $\alpha + \beta$ *is odd then,* $\chi_{\alpha,\beta}$ *has a quatonion structure.*

iii) $\chi_{\alpha,0} \otimes_{\mathbf{C}} \chi_{0,\beta} \cong \chi_{\alpha,\beta}$

iv) $\chi_{\alpha,0} \otimes_{\mathbf{C}} \chi_{\alpha',0} \cong \sum_{j=0}^{\min(\alpha,\alpha')} \chi_{\alpha+\alpha'-2j,0}$

$$\chi_{0,\beta} \otimes_{\mathbf{C}} \chi_{0,\beta'} \cong \sum_{j=0}^{\min(\beta,\beta')} \chi_{0,\beta+\beta'-2j}$$

PROOF. See for example [BtD]

Now we prove:

PROPOSITION 5.5. *Let G be compact Lie group and H a closed connected subgroup and assume* $\dim G/H = 4$. *Choose a homogeneous metric, and an orientation on G/H.*

If the rank of $\lambda(H)$ *is 2, where* $\lambda\colon H \to \mathrm{SO}(4)$ *is the linear isotropy representation (see 2.4), then for any non-negative even integer* β, *the principal* $\mathrm{SO}(\beta + 1)$*-bundle:*

$$G \times_{\chi_{0,\beta}\circ\lambda} \mathrm{SO}(\beta + 1) \to G/H \tag{5.6}$$

has a unique G-invariant connection, and this is anti-self-dual, with respect to the homogeneous metric and orientation.

PROOF. By proposition 2.8 the G-invariant connections are in 1-1 correspondence with the vector space $\mathrm{Hom}_H(\lambda, \lambda^*(\mathrm{Ad} \circ \chi_{0,\beta}))$. First we calculate $\mathrm{Ad} \circ \chi_{0,\beta}$

for β even, using proposition 5.4:

$$\mathrm{Ad} \circ \chi_{0,\beta} \cong \sum_{j=0}^{[\frac{\beta-1}{2}]} \chi_{0,2(\beta-1)-4j} \tag{5.7}$$

since $\chi_{1,1}$ is the identity map we have:

$$\begin{aligned} &\mathrm{Hom}_H(\lambda, \lambda^*(\mathrm{Ad} \circ \chi_{0,\beta})) \\ \cong &\sum_{j=0}^{[\frac{\beta-1}{2}]} \mathrm{Hom}_H(\lambda^*(\chi_{1,1}), \lambda^*(\chi_{0,2(\beta-1)-4j})) \end{aligned}$$

Now by assumption $\lambda(H)$ has the same maximal torus as SO(4), so the vector space above must be zero by proposition 5.4 and Shur's lemma. This proves that there is precisely one G-invariant connection in the bundle $G \times_{\chi_{0,\beta}\circ\lambda} \mathrm{SO}(\beta + 1) \to G/H$.

We now prove that this G-invariant connection is anti-self-dual. By proposition 2.8 and since π: $G \times_{\chi_{2,0}\circ\lambda} \mathrm{SO}(3) \to G/H$ is the SO(3)-bundle of the vector bundle of skew self-dual 2-tensors π: $\Lambda^2_+ TM \to M$ (lemma 2.6); the self-dual part of the curvature form for this connection evaluated in the point 1 is an element in $\mathrm{Hom}_H(\lambda^*(\chi_{2,0}), \lambda^*(\mathrm{Ad} \circ \chi_{0,\beta}))$. We have:

$$\begin{aligned} &\mathrm{Hom}_H(\lambda^*(\chi_{2,0}), \lambda^*(\mathrm{Ad} * \chi_{0,\beta})) \\ \cong &\sum_{j=0}^{[\frac{\beta-1}{2}]} \mathrm{Hom}_H(\lambda^*(\chi_{2,0}), \lambda^*(\chi_{0,2(\beta-1)-4j})) \end{aligned}$$

again by proposition 5.4 and Shur's lemma this is zero, we conclude that the self-dual part of the curvature form vanish and we are done.

REMARK 5.8. i) The above proposition applies in the 3 cases:

$$S^4 \simeq \frac{\mathrm{SO}(5)}{1 \oplus \mathrm{SO}(4)}$$

$$\mathrm{CP}^2 \simeq \frac{\mathrm{SU}(3)}{S(U(1) \times U(2))}$$

$$S^2 \times S^2 \simeq \frac{\mathrm{SU}(4)}{\mathrm{SO}(2) \times \mathrm{SO}(2)}$$

ii) From the proof of the proposition above it is obvious that we could replace $\chi_{0,\beta}$ and anti-self-dual connections with $\chi_{\alpha,0}$ and self-dual connections.

To be able to calculate the dimension of the moduli space of anti-self-dual connections we include the following lemma, the proof is an exercise in calculating weights of the representations of SO(4):

LEMMA 5.9. *With the same assumptions as in proposition 5.5, the first Pontrjagin class of the adjoint bundle of the bundle in 5.6 is:*

$$\left(\sum_{\{\varepsilon \mid 0 \leq \varepsilon \leqq 2\beta, \varepsilon \equiv_4 2\}} \sum_{j=0}^{\frac{\varepsilon}{2}} j^2\right)(3\tau - 2\chi)$$

where τ is the signature of G/H and χ is the Euler characteristic of G/H.

REMARK 5.10. i) In the case where $\beta = 2$, the bundle 5.6 is the SO(3)-frame bundle coming from the vector bundle, π: $\Lambda^2_- TM \to M$, of anti-self-dual skew 2-tensors

ii) In the case where $\beta = 2$ the bundle 5.6 and its adjoint bundle are isomorphic, since the natural representation of SO(3) on $\mathbf{R}^3$ and the adjoint action of SO(3) on its Lie algebra are isomorphic representations.

iii) If we use the above formula in the 3 cases in remark 5.8 we get:

On S^4:

$$p_1(\Lambda^2_-) = p_1(\chi_{0,2}) = -4$$
$$p_1(\Lambda^2_+) = p_1(\chi_{2,0}) = 4$$

On $\mathbf{CP}^2$:

$$p_1(\Lambda^2_-) = p_1(\chi_{0,2}) = -3$$
$$p_1(\Lambda^2_+) = p_1(\chi_{2,0}) = 9$$

On $S^2 \times S^2$:

$$p_1(\Lambda^2_-) = p_1(\chi_{0,2}) = -8$$
$$p_1(\Lambda^2_+) = p_1(\chi_{2,0}) = 8$$

COROLLARY 5.11. *The dimension of the moduli space of anti-self-dual connections on the* SO(3)*-bundle on S^4 (standard metric) with $p_1 = -4$ and $w_2 = 0$ is a non-empty smooth manifold of dimension* 5.

The dimension of the moduli space of anti-self-dual connections on the SO(3)*-bundle on* $\mathbf{CP}^2$ *(Study-Fubini metric) with $p_1 = -3$ and $w_2 \neq 0$ is a non-empty smooth manifold of dimension* 0.

PROOF. The S^4 case: By 4.6 there are no reducible connections in the moduli space, since $p_1 = -4$ and $H^2(S^4; \mathbf{Z}) = 0$, so by theorem 4.2 and remark 5.10, the moduli space is a smooth 5-manifold. It is non-empty by proposition 5.5.

The $\mathbf{CP}^2$ case: Again by 4.6 there are no reducible connections in the moduli space, since $p_1 = -3$ and $\mathbf{CP}^2$ is positive definite. By theorem 4.4 and remark 5.10, the moduli space is a smooth 0-manifold, and it is non-empty by proposition 5.5

REMARK 5.12. Since $S^2 \times S^2$, with the standard metric, is not self-dual we can not apply theorem 4.2 in this case, in fact we do not know of the moduli space of

SO(3)-connections on $\Lambda^2_- \to S^2 \times S^2$ is a smooth manifold. We know it is non-empty (by proposition 5.5).

Proof of theorem 5.1. The bundle we consider is $\mathrm{SO}(5) \times_{\chi_{0,2}} \mathrm{SO}(3) \to \mathrm{SO}(5)/\mathrm{SO}(4)$; for short we denote it $P \to S^4$. The group of orientation preserving conformal map, of S^4 is denoted $C^+(S^4)$ this group acts on the moduli space of anti-self-dual connections, since conformal maps preserve the Hodge star-operator. Let us fix a class $[\omega] \in \mathcal{M}^-$ we want to calculate the isotropy group $C^+_{[\omega]}$.

Let U denote an open connected subset of S^4, on which the point-wise norm of the curvature-form for ω is non-zero:

$$U \subset \{x \in S^4 \mid |\Omega^\omega|_x > 0\}$$

U is non-empty. To see this, let vol denote the volume on S^4 then observe:

$$\begin{aligned} \operatorname{tr}(\Omega^\omega \wedge \Omega^\omega) &= -\operatorname{tr}(\Omega^\omega \wedge *\,\Omega^\omega) \\ &= \operatorname{tr}(\Omega^{\omega *} \wedge *\,\Omega^\omega) \\ &= |\Omega^\omega|^2 \operatorname{vol} \end{aligned}$$

The left-hand side of this expression is a 4-form, representing the first Pontrjagin class in de Rham cohomology, thus:

$$-\frac{1}{8\pi^2}\int_{S^4} |\Omega^\omega|^2 \operatorname{vol} = -4$$

We deduce $|\Omega^\omega|$ must be non-zero on an open subset of S^4 (alternatively, observe Ω^ω is real-analytic, so if it is zero on an open set it is identically zero).

Now define a new metric $\tilde{g}$ on U, conformal to the standard metric g on U:

$$\tilde{g} = |\Omega^\omega| g \quad \text{on } U$$

From the definition of $\tilde{g}$ it follows that the isotropy subgroup of $[\omega]$; fixes $\tilde{g}$, thus $C^+_{[\omega]}$ acts on $(U, \tilde{g})$ by isometries. Now recall that a riemannian manifold of dimension n, has isometry group of dimension at most $\frac{1}{2}n(n+1)$ and if the dimension is maximal then the manifold is a space of constant curvature [KNI p. 238]. By corollary 5.11 and since the group $C^+(S^4)$ has dimension 15, we have that:

$$\dim C^+_{[\omega]} \geqq \dim C^+(S^4) - \dim \mathcal{M}^- = 15 - 5 = 10$$

Thus $(U, \tilde{g})$ has to be isometric to (S^4, g) [KNI, p. 308] and we have an isometry σ:

$$\sigma\colon (U, \tilde{g}) \to (S^4, g)$$

or:

$$\sigma\, \mathrm{SO}(5)\, \sigma^{-1} = C^+_{[\omega]} \quad \sigma \in C^+(S^4)$$

We conclude that $[\sigma^*(\omega)]$ is SO(5)-invariant.

We now want to apply corollary 2.18, we want to show that $\sigma^*(\omega)$ is gauge equivalent to the SO(5)-invariant connection in proposition 5.5. In our case we see that:

$$\mathrm{Hom}^P(G_{x_0}; K) = \mathrm{Hom}^P(\mathrm{SO}(4); \mathrm{SO}(3))$$

P does not have a reduction (else there would be a class in $H^2(S^4; \mathbf{Z})$ with square -4) so if $\chi \in \mathrm{Hom}^P(\mathrm{SO}(4); \mathrm{SO}(3))$ then χ has to be surjective (remark 1.6). From proposition 5.4 it follows that:

$$\mathrm{Hom}^P(\mathrm{SO}(4); \mathrm{SO}(3))/\mathrm{Inner}(\mathrm{SO}(3)) = \{\chi_{0,2}\}$$

so there is only 1 equivalence class of lifts of the usual SO(5)-action on S^4 to the bundle $P \to S^4$, namely the action considered in proposition 5.5. Corollary 2.18 now gives that $\sigma^*(\omega)$ is gauge equivalent to the connection in proposition 5.5, and we are done.

Proof of theorem 5.2. The bundle we consider is $\mathrm{SU}(3) \times_{\chi_{0,2}\circ\lambda} \mathrm{SO}(3) \to \mathrm{SU}(3)/U(2)$ for short we denote it $P \to \mathbf{CP}^2$.

The group SU(3) acts on $\mathbf{CP}^2$ by orientation preserving isometries, hence it acts on the moduli space of anti-self-dual connections, since isometries preserve the Hodge star-operator.

Let us fix a class $[\omega] \in \mathcal{M}^-$ we want to calculate the isotropy group $\mathrm{SU}(3)_{[\omega]}$. By corollary 5.11 the dimension of $\mathcal{M}^-$ is 0, hence $\mathrm{SU}(3)_{[\omega]}$ is a closed subgroup of SU(3) of the same dimension as SU(3), since SU(3) is connected we conclude:

$$\mathrm{SU}(3)_{[\omega]} = \mathrm{SU}(3)$$

We will use the same type of argument as in the proof of theorem 5.1. In our case we see that:

$$\mathrm{Hom}^P(G_{x_0}; K) = \mathrm{Hom}^P(U(2); \mathrm{SO}(3))$$

again P does not have a reduction (else there would be a class in $H^2(\mathbf{CP}^2; \mathbf{Z})$ with square -3) so if $\chi \in \mathrm{Hom}^P(U(2); \mathrm{SO}(3))$ then χ has to be surjective (remark 1.6). It is not hard to see that there is only one surjective real representation of $U(2)$ in SO(3), and it is the linear isotropy representation λ: $U(2) \to \mathrm{SO}(4)$ composed with $\chi_{0,2}$:

$$\mathrm{Hom}^P(U(2); \mathrm{SO}(3))/\mathrm{Inner}(\mathrm{SO}(3)) = \{\chi_{0,2} \circ \lambda\}$$

so there is only one equivalence class of lifts of the usual SU(3)-action on $\mathbf{CP}^2$ to the bundle $P \to \mathbf{CP}^2$, namely the action considered in proposition 5.5.

Hence ω has to be gauge equivalent to a SU(3)-invariant connection. By proposition 5.5 there is only one SU(3)-invariant connection on the our bundle.

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