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WAVELET TRANSFORM AND TOEPLITZ-HANKEL TYPE OPERATORS*

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§1. Introduction.

Let G denote the affine group. It consists of $\{(x,y)\colon y > 0, x \in R\}$ with the group law $(x',y')(x,y) = (y'x + x', y'y)$. It is a locally compact nonunimodular group with right Haar measure $d\mu_R(x,y) = dx\,dy/y$ and left Haar measure $d\mu_L(x,y) = dx\,dy/y^2$. It can be identified as the quotient group of SL$(2,R)$ by SO$(2,R)$ (see [17]). The identification is made by

$$g = (x,y) \Leftrightarrow \begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix}.$$

And we have

$$\begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix}^{-1} = \begin{pmatrix} 1/\sqrt{y} & -x/\sqrt{y} \\ 0 & \sqrt{y} \end{pmatrix}.$$

We consider the representation U of G on $L^2(R)$ defined by

$$U_g f(x') = \frac{1}{\sqrt{y}} f\left(\frac{x'-x}{y}\right). \tag{1.1}$$

Then U is reducible on $L^2(R)$, but irreducible on the Hardy space $H^2(R)$.

Following Paul [17] (cf. Grossmann et al [9] and Meyer [13]), we call function ψ to be an admissible wavelet if it satisfies $0 < \|\psi\|_{L^2} < \infty$ and

$$\int_G |(\psi, U_g\psi)|^2 \, d\mu_K(g) < \infty, \tag{1.2}$$

where $(.,.)$ is the scalar product on $L^2(R)$.

For an admissible wavelet ψ, we say it is an admissible analyzing wavelet if its

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Fourier transform is supported in $[0, +\infty)$ and we let AAW denote the space consisting of all such functions, and let $\overline{\text{AAW}} = \{\psi\colon \bar{\psi} \in \text{AAW}\}$.

Let U be the upper half-plane, $\{x + iy, y > 0\}$. The space $L^{2,-2}(U)$ consists of all functions on U for which the integral $\|f\|_2^2 = \int_U |f(x,y|^2\, dx\, dy/y^2$ is finite, i.e. $L^2(G, d\mu_L)$. For $\psi \in \text{AAW}$, write

$$c_\psi = \frac{\int |(\psi, U_g\psi)|^2\, d\mu_L(g)}{(\psi, \psi)},$$

we define the operator T from H^2 onto a subspace (denoted by $L_0^{2,-2}$) of $L^{2,-2}(U)$ by

$$(Tf)(g) = c_\psi^{-\frac{1}{2}}(f, U_g\psi). \tag{1.3}$$

Then (see [17])

$$\int (Tf)(g)\overline{(Tf)(g)}\, d\mu_L(g) = (f,f).$$

By (1.1), we can write T as

$$(Tf)(g) = c_\psi^{-\frac{1}{2}}\, \tilde{\psi}_y * f(x),$$

where $\psi_y(x) = y^{-\frac{1}{2}}\psi(y^{-1}x)$ and $\tilde{\psi}(x) = \overline{\psi(-x)}$. Thus T is a "continuous wavelet transform" (see [4]).

Let τ denote the operator from $L_0^{2,-2}$ onto H^2 defined by

$$(\tau F)(x) = c_\psi^{-\frac{1}{2}} \int_0^\infty (\psi_y * F(.,y))(x)\frac{dy}{y^2}, \tag{1.4}$$

then τT is the identity on H^2. More explicity,

$$f(x) = c(\psi)^{-1} \int_0^\infty \tilde{\psi}_y * \psi_y * f(x)\frac{dy}{y^2} \tag{1.5}$$

for all $f \in H^2$. (1.5) is the well-known Calderon reproducing formula (see [14]). It can be used as starting points for the construction of time frequency localization or filter operators and be used in many other fields of science or technology (see [2], [3], [5], [6]). The discrete version of (1.5) is $f(x) = \sum_{\lambda\in\Lambda} c(\lambda)\psi_\lambda(x)$, where λ is a suitable discrete set. There are many works about this problem ([3], [4], [7], [8], [14]).

Nowak and Rochberg considered the following interesting problem. Let P denote the orthogonal projection from $L^{2,-2}$ onto $L_0^{2,-2}$, they defined the Toeplitz operator $T_b = PM_bP$, and the Hankel operator $H_b = (I - P)M_bP$, then studied the boundedness, compactness and membership in the Schatten-von

Neumann class of the above operators. In this paper using a decomposition of AAW and $\overline{\text{AAW}}$ by Laguerre polynomials, we decompose $L^{2,-2}$ to be the orthogonal sum $\oplus_{k=0}^{\infty}(A_k \oplus \bar{A}_k)$. Let P_k (resp. $\bar{P}_k$) be the orthogonal projection from $L^{2,-2}$ onto A_k (resp. $\bar{A}_k$). Then we define the Toeplitz type operators $T_b^{(k,l)} = P_k M_b P_l$, the small and big Hankel type operators $h_b^{(k,l)} = \bar{P}_k M_b P_l$, $H_b^{(k,l)} = (I - \sum_{v=0}^{k} P_v) M_b P_l$ with anti-analytic symbol $b(z)$ on U. They are called the Ha-plitz operators (using the terminology of Nilkol'skii [15]). We then study the boundedness and membership in the Schatten-von Neumann class of the above Ha-plitz operators.

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§2. The decomposition of $L^{2,-2}$ and the main results.

By computing the admissibility condition (1.2), we easily get (or see [9])

$$\text{AAW} = \left\{\psi\colon \int_0^\infty |\psi(\xi)|^2 \frac{d\xi}{\xi} < \infty, 0 < \|\psi\|_2 < \infty, \operatorname{supp}\hat{\psi} \subset [0,\infty)\right\}.$$

Let $L_n^{(\alpha)}(x) = \sum_{v=0}^{n} \binom{n+\alpha}{n-v}(-x)^v/v!$ be the Laguerre polynomials, where $\alpha > -1$. They satisfy the following conditions of orthogonality and normalization (see [21]):

$$\int_0^\infty e^{-x} x^\alpha L_n^{(\alpha)}(x) L_m^{(\alpha)}(x)\,dx = \Gamma(\alpha+1)\binom{n+\alpha}{n}\delta_{nm}.$$

And for $k \in Z^+$, let ψ^k, $\bar{\psi}^k$ be functions on R, their Fourier transforms are defined by

$$\hat{\psi}^k(\xi) = \begin{cases} (k+1)^{-\frac{1}{2}}(2\xi)e^{-\xi}L_k^{(1)}(2\xi), & \text{for } \xi \ge 0 \\ 0, & \text{for } \xi < 0 \end{cases}$$

and $\hat{\bar{\psi}}^k(\xi) = \hat{\psi}^k(-\xi)$. We can get

$$\psi^k(x) = -\frac{2}{\pi}(k+1)^{\frac{1}{2}}\left(\frac{x-i}{x+i}\right)^k \frac{1}{(x+i)^2}$$

and $\bar{\psi}^k(x) = \overline{\psi^k(x)}$. Clearly, for each $k \in Z^+$, $\psi^k \in \text{AAW}$, $\tilde{\psi}^k = \psi^k$ and $\bar{\psi}^k \in \overline{\text{AAW}}$, $\tilde{\bar{\psi}}^k = \bar{\psi}^k$. Thus by (1.5)

$$f(x) = \int_0^\infty \psi_y^k * \psi_y^k * f(x)\frac{dy}{y^2}, \tag{2.1}$$

$$h(x) = \int_0^\infty \bar{\psi}_y^k * \bar{\psi}_y^k * h(x)\frac{dy}{y^2}$$

for all $f \in H^2$ and $h \in \bar{H}^2$, where $H^2(R)$ and $\bar{H}^2(R)$ are the usual Hardy space and conjugate Hardy spaces on R, i.e.

$$H^2(R) = \{f \in L^2(R): \operatorname{supp} \hat{f} \subset [0, \infty)\},$$
$$\bar{H}^2(R) = \{f \mathrm{M}\, \mathrm{L}^2(R): \operatorname{supp} \hat{f} \subset (-\infty, 0]\}.$$

By Theorem 5.7.1 in [21], i.e., $\{x^{\frac{1}{2}} e^{-\frac{x}{2}} L_k^{(1)}(x)\}_{k=0}^{\infty}$ is complete in $L^2(0, \infty)$, we have

$$\mathrm{AAW} = \operatorname{span}\{\psi^k\}_{k \geqq 0}, \overline{\mathrm{AAW}} = \operatorname{span}\{\bar{\psi}^k\}_{k \geqq 0}$$

We define the subspaces A_k and $\bar{A}_k$ of $L^{2,-2}$ by

$$A_k = \{f * \psi_y^k(x): f \in H^2\},$$
$$\bar{A}_k = \{f * \psi_y^k(x): f \in \bar{H}^2\}.$$

Then we can prove the following theorem

THEOREM 1. *Let A_k and $\bar{A}_k$ be defined as above, then*

$$L^{2,-2} = \oplus_{k=0}^{\infty} (A_k \oplus \bar{A}_k).$$

Now let us give the bases of A_k and $\bar{A}_k$. Let φ_n be functions defined by

$$\hat{\varphi}_n(\xi) = \begin{cases} e^{-\xi} L_n^{(0)}(2\xi), & \text{for } \xi \geq 0 \\ 0, & \text{for } \xi < 0. \end{cases}$$

and $\bar{\varphi}(x) = \overline{\varphi_n(x)}$, and let $e_{nk}(x, y)$ be functions whose Fourier transforms of the first variable satisfy

$$\hat{e}_{nk}(\xi, y) = y^{\frac{1}{2}} \hat{\varphi}_n(\xi) \hat{\psi}^k(y\xi).$$

Then $A_k = \operatorname{span}\{e_{nk}(x, y)\}_{n \geqq 0}$ and $\bar{A}_k = \operatorname{span}\{\overline{e_{nk}(x, y)}\}_{n \geqq 0}$. An easy compution gives that

$$e_{nk}(x, y) = \frac{(k+1)^{\frac{1}{2}}}{2\pi} \sum_{v=0}^{n} \sum_{j=0}^{k} \binom{v+j+1}{v} \binom{k}{j} \binom{n}{v} \frac{(-2)^{v+j+1} y^{j+\frac{3}{2}}}{(y+1-ix)^{v+j+2}}.$$

In the definitions of T and τ in §1, letting $\psi = \psi^k$, we get the corresponding T_k and τ_k. As we mentioned in §1, T_k and τ_k give the isometries: $A_k \cong H^2$ for all $k \in Z^+$. Similarly, define $\bar{T}_k$ and $\bar{\tau}_k$ from $\overline{\mathrm{H}^2}$ to $\bar{A}_k$ and from $\bar{A}_k$ to $\bar{H}^2$ respectively, we then have $\bar{A}_k \cong \bar{H}^2$.

We now give the reproducing kernel of A^k, denoted by $K^{(k)}(z, w)$. Namely, $F(z) = \langle F, K_z^{(k)} \rangle$ for all $F \in A_k$, where $\langle .,. \rangle$ is scalar product on $L^{(2,-2)}(U)$ and $K_z^{(k)}(w) = K^{(k)}(w, z)$. For $f \in H^2$, by (2.1), $f * \psi_y^k(x) = \psi_y^k * \int_0^\infty \psi_v^k * \psi_v^k * f(x)\, dv/v^2 = \int_U (\psi_y^k * \psi_v^k)(x-u)(f * \psi_v^k)(u)\, du\, dv/v^2$, thus

$$K^{(k)}(z,w) = \psi_y^k * \psi_v^k(x-u) = (\psi_z^k, \psi_w^k), \tag{2.2}$$

where $z = x + iy$, $w = u + iv$, $\psi_y^k(\cdot) = (1/\sqrt{y})\psi^k(\cdot - x/y) = U_z\psi^k(\cdot)$.

Similarly we get easily the reproducing kernel (denoted by $\tilde{K}^{(k)}(z,w)$) of $\bar{A}_k$:

$$\tilde{K}^{(k)}(z,w) = \overline{K^{(k)}(z,w)} = K^{(k)}(w,z).$$

For fixed $w \in U$, the Fourier transform of $K_w^{(k)}(z)$ about the x variable is

$$\hat{K}_w^{(k)}(\xi, y) = y^{\frac{1}{2}} v^{\frac{1}{2}} e^{-i\xi u} \hat{\psi}^k(\xi y)\hat{\psi}^k(\xi v). \tag{2.3}$$

By (2.3) and the Fourier inversion formula, we have (omitting the details of calculation)

$$K_w^{(k)}(z) = \frac{1}{2\pi(k+1)} \sum_{s=0}^{k} \sum_{j=0}^{k} \binom{k+1}{k-s}\binom{k+1}{k-j} \frac{(s+j+2)!}{s!j!} \times$$
$$\times \left(\frac{2iy}{\bar{w}-z}\right)^s \left(\frac{2iv}{\bar{w}-z}\right)^j \frac{4iy^{\frac{1}{2}}v^{\frac{1}{2}}}{(\bar{w}-z)^3}.$$

If $z, w \in U$, then $y \leqq |\bar{w} - z|$, $v \leqq |\bar{w} - z|$, thus we have

PROPOSITION 1. *For all* $k \in Z^+$,

$$|K^{(k)}(z,w)| \leq c_k \frac{(yv)^{\frac{1}{2}}}{|\bar{w}-z|^3},$$

where c_k *are constants depending only on* k.

Let P_k (resp. $\bar{P}_k$) denote the orthogonal projection from $L^{2,-2}$ onto A_k (resp. $\bar{A}_k$). Again by (2.1), we have the following

PROPOSITION 2. *For* $F \in L^{2,-2}$,

$$P_k(F)(x,y) = \int_0^\infty \psi_y^k * \psi_v^k * F(\cdot, v)(x) \frac{dv}{v^2} = \langle F, K_z^{(k)} \rangle, \tag{2.5}$$

$$\bar{P}_k(F)(x,y) = \int_0^\infty \bar{\psi}_v^k * \bar{\psi}_v^k * F(\cdot, v)(x) \frac{dv}{v^2} = \langle F, \tilde{K}_z^{(k)} \rangle. \tag{2.6}$$

We define the Toeplitz type operators $T_b^{(k,l)} = P_k M_b P_l$, the small and big Hankel type operators $h_b^{(k,l)} = \bar{P}_k M_b P_l$, $H_b^{(k,l)} = (I - \sum_{v=0}^{k} P_v) M_b P_l$ with anti-analytic symbol $b(z)$ on U, here M_b is the operator of multiplication by b.

In this paper we will consider the analytic Besov spaces $B_p(U)$ on U and $B_p(R)$ on R. The space $B_p(U)$ $(0 < p < \infty)$ consists of all analytic functions on U for which the integral $\|F\|_{B_p}^p = \int_U |y^m F^{(m)}(z)|^p y^{-z} \, dx\, dy$ is finite and $B_\infty(U)$ is the Bloch space, i.e., $F(z)$ analytic on U and $\|F\|_{B_\infty} = \sup_{z \in U} |y^m F^{(m)}(z)|$ is finite, here m is any integer such that $m > 1/p$. The space $B_p(R)$, consists of all functions on R such that

$$\|f\|_{B_p}^p = \sum_{j=-\infty}^{\infty} 2^j \|\psi_j * f\|_{L^p}^p < \infty$$

where $\psi_j(x) = 2^j\psi(2^j x)$ and $\psi \in S(R)$ is a function such that $\hat{\psi}(\xi) = 1$ for $\xi \in \{\xi : 1 \leqq |\xi| \leqq 2\}$ and $\operatorname{supp} \hat{\psi} \subset \{\xi : 1/2 \leqq |\xi| \leqq 4\}$. If $F(z)$ is analytic on U and it can be writen as $\hat{F}(\xi, y) = \hat{f}(\xi) \cdot e^{-|\xi| y}$, then $F(Z) \in B_p(U)$ iff $f \in B_p(R)/\mathscr{P}$ and $\operatorname{supp} \hat{f} \subset [0, \infty)$ with equivalent norms (see [18]), where $\mathscr{P}$ is the set of all polynomials.

Let $S_p(H_1, H_2)$ denote the Schatten-von Neumann class from one Hilbert space H_1 to another H_2 ($S_\infty(H_1, H_2)$ denotes the set of bounded operators).

The main results about the above operators is the following

THEOREM 2. *Let* $0 < p \leqq \infty$, *then* $h_b^{(k,l)} \in S_p$ *iff* $\overline{b(z)} \in B_p(U)$.

THEOREM 3. *Let* $T_b^{(k,l)}$ *be defined as above, then*

(1) *If* $k < l$, *then* $T_b^{(k,l)}$ *is a zero operator;*

(2) *If* $k = l$, *then* $T_b^{(k,l)}$ *is bounded iff* $b(z) \in L^\infty$; *and* $T_b^{(k,l)}$ *is never compact unless* $b \equiv c$;

(3) *If* $k > l$, $\dfrac{1}{k-l} < p \leqq \infty$, *then* $T_b^{(k,l)} \in S_p$ *iff* $\overline{b(z)} \in B_p(U)$;

(4) *If* $k > l, 0 < p \leqq \dfrac{1}{k-l}$ *and* $T_b \in S_p$, *then* $b(z) \equiv c$.

THEOREM 4. *For* $k \geqq l$,

(1) *If* $\dfrac{1}{k-l+1} < p \leqq \infty$ *and* $H_b^{(k,l)} \in S_p$ *iff* $\overline{b(z)} \in B_p(U)$;

(2) *If* $0 < p \leqq \dfrac{1}{k-l+1}$ *and* $H_b^{(k,l)} \in S_p$, *then* $b(z) \equiv c$.

The phenomena in (4) of Theorem 3 and (2) of Theorem 4 are called the cut-off. Theorem 3 (resp. Theorem 4) says that $T_b^{(k,l)}$ has the cut-off at $1/(k-l)$ for $k > l$ (resp. $H_b^{(k,l)}$ at $1/(k-1+1)$ for $k \geqq l$). We will prove Theorem 2 and Theorem 3 in §3 and §4 respectively, and prove Theorem 4 for $p = \infty$ in §5 and for $0 < p < \infty$ in §6 respectively.

Let $-1 < \alpha < \infty$ and $d\mu_\alpha = y^\alpha \, dx \, dy$ be the weighted measure. One can consider the space $L^{\alpha,2}(U)$ consisting the square integrable functions on U with respect to $d\mu_\alpha$. In [12], an orthogonal decomposition of $L^{\alpha,2}(U)$ is given to be $\oplus_{k=0}^\infty (A_k \oplus \bar{A}_k)$ (A_0 is just the Bergman space), then operators of more general types $h_b^{(k,l,k')}$, $T_b^{(k,l,k')}$ and $H_b^{(k,l,k')}$ are defined and studied.

§3. The operator $h_b^{(k,l)}$.

Recall $h_b^{(k,l)} = \bar{P}_k M_b P_l$. For $F(x,y) = f * \psi_y^l(x) \in A_l$, by Proposition 2 in §1, we have

$$(h_b^{(k,l)}F)(x,y) = \int_0^\infty \bar{\psi}_y^k * \bar{\psi}_v^k * [b(\cdot + iv) f * \psi_v^l(\cdot)](x) \frac{dv}{v^2}.$$

Taking Fourier transform about the first variable

$$\begin{aligned}(h_b^{(k,l)}F)^\wedge(\xi,y) &= y^{\frac{1}{2}}\hat{\psi}^k(-\xi y)\int_0^\infty v^{\frac{1}{2}}\hat{\psi}^k(-\xi v)[b(\cdot + iv) f * \psi_v^l]^\wedge(\xi)\frac{dv}{v^2} \\ &= y^{\frac{1}{2}}\hat{\psi}^k(-\xi y)\int_0^\infty \hat{\psi}^k(-\xi v)\frac{1}{2\pi}\int \hat{b}(\cdot + iv)(\xi - \eta)\hat{f}(\eta)\hat{\psi}^l(v\eta)\,d\eta\frac{dv}{v} \\ &= \frac{y^{\frac{1}{2}}\hat{\psi}^k(-\xi y)}{2\pi}\int_0^\infty\int_{-\infty}^\infty \hat{b}(\xi-\eta)e^{-v(\eta-\xi)}\hat{f}(\eta)\hat{\psi}^k(-\xi v)\hat{\psi}^l(v\eta)\,d\eta\frac{dv}{v}.\end{aligned}$$

By direct calculation, for $\xi < 0$, we have

$$\int_0^\infty e^{-v(\eta-\xi)}\hat{\psi}^k(-\xi v)\hat{\psi}^l(\eta v)\frac{dv}{v} = c(k,l)\left(\frac{\eta}{\eta-\xi}\right)^{k+2}\left(\frac{-\xi}{\eta-\xi}\right)^{l+2}$$

where $c(k,l) = (k+1)^{\frac{1}{2}}(l+1)^{-\frac{1}{2}}\sum_{0 \leqq j \leqq \min(k,l)}\binom{k}{j}\binom{l+1}{j+1}$. And for $\xi \geqq 0$, $(h_b^{(k,l)}F)^\wedge(\xi,y) = 0$. Thus

$$(h_b^{(k,l)}F)^\wedge(\xi,y) = \frac{1}{2\pi}\int_0^\infty \hat{b}(\xi-\eta)\hat{f}(\eta)a_y^{(k,l)}(\xi,\eta)\,d\eta, \tag{3.1}$$

where

$$a_y^{(k,l)}(\xi,\eta) = \begin{cases} c(k,l)\left(\dfrac{\eta}{\eta-\xi}\right)^{k+2}\left(\dfrac{-\xi}{\eta-\xi}\right)^{l+2} y^{\frac{1}{2}}\hat{\psi}^k(-\xi y), & \text{for } \xi \leqq 0,\ \eta \geqq 0 \\ 0, & \text{elsewhere.}\end{cases}$$

From (3.1), we know $h_b^{(k,l)}$ are vector-valued paracommutators (see [1]). Since $A_k \cong H^2$ and $\bar{A}_k \cong \bar{H}^2$, we can change these vector-paracommutators into usual paracommutators.

Let $\tilde{h}_b^{(k,l)}$ be operator from H^2 to $\bar{H}^2$ defined by

$$\tilde{h}_b^{(k,l)} = \bar{\tau}_k h_b^{(k,l)} T_l$$

where $\bar{\tau}_k$ and T_l are operators defined in §2. Then for $f \in H^2$,

$$(h_b^{(k,l)}f)^\wedge(\xi) = \frac{1}{2\pi}\int_0^\infty \hat{b}(\xi - \eta)\hat{f}(\eta)a^{(k,l)}(\xi,\eta)\,d\eta,$$

where

$$a^{(k,l)}(\xi,\eta) = \begin{cases} c(k,l)\left(\dfrac{\eta}{\eta-\xi}\right)^{k+2}\left(\dfrac{-\xi}{\eta-\xi}\right)^{l+2}, & \text{for } \xi \leqq 0,\ \eta \geqq 0 \\ 0, & \text{elsewhere.}\end{cases}$$

Thus by the theory of paracommutator and the fact that $a^{(k,l)}$ satisfies the conditions A_0, A_1, $A_3(\infty)$, A_4, $A_{4\frac{1}{2}}$, we know that Theorem 2 is true (cf. [11], [19]).

§4. The operator $T_b^{(k,l)}$.

Recall $T_b^{(k,l)} = P_k M_b P_l$. For $F(x,y) = f * \psi_y^l(x) \in A_l$, as we did in §3, we know that $T_b^{(k,l)}$ are also vector-valued paracommutators:

$$(T_b^{(k,l)}F)^\wedge(\xi,y) = \frac{1}{2\pi}\int_0^\infty \hat{b}(\xi-\eta)\hat{f}(\eta)A_y^{(k,l)}(\xi,\eta)\,d\eta, \tag{4.1}$$

and a similar calculation gives that

$$A_y^{(k,l)}(\xi,y) = \begin{cases} (k+1)^{\frac{1}{2}}(l+1)^{-\frac{1}{2}}y^{\frac{1}{2}}\hat{\psi}^k(\xi y)\dfrac{\xi}{\eta}c^{(k,l)}\left(\dfrac{\xi}{\eta}\right), & \text{for } 0 \leqq \xi \leqq \eta \\ 0, & \text{elsewhere}\end{cases}$$

where $c^{(k,l)}(t) = \int_0^\infty xe^{-x}L_k^{(1)}(tx)L_l^{(1)}(x)\,dx$, equaling to 0 for $l > k$ and to $(k+1)!/(l!(k-l)!)(1-t)^{k-l}$ for $k \geqq l$.

Thus if $l > k$, $T_b^{(k,l)}$ is the zero operator. For $k \geqq l$, we also can change the above vector-valued paracommutators into usual paracommutators. Let $t_b^{(k,l)}$ be the operator from H^2 to itself defined by $t_b^{(k,l)} = \tau_k T_b^{(k,l)} T_k$, where τ_k and T_k are the operators defined in §2. Thus $t_b^{(k,l)} \in S_p$ if $T_b^{(k,l)} \in S_p$. For $f \in H^2$, we have

$$(t_b^{(k,l)}f)^\wedge(\xi) = \frac{1}{2\pi}\int_0^\infty \hat{b}(\xi-\eta)\hat{f}(\eta)A^{(k,l)}(\xi,\eta)\,d\eta$$

where

$$A^{(k,l)} = \begin{cases} \left[\dbinom{k+1}{l+1}\dbinom{k}{l}\right]^{\frac{1}{2}}\left(\dfrac{\xi}{\eta}\right)^{k+1}\left(1-\dfrac{\xi}{\eta}\right)^{k-l}, & \text{for } 0 \leqq \zeta \leqq \eta \\ 0, & \text{elsewhere.}\end{cases}$$

By the theory of paracommutator and the fact that $A^{(k,l)}$ satisfies the conditions A_0, A_1, $A_3(k-l)$, A_4, $A_{4\frac{1}{2}}$ and that $A^{(k,k)}|_{\xi=\eta>0} \neq 0$, we get (1), (2) and (3) of Theorem 3 (cf. [11], [19]). For (4) of Theorem 3, since $T_b^{(k,l)} \in S_p \subset S_2$, by (3) of

Theorem 3, $\overline{b(z)} \in B_2(U)$. We write $\overline{b(z)} = b * P_y(x)$, where $P_y(x)$ is the Poisson kernel and $\bar{b}(x)$ is the boundary value of $\overline{b(z)}$. We first prove that $\bar{b}(x)$ is a polynomial. If it is not, then there exists a $\theta \neq 0, \theta \in \operatorname{supp} \hat{\bar{b}}$. Without loss of generality, we assume that $\theta = -1$, then there exist two functions g and h such that

$$\left| \int\int \bar{b}(\xi - 1 - \eta)\hat{g}(\xi)\hat{h}(\eta)\, d\xi\, d\eta \right| > c \neq 0$$

and $\|g\|_{L^2} = \|h\|_{L^2} = 1$, $\operatorname{supp} \hat{g}$, $\operatorname{supp} \hat{h} \subset B(0, \delta)$, here σ is a constant such that $0 < \delta < \frac{1}{2}$. We let $B_n = B(n, \delta)$, $\tilde{B}_n = B(n + 1, \delta)$ and set $\hat{g}_n(\xi) = \hat{g}(\xi - n)$, $\hat{h}_n(\xi) = \hat{h}(\xi - n - 1)$, thus we have $\operatorname{supp} \hat{g}_n \subset B_n$, $\operatorname{supp} \hat{h}_n \subset \tilde{B}_n$ and $\|g_n\|_2 = \|h_n\|_2 = 1$. We have

$$\|T_b^{(k,l)}\|_{S_p}^p \geqq \sum_{n=2}^{\infty} \|T_b^{(k,l)}\|_{S_p(\tilde{B}_n \times B_n)}^p \geqq \sum_{n=2}^{\infty} \|T_b^{(k,l)}\|_{S_\infty(\tilde{B}_n \times B_n)}^p \tag{4.2}$$

and

$$\|T_b^{(k,l)}\|_{S_\infty(\tilde{B}_n \times B_n)} = \sup |\langle T_b^{(k,l)}(\tilde{\varphi}_n), \varphi_n \rangle|$$

the sup being taken over all functions φ_n, $\tilde{\varphi}_n$ such that $\|\varphi_n\|_2$, $\|\tilde{\varphi}_n\| \leqq 1$ and $\operatorname{supp} \varphi_n \subset B_n$, $\operatorname{supp} \tilde{\varphi}_n \subset \tilde{B}_n$. Then

$$\begin{aligned}
&\left| \int\int \bar{b}(\xi - 1 - \eta)\hat{h}(\eta)\hat{g}(\xi)\, d\xi\, d\eta \right| \\
&= \left| \int\int \bar{b}(\xi - \eta)\hat{h}(\eta - n - 1)\hat{g}(\xi - n)\, d\xi\, d\eta \right| \\
&= c\left| \int\int \bar{b}(\xi - \eta)\hat{h}_n(\eta)\hat{g}_n(\xi) A^{(k,l)}(\xi, \eta) \left(\frac{\eta}{\xi}\right)^{l+1} \left(1 - \frac{\xi}{\eta}\right)^{l-k} d\xi\, \xi\eta \right| \\
&= c\left| \int\int \bar{b}(\xi - \eta)\hat{h}_n(\eta)\hat{g}_n(\xi) A^{(k,l)}(\xi, \eta) \sum_{v=0}^{\infty} \frac{(v + k - l - 1)!}{(k - l - 1)!} \left(\frac{\xi}{\eta}\right)^{v-l-1} d\xi\, d\eta \right| \\
&\leqq c\, \|T_b^{(k,l)}\|_{S_\infty(\tilde{B}_n \times B_n)} \sum_{v=0}^{\infty} \frac{(v + k - l - 1)!}{(k - l - 1)!} \|\xi^{v-1-1}\hat{g}_n(\xi)\|_2\, \|\eta^{l-v+1}\hat{h}_n(\eta)\|_2 \\
&\leq c\, \|T_b^{(k,l)}\|_{S_\infty(\tilde{B}_n \times B_n)} \sum_{v=0}^{\infty} \frac{(v + k - l - 1)!}{(k - l - 1)!} \left(\frac{n + \delta}{n + 1 - \delta}\right)^{v} \\
&= c\, \|T_b^{(k,l)}\|_{S_\infty(\tilde{B}_n \times B_n)} \left(\frac{n + 1 - \delta}{1 - 2\delta}\right)^{k-l}.
\end{aligned}$$

Thus $\|T_b^{(k,l)}\|_{S_\infty(\tilde{B}_n \times B_n)} \geqq cn^{l-k}$, and by (4.2)

$$\|T_b^{(k,l)}\|_{S_p}^p \geqq c \sum_{n=2}^{\infty} \frac{1}{n^{(k-l)p}} = +\infty$$

this contradicts $T_b^{(k,l)} \in S_p$. This contradiction shows that $\bar{b}(x)$ must be a polynomial.

If $\overline{b(z)} = \bar{b} * P_y$ is analytic on U and $\bar{b}(x)$ is a polynomial, then $\overline{b(z)}$ must be a constant. Hence $b(z) \equiv c$ and (4) of Theorem 3 is true.

§5. The operator $H_b^{(k,l)}$ for $p = \infty$.

Recall $H_b^{(k,l)} = (I - \sum_{v=0}^{k} P_v)M_b P_l = M_b P_l - \sum_{v=0}^{k} T_b^{(k,l)}$. By Theorem 3, if $l > k$, then $H_b^{(k,l)} = M_b P_l$ and $H_b^{(k,l)} \in S_\infty$ iff $b \in L^\infty$. Thus from now on we assume $k \geqq l$.

We define $T_1 \prec T_2$, if $T_1^* T_1 \leqq T_2^* T_2$. We note that $h_b^{(k,l)} \prec H_b^{(k,l)}$, thus by Theorem 2, the converse part of (1) in Theorem 4 is true. And we note $H_b^{(k,l)} - H_b^{(k+1,l)} = T_b^{(k+1,l)} \prec H_b^{(k,l)}$, thus if $H_b^{(k,l)} \in S_p$ for $0 < p \leqq 1/(k+1-l)$. Thus by Theorem 3, we know (2) of Theorem 4 is true. So that we only need prove the direct part of (1), i.e. if $1/(k-l+1) < p \leqq \infty$ and $\overline{b(z)} \in B_p(U)$, then $H_b^{(k,l)} \in S_p$. We will prove the case $p = \infty$ in this section and $1/(k+1-l) < p < \infty$ in §6.

Note that $H_b^{(k,l)} \prec (I - P_l)M_b P_l$, it suffices to prove that if $\overline{b(z)} \in B_\infty(U)$, then $(I - P_l)M_b P_l \in S_\infty$. For $F(z) \in A_l$,

$$(5.1) \qquad (I - P_l)M_b P_l F(z) = \int_U (b(z) - b(w)) K^{(l)}(z,w) F(w)\, d\mu_{-2}(w)$$

where $d\mu_{-2}(w) = du\, dv/v^2$ and $K^{(l)}(z,w)$ is the reproducing kernel of A_l.

Proposition 3. *Let* $1 \leqq q \leqq \infty$, $b(z) \in B_\infty(U)$ *and* $K(z,w) = |b(z) - b(w)|\,(yv)^{\frac{3}{2}}/|\bar{z} - w|^3$, *then the operators* $\mu \to \int_U K(z,w)\mu(w)\, d\mu_{-2}(w)$ *and* $\mu \to \int_U K(z,w)\mu(z)\, d\mu_{-2}(z)$ *are bounded from* $L^q(U)$ *to itself, where* $L^q(U) = \{F(z): \int_U |F(z)|^q\, d\mu_{-2}(z) < \infty\}$.

If Proposition 3 is true, then by Proposition 2 in §1, Proposition 3 for $q = 2$ and (5.1), we know $I - P_l M_b P_l \in S_\infty$. Thus the direct part of (1) for $p = \infty$ is true. So that it suffices to prove Proposition 3.

Proof of Proposition 3. We will prove

$$(5.2) \qquad \int_U K(z,w)\, d\mu_{-2}(z) \leqq c\, \|b\|_{B_\infty},$$

$$(5.3) \qquad \int_U K(z,w)\, d\mu_{-2}(w) \leqq c\, \|b\|_{B_\infty},$$

then we get Proposition 3 for $q = 1, q = \infty$. By interpolation, we get the desired result.

We only need prove (5.2) and assume $\|b\|_{B_\infty} = 1$, i.e., $|yb'(z)| \leqq 1$ for all $z \in U$.

We also assume $w = iv$ in (5.2) since for any constant $a \in R$, $b(a + \cdot) \in B_\infty(U)$ iff $b(\cdot) \in B_\infty(U)$ and they have the same B_∞ norms.

Let $\int_U K(z, w)\, d\mu_{-2} = \sum_{i=1}^4 \int_{U_i}$, where

$$U_1 = \{(x, y): |x| \leqq y < v\},\ U_2 = \{(x, y): |x| > y, y \leqq v\}$$

$$U_3 = \{(x, y): |x| \leqq y, v \leqq y\},\ U_4 = \{(x, y): |x| \geqq y > v\}.$$

On U_1,

$$\begin{aligned} &|b(z) - b(iv)| \\ &\leqq |b(x + iy) - b(iy)| + |b(iy) - b(iv)| \\ &\leqq \frac{|x|}{y} + \int_y^v |b'(is)|\, ds \leqq 1 + \log\frac{v}{y}. \end{aligned}$$

On U_2, let $a = \max(|x|, v)$,

$$\begin{aligned} &|b(z) - b(iv)| \\ &\leqq |b(x + iy) - b(x + ia)| + |b(x + ia) - b(ia)| + |b(ia) - b(iv)| \\ &\leqq \int_y^a \frac{ds}{s} + \frac{|x|}{a} + \int_v^a \frac{ds}{s} \leqq \log\frac{a}{y} + \log\frac{a}{v} + 1 \\ &\leqq 2\left(\log\frac{|x|}{y} + \log\frac{v}{y}\right) + 1. \end{aligned}$$

Similarly, on U_3,

$$|b(z) - b(iv)| \leqq \log\frac{y}{v} + 1$$

and on U_4,

$$|b(z) - b(iv)| \leqq \log\frac{|x|}{y} + \log\frac{|x|}{v} + 1.$$

Thus

$$\begin{aligned} \int_{U_1} &= \int_{0<y<v} \int_{|x| \leqq y} \frac{v^{\frac{1}{2}}\, |b(x + iy) - b(iv)|}{|x + iy + iv|^3}\, dx\, dy/y^{\frac{1}{2}} \\ &\leqq 2v^{\frac{1}{2}} \int_{0<y<v} \int_{0 \leqq x \leqq y} \left(1 + \log\frac{v}{y}\right)[x^2 + (y + v)^2]^{-\frac{3}{2}}\, dx\, dy/y^{\frac{1}{2}}. \end{aligned}$$

The change of variables $y = tv$, $x = tvs$ gives that

$$\int_{U_1} \leqq 2\int_{0<t<1}\int_{0\leqq s\leqq 1}(1-\log t)t^{\frac{1}{2}}[s^2t^2+(t+1)^2]^{-\frac{1}{2}}\,ds\,dt$$

$$\leqq 2\int_{0<t<1}\int_{0\leqq s\leqq 1}(1-\log t)t^{\frac{1}{2}}\,ds\,dt = c < \infty.$$

We can obtain similarly (omitting the details) for $i = 2, 3, 4$

$$\int_{U_i} \leqq c.$$

Thus we have proved Proposition 3.

§6. The operators $H_b^{(k,l)}$ for $1/(k-l+1) < p < \infty$.

If $k > l$, we will prove in this section the direct part of (1) in Theorem 4 for $1/(k-l+1) < p \leqq 1$. Since we have proved for $p = \infty$ in §5, by interpolation, we get the desired result for all $1/(k-l+1) < p \leqq \infty$ in the case $k > l$. For the case $k = l$, we can consider the $H_b^{(k,l)}$ type operators between the spaces $L^{2,\beta}(U)$ and $L^{2,-2}(U)$, where $L^{2,\beta}(U) = \{F(z): \int_U |F(z)|^2 y^\beta\,dx\,dy < \infty\}$. Just copying the method in §7 of S. Janson's [10], we also can get the direct part of (1) in Theorem 4 for $1 < p < \infty$. In this paper we consider the case $k > l$.

Recall $H_b^{(k,l)} = (I - \sum_{v=0}^k P_v)M_bP_l = M_bP_l - \sum_{v=0}^k T_b^{(k,l)}$. Similarly for $F(x,y) = f * \psi_y^l \in A_l$, we calculate the Fourier transform of the first variable x of $(H_b^{(k,l)}F)(x,y)$:

$$(bP_lF)^\wedge(\xi,y) = \frac{1}{2\pi}\int \hat{b}(\cdot + iy)(\xi-\eta)(P_lF)^\wedge(\eta,y)\,d\eta$$

$$= \frac{1}{2\pi}\int_0^\infty \hat{b}(\xi-\eta)\hat{f}(\eta)(l+1)^{-\frac{1}{2}}e^{-(2\eta-\xi)y}y^{\frac{1}{2}}(2y\eta)L_l^{(1)}(2y\eta)\,d\eta.$$

Thus

$$(H_b^{(k,l)}F)^\wedge(\xi,y) = \frac{1}{2\pi}\int_0^\infty \hat{b}(\xi-\eta)\hat{f}(\eta)B_y^{(k,l)}(\xi,\eta)\,d\eta,$$

where

$$B_y^{(k,l)}(\xi,\eta)$$

$$= \begin{cases} D_y^{(l)}(\xi,\eta) - \sum_{s=0}^k \dfrac{y^{\frac{1}{2}}}{(s+1)^{\frac{1}{2}}}\dfrac{\xi}{\eta}y^{\frac{1}{2}}\hat{\psi}^s(y\xi)c^{(s,l)}\left(\dfrac{\xi}{\eta}\right), & \text{for } 0 \leqq \xi \leqq \eta \\ D_y^{(l)}(\xi,\eta), & \text{for } \xi < 0, \eta \geqq 0 \\ 0, & \text{elsewhere,} \end{cases}$$

where $D_y^{(l)}(\xi,\eta) = y^{\frac{1}{2}}/(l+1)^{\frac{1}{2}}(2y\eta)L_l^{(1)}(2y\eta)e^{-(2\eta-\xi)y}$. Thus we can consider $H_b^{(k,l)}$ as vector-valued paracommutators, but we can not change directly these vector-valued paracommutators into usual paracommutators as we did in §3 and §4. However, we can change them into multi-fold paracommutators, which were studied by Peng [20].

Let $S_b^{(k,l)}$ be operator from H^2 to $L^{2,-2}$ defined by $S_b^{(k,l)} = H_b^{(k,l)}T_l$, where T_l is the operator defined in §2. Let T^* be the adjoint operator of T, then $(S_b^{(k,l)})*S_b^{(k,l)}$ become a two-fold paracommutator:

$$((S_b^{(k,l)})^*S_b^{(k,l)}f)\wedge(\eta_2)$$
$$= \frac{1}{(2\pi)^2}\iint \hat{b}(\eta_1-\eta_0)\hat{\bar{b}}(n_2-\eta_1)B^{(k,l)}(\eta_0,\eta_1,\eta_2)\hat{f}(\eta_0)\,d\eta_1\,d\eta_0$$

where

$$B^{(k,l)}(\eta_0,\eta_1,\eta_2) = \int_0^\infty B_y^{(k,l)}(\eta_1,\eta_0)B_y^{(k,l)}(\eta_1,\eta_2)\,dy/y^2.$$

We can calculate (omitting the details)

$$(6.1)\qquad B^{(k,l)}(\eta_0,\eta_1,\eta_2) = \begin{cases} I_1 - I_2, & \text{for } 0 \leqq \eta_1 \leqq \min(\eta_0,\eta_2) \\ I_1, & \text{for } \eta_1 \leqq 0,\ \eta_0,\eta_2 \geqq 0 \\ 0, & \text{elsewhere,} \end{cases}$$

where

$$I_1 = \frac{1}{l+1}\int_0^\infty e^{-2(\eta_0+\eta_2-\eta_1)}(2y\eta_0)L_l^{(1)}(2y\eta_0)(2y\eta_2)L_l^{(1)}(2y\eta_2)\frac{dy}{y}$$
$$= \frac{\eta_0\eta_2}{(\eta_0+\eta_2-\eta_1)^2}\sum_{j=0}^{l}\binom{l}{y}\binom{l+1}{j+1}\frac{(\eta_0\eta_2)^j(\eta_0-\eta_1)^{l-j}(\eta_2-\eta_1)^{l-j}}{(\eta_0+\eta_2-\eta_1)^{2l}}$$

and

$$I_2 = \frac{1}{l+1}\sum_{s=0}^{k}\frac{1}{s+1}\frac{\eta_1^2}{\eta_0\eta_2}c^{(s,l)}\left(\frac{\eta_1}{\eta_2}\right)$$
$$= \sum_{s=1}^{k}\binom{s}{l}\binom{s+1}{l+1}\left(\frac{\eta_1^2}{\eta_0\eta_2}\right)^{s+1}\left(1-\frac{\eta_1}{\eta_0}\right)^{s-l}\left(1-\frac{\eta_1}{\eta_2}\right)^{s-l}.$$

We now consider $T_b^{(v,l)}$ (for $v \geqq l$) similarly as $H_b^{(k,l)}$. Define $R_b^{(v,l)} = T_b^{(v,l)}T_l$ from H^2 onto A_k, then $(R_b^{(v,l)})^*R_b^{(v,l)}$ is also a two-fold paracommutator:

$$((R_b^{(v,l)*}R_b^{(v,l)}f)\wedge(\eta_2)$$
$$= \frac{1}{(2\pi)^2}\iint \hat{b}(\eta_1-\eta_0)\hat{\bar{b}}(\eta_2-\eta_1)A^{(v,l)}(\eta_0,\eta_1,\eta_2)\hat{f}(\eta_0)\,d\eta_1\,d\eta_0$$

where supp $A^{(v,l)}$ is in the domain $\{(\eta_0,\eta_1,\eta_2)\colon 0 \leqq \eta_1 \leqq \min(\eta_0,\eta_2)\}$, on which it equals to $\binom{v}{l}\binom{v+1}{l+1}(\eta_1^2/\eta_0\eta_2)^{l+1}(1-\eta_1/\eta_0)^{v-l}(1-\eta_1/\eta_2)^{v-l}$.

By the equality $T^{(v,l)}b = H_b^{(v-1,l)} - H_b^{(v,l)}$.

$$\sum_{v=k+1}^{\infty} T_b^{(v,l)} = H_b^{(k,l)} - C_b^{(0,l)}. \tag{6.2}$$

where

$$C_b^{(0,l)} = \lim_{N\to\infty} H_b^{(N,l)} = \lim_{N\to\infty}\left(I - \sum_{v=0}^{N} P_v\right) M_b P_l = \sum_{v=0}^{\infty} \bar{P}_v M_b P_l.$$

Let $c_b^{(0,l)} = C_b^{(0,l)} T_l$, then $(c_b^{(0,l)})^* c_b^{(0,l)}$ is also a two-fold paracommutator:

$$\begin{aligned}
&((c_b^{(0,l)})^* c_b^{(0,l)} f)^\wedge(\eta_2)\\
&= \frac{1}{(2\pi)^2}\iint \hat{b}(\eta_1-\eta_0)\bar{\hat{b}}(\eta_2-\eta_1) A_0^{(0,l)}(\eta_0,\eta_1,\eta_2)\hat{f}(\eta_0)\,d\eta_1\,d\eta_0
\end{aligned}$$

where supp $A_0^{(0,l)}$ is in the domain $\{(\eta_0,\eta_1,\eta_2)\colon \eta_1 \leqq 0, \eta_0,\eta_2 \geqq 0\}$, on which it equals I_1. We can get easily (cf. [19]):

LEMMA 6.1. *If $\bar{b} \in B_p(U)$, then $C_b^{(0,l)} \in S_p$ for $0 < p \leqq \infty$.*

We now prove $H_b^{(k,l)} \in S_p$ if $\bar{b} \in B_p$ and $1/(k-l+1) < p \leqq 1$.

Let $\psi, \psi' \in S(R)$ be functions such that supp$\hat{\psi}' \subset \{\frac{1}{4} \leqq |\xi| \leqq 4\}$, $\hat{\psi}'(\xi) = 1$ on $\{\frac{1}{2} \leqq |\xi| = 2\}$ and supp $\hat{\psi} \subset \{\frac{1}{8} \leqq |\xi| \leqq 8\}$, $\hat{\psi}(\xi) = 1$ for $\xi \in$ supp $\hat{\psi}'$. Let $\hat{\psi}_j'(\xi) = \hat{\psi}'(2^{-j}\xi)$ and $\hat{\psi}_j(\xi) = \hat{\psi}(2^{-j}\xi)$. Thus $b = \sum_{j=-\infty}^{\infty} b_j$, where $b_j(\xi) = \hat{b}(\xi)\hat{\psi}_j'(\xi) = \hat{b}(\xi)\hat{\psi}_j'(\xi)\cdot\hat{\psi}_j(\xi)$, and b is the boundary value of $b(z)$. By the properties of "S_p-norm",

$$\|T_b\|_{S_p}^p \leq \sum_{j=-\infty}^{\infty} \|T_{b_j}\|_{S_p}^p \tag{6.3}$$

here $T_b = \sum_{v=k+1}^{\infty} R_b^{(v,l)}$.

Let $(\hat{b}_j)_e$ denote the periodic extension of $\hat{b}_j$ with the period $2\pi\cdot 2^{j+2}$, for $2^{j-1} \leqq |\xi-\eta| \leqq 2^{j+2}$, we have

$$\begin{aligned}
\hat{b}_j(\xi-\eta) &= (\hat{b}_j)_e(\xi-\eta)\hat{\psi}_j(\xi-\eta)\\
&= \sum_{k=-\infty}^{\infty} a_k e^{ik2^{-(j+2)}(\xi-\eta)}\hat{\psi}_j(\xi-\eta)\\
&= \sum_{k=-\infty}^{\infty} a_k e^{ik2^{-(j+2)}\xi}\hat{\psi}_j(\xi-\eta)e^{-ik2^{-(j+2)}\eta}.
\end{aligned}$$

Thus $T_{b_j} = \sum_{k=-\infty}^{\infty} a_k U_k T_{\psi_j} V_k$, where U_k and V_k are unitary operators, and by Lemma 6 and Lemma 7 in [19], we have

$$\sum_{k=-\infty}^{\infty} |a_k|^p \approx 2^{j(1-p)} \|b_j\|_p^p. \tag{6.4}$$

Thus

$$\|T_{b_j}\|_{S_p}^p \leqq c \sum_{k=-\infty}^{\infty} |a_k|^p \|T_{\psi_j}\|_{S_p}^p \leqq c 2^{j(1-p)} \|b_j\|_p^p \|T_{\psi_j}\|_{S_p}^p \tag{6.5}$$

Now we estimate the "S_p-norm" of T_{ψ_j}. Note that (cf. [19]) for $0 < q$

$$\|\hat{\psi}_j\|_{S_\infty(W_0 \times W_1)} \leqq \|\hat{\psi}_j\|_{S_q(W_0 \times W_1)} \leqq c \|\hat{\psi}\|_q |W_0|^{\frac{1}{2}} |W_1|^{\frac{1}{2}}. \tag{6.6}$$

By the orthogonality of projections P_v, we have

$$\begin{aligned}
\|T_{\psi_j}\|_{S_p}^p &= \|T_{\psi_j}^* T_{\psi_j}\|_{S_{\frac{p}{2}}}^{\frac{p}{2}} \\
&= \left\| \left(\sum_{v=k+1}^{\infty} R_{\psi_j}^{(v,l)} \right)^* \left(\sum_{v=k+1}^{\infty} R_{\psi_j}^{(v,l)} \right) \right\|_{S_{\frac{p}{2}}}^{\frac{p}{2}} \\
&= \left\| \sum_{v=k+1}^{\infty} (R_{\psi_j}^{(v,l)})^* (R_{\psi_j}^{(v,l)}) \right\|_{S_{\frac{p}{2}}}^{\frac{p}{2}} \\
&= c \left\| \int_0^\infty \hat{\psi}_j(\eta_1 - \eta_0) \hat{\bar{\psi}}_j(\eta_2 - \eta_1) \sum_{v=k+1}^{\infty} A^{(v,l)}(\eta_0, \eta_1, \eta_2)\, d\eta_1 \right\|_{S_{\frac{p}{2}}}^{p-2}.
\end{aligned}$$

Let I_i^j be interval with center $(i + \frac{1}{2})2^{j+3}$ and length 2^{j+3}, then $R_+ = [0, \infty) = \bigcup_{i=0}^{\infty} I_i^j$. If $\eta_2 \in I_i^j$ and $\hat{\psi}_j(\eta_1 - \eta_0)\hat{\bar{\psi}}_j(\eta_2 - \eta_1) \neq 0$, then $\eta_1 \in 2I_i^j$ and $\eta_0 \in 4I_i^j$. Thus

$$\begin{aligned}
\|T_{\psi_j}\|_{S_p}^p &\leqq \sum_{i=0}^{\infty} \left\| \int_{2I_i^j} \hat{\psi}_j(\eta_1 - \eta_0) \hat{\bar{\psi}}_j(\eta_2 - \eta_1) \times \right. \\
&\qquad \left. \times \sum_{v=k+1}^{\infty} A^{(v,l)}(\eta_0, \eta_1, \eta_2)\, d\eta_1 \right\|_{S_{\frac{p}{2}}(4I_i^j \times I_i^j)}^{\frac{p}{2}} \\
&= \sum_{i>8} \sum_{i \leqq 8} = A + B \\
&\leqq \sum_{i>8} \sum_{v=k+1} a_{iv} + B,
\end{aligned}$$

where

$$
\begin{aligned}
a_{iv} &= \left\| \int_{2I_i^j} \hat{\psi}_j(\eta_1 - \eta_0)\hat{\psi}_j(\eta_2 - \eta_1) \times \right.\\
&\quad \left. \times \left(\frac{\eta_1^2}{\eta_0\eta_2}\right)^{l+1} \left(1 - \frac{\eta_1}{\eta_0}\right)^{v-l} \left(1 - \frac{\eta_1}{\eta_2}\right)^{v-l} d\eta_1 \right\|^{\frac{p}{2}}_{S_{\frac{p}{2}}(4I_i^j \times I_i^j)}\\
&\leqq cv^{lp} \left\| \hat{\psi}_j(\eta_1 - \eta_0)\left(\frac{\eta_1}{\eta_0}\right)^{l+1} \left(1 - \frac{\eta_1}{\eta_0}\right)^{v-l} \right\|^{\frac{p}{2}}_{S_\infty(4I_i^j \times 2I_i^j)} \times\\
&\quad \times \left\| \hat{\psi}_j(\eta_2 - \eta_1)\left(\frac{\eta_1}{\eta_2}\right)^{l+1} \left(1 - \frac{\eta_1}{\eta_2}\right)^{v-l} \right\|^{\frac{p}{2}}_{S_{\frac{p}{2}}(2I_i^j \times I_i^j)}\\
&\leqq cv^{lp} i^{(l-v)p} \|\hat{\psi}_j(\eta_1 - \eta_0)\|^{\frac{p}{2}}_{S_\infty(4I_i^j \times 2I_i^j)} \|\hat{\psi}_j(\eta_2 - \eta_1)\|^{\frac{p}{2}}_{S_{\frac{p}{2}}(2I_i^j \times I_i^j)}\\
&\leqq cv^{lp} i^{(l-v)p} 2^{jp},
\end{aligned}
$$

the last inequality is obtained by (6.6). Thus

$$
A \leqq \sum_{i>8} \sum_{v=k+1}^{\infty} a_{iv} \leqq c2^{jp} \sum_{v=k+1}^{\infty} v^{lp} \sum_{i>8} \left(\frac{1}{i}\right)^{p(v-l)}.
$$

The series $\sum_{i>8}(1/i)^{p(v-l)}$ converges iff $p(v-l) > 1$, *i.e.* $1/(v-l) < p$ for all $v \geqq k+1$. This is just the condition that p satisfies in (1) of Theorem 4. Thus

$$
A \leqq c^{2jp} \sum_{v=k+1}^{\infty} \frac{v^{lp}}{8^{vp}} = c2^{jp}.
$$

About B, by the equality (6.2), we have

$$
\sum_{v=k+1}^{\infty} A^{(v,l)}(\eta_0, \eta_1, \eta_2) = B^{(k,l)}(\eta_0, \eta_1, \eta_2) - A_0^{(0,l)}(\eta_0, \eta_1, \eta_2).
$$

Thus

$$
\begin{aligned}
B &= \sum_{i \leqq 8} \left\| \int \hat{\psi}_j(\eta_1 - \eta_0)\hat{\psi}_j(\eta_2 - \eta_1) \times \right.\\
&\quad \left. \times (B^{(k,l)}(\eta_0, \eta_1, \eta_2) - A_0^{(0,l)}(\eta_0, \eta_1, \eta_2)) \right\|^{\frac{p}{2}}_{S_{\frac{p}{2}}(4I_i^j \times I_i^j)} = \sum_{i \leqq 8} b_i.
\end{aligned}
$$

Let us prove that $B \leqq c2^{jp}$. We need consider one term b_i in the sum B. Let $\Delta_k = \{\xi: 2^{-k} \leqq |\xi| \leqq 2^{-k+1}\}$. Then for $i \leqq 8$, there exists a fixed integer j_0 such that

$$4I_i^j \subset \bigcup_{k=-\infty}^{j+j_0} \Delta_k.$$

By the similar method in [19], we can show that

$$\|B^{(k,l)}(\eta_0,\eta_1,\eta_2)\|_{V_{\frac{p}{2}}(\Delta_{k_0}\times\Delta_{k_1}\times\Delta_{k_2})} \leqq c,$$
$$\|A_0^{(0,l)}(\eta_0,\eta_1,\eta_2)\|_{V_{\frac{p}{2}}(\Delta_{k_0}\times\Delta_{k_1}\times\Delta_{k_2})} \leqq c.$$

for all k_0, k_1, $k_2 \in Z^+$.

Thus we have

$$b_i \leqq \Bigg\| \int_{\cup_{k_1=-\infty}^{j+j_0}\Delta_{k_1}} \hat{\psi}_j(\eta_0-\eta_1)\hat{\bar{\psi}}_j(\eta_1-\eta_2)(B^{(k,l)}(\eta_0,\eta_1,\eta_2) -$$
$$- A_0^{(0,l)}(\eta_0,\eta_1,\eta_2))\, d\eta_1 \Bigg\|^{\frac{p}{2}}_{S_{\frac{p}{2}}(\cup_{k_0=-\infty}^{j+j_0}\Delta_{k_0}\times\cup_{k_2=-\infty}^{j+j_0}\Delta_{k_2})}$$
$$\leqq \sum_{\{k_i\leqq j+j_0:\, i=1,2,3\}} \|B^{(k,l)}(\eta_0,\eta_1,\eta_2) -$$
$$- A_0^{(0,l)}(\eta_0,\eta_1,\eta_2)\|^{\frac{p}{2}}_{V_{\frac{p}{2}}(\Delta_{k_0}\times\Delta_{k_1}\times\Delta_{k_2})}(|\Delta_{k_0}| \times |\Delta_{k_1}|^2 \times |\Delta_{k_2}|)^{\frac{p}{4}}$$
$$\leqq c \sum_{\{k_i\leqq j+j_0:\, i=1,2,3\}} 2^{-\frac{p}{4}(k_0+2k_1+k_2)} = c2^{jp}.$$

The second inequality is obtained by (6.6). Thus we have proved $\|T_{\psi_j}\|^{\frac{p}{2}}_{S_p} \leqq c2^{jp}$, where c is a constant independent of j. By (6.3) and (6.5),

$$\|T_b\|^{\frac{p}{2}}_{S_p} \leqq c \sum_{s=-\infty}^{\infty} 2^j \|b_j\|^{\frac{p}{p}}_p = c\,\|b\|^{\frac{p}{2}}_{B_p}. \tag{6.7}$$

By (6.2), (6.7) and Lemma 6.1,

$$\|H_b^{(k,l)}\|_{S_p} \leqq c\,\|b\|_{B_p}.$$

Thus we complete the proof of Theorem 4.

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