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VALUED GROUPS

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1. Introduction.

For ordered groups the theories of convergence and extensions are well-known even in the non-commutative case; see e.g. [2], [4] and [9]. In these theories the metric notion comes from the order in question, which also induces the natural valuation. On the other hand, a notion of a general group valuation extending the natural valuation of an ordered group has been introduced, for the Abelian case see [3] and for the non-Abelian case see [9]. But, almost nothing has been done in studying the metric properties of groups with this general valuation. Here we start a systematic study of these valued groups. In this paper we consider fundamental topological properties of group valuations, sequences and pseudo-convergence. This article forms a ground work for the study of continuous and immediate extensions of valued groups and their embeddings into Hahn-like products; see [13] and [14].

Our definition of a valuation is basically the same as in [9]. It is a definite function from a (generally non-Abelian) group to an ordered set with a least element mapping the neutral element to the least element and satisfying the ultrametric inequality. Contrary to [9] a valuation is not required to be surjective. Some ideas used in Section 2 and 3 have been derived from [1], where real-valued ultrametric functions on commutative groups are considered.

Our concept of group valuation is much more general than that of [5], where valuations are real-valued functions on finite groups satisfying a special condition on commutators. In spite of certain similarity of the names "valued groups" used in this paper and "valuated groups" used in [11] these notions have very little in common. So-called p-valuations considered also in [11] are special cases of our valuation.

Since the group operation is allowed to be non-commutative and the valuation is not necessarily a natural valuation of an ordered group, it is clear that this general valuation theory brings new difficulties. Even basic valuation theoretical results require thus a careful re-examination. The non-commutativity brings also

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some side dependencies. Such is for example the fact that the topology, which is also now naturally induced by the ultrametric valuation on the base group, need not be a group topology. An analogous situation occurs in considering the topology induced by a valuation of a skew field; see [6] or [7].

In Section 2 the definition of a general group valuation is given and the basic properties of these valuations are studied. The main result is Theorem 2.4 which gives several different criteria for a valuation to make the group into a topological group. Furthermore, we introduce various types of valuations, such as topological, uniformly topological, normal, conjugation isotonic or conjugation invariant and study the relations between them. The quotient groups are examined in Section 3. Here the non-surjectivity of the valuation allows for certain natural results which may not been obtained otherwise. Sequences and subsequences are introduced in Section 4. Here we also present some probably new results on cofinality types of ordered sets; see especially Theorem 4.8. We use them in Corollary 4.9 and Theorem 4.10 to characterize subsequences and index sets of Cauchy sequences. The theory of pseudo-convergence is set forth in Section 5. Particularly it is shown in Theorem 5.7 that a right proper Cauchy sequence has a right pseudo-Cauchy subsequence. Herewith we not only generalize the commutative case in [9], Lemma 11, p. 44, but also rectify its incomplete proof. In the course of the proof the earlier results about the cofinality types are needed.

2. Valuations.

Let G denote a not necessarily commutative group, which will be written additively, and let Γ denote an ordered set, which has a least element 0. Denote the set $\Gamma \setminus \{0\}$ by Γ^*.

2.1. DEFINITION. A mapping $|\cdot|: G \to \Gamma$ is a *semi-valuation*, if

(1) $|0| = 0$;

(2) $|g - h| \leqq \max\{|g|, |h|\}$ for all $g, h \in G$.

A semi-valuation $|\cdot|$ is called *definite*, if $|g| = 0$ implies $g = 0$. A *valuation* is a definite semi-valuation. If $|\cdot|$ is a (semi-)valuation, the triple $(G, \Gamma, |\cdot|)$, or simply G, is called a *group with (semi-)valuation*.

Note that we do not suppose that semi-valuations or valuations are surjective.

A natural example of a group with valuation is an ordered group with the natural valuation, which maps every element to its Archimedean class; see e.g. [9], §I.4. A general method for forming new groups with valuations is the *Hahn product*: Let A be an ordered set and for each $\alpha \in A$ let $(G_\alpha, \Gamma_\alpha, |\cdot|_\alpha)$ be a group with semi-valuation. For $g \in \Pi_A G_\alpha$ set $\mathrm{supp}(g) = \{\alpha \in A: |g(\alpha)|_\alpha \neq 0\}$. Define further:

$$G = \{g \in \prod_A G_\alpha : \operatorname{supp}(g) \text{ is dually well ordered in } A\},$$

$$\Gamma = \{0\} \oplus (\sum_A \Gamma_\alpha^*) \text{ (Ordered sum)},$$

$$|g| = \begin{cases} 0, & \text{if } \operatorname{supp}(g) = \emptyset, \\ |g(\alpha)|_\alpha, & \text{if } \operatorname{supp}(g) \neq \emptyset \text{ and } \alpha = \max \operatorname{supp}(g). \end{cases}$$

The set $\mathbf{H}(A; G_\alpha) = \mathbf{H}(A; G_\alpha, \Gamma_\alpha, |\cdot|_\alpha) = (G, \Gamma, |\cdot|)$ is a group with semi-valuation; it is called a *Hahn group* (*with semi-valuation*).

In the following we represent basic properties of semi-valuations; the proofs follow in the same way as in [9], §I.4, or [1], 1.1.

2.2. PROPOSITION. *A semi-valuation has the following properties*:

(1) $|-g| = |g|$ *for all* $g \in G$;

(2) $|g + h| \leqq \max\{|g|, |h|\}$ *for all* $g, h \in G$;

(3) $|g + h| = \max\{|g|, |h|\}$, *if* $|g| \neq |h|$;

(4) $|\Sigma_{k=1}^n g_k| \leqq \max\{|g_k| : k = 1, 2, \ldots, n\}$; *furthermore, equality holds if there is exactly one index j such that* $|g_j| = \max\{|g_k| : k = 1, 2, \ldots, n\}$.

2.3. COROLLARY. *A mapping* $|\cdot| : G \to \Gamma$ *with* $|0| = 0$ *is a semi-valuation if and only if it satisfies the first two items in Proposition* 2.2.

For $\gamma \in \Gamma$ we define the *open* (resp. *closed*) *ball* with *center* 0 and *radius* γ by the formulae

$$B_\gamma = B(0; \gamma) = \{g \in G : |g| < \gamma\},$$

$$B'_\gamma = B'(0; \gamma) = \{g \in G : |g| \leqq \gamma\}.$$

The sets B_γ for $\gamma \in \Gamma^*$ and B'_γ for $\gamma \in \Gamma$ are subgroups of G. The open and closed balls determine the semi-valuation completely in the following way: Let G be a group, Γ an ordered set with a least element 0 and let $(B_\gamma)_{\gamma \in \Gamma^*}$ and $(B'_\gamma)_{\gamma \in \Gamma}$ be increasing chains of subgroups of G such that

(1) $B_\gamma \subset B'_\gamma$ for every $\gamma \in \Gamma^*$;

(2) $B'_\gamma \subset B_\delta$ for every $\gamma, \delta \in \Gamma$ with $\gamma < \delta$;

(3) for every $g \in G$ there exists $\gamma \in \Gamma$ such that $g \in B'_\gamma \setminus B_\gamma$ ($B_0 = \emptyset$).

For each $g \in G$ choose the element $\gamma \in \Gamma$ according to the condition (3) and define $|g| = \gamma$. Then $|\cdot|$ is a semi-valuation of G and the subgroups B_γ and B'_γ are exactly the 0-centered open and closed balls with respect to this semi-valuation.

If the center of a ball is not 0, then we get right and left hand side balls:

$$B_r(g; \gamma) = \{h \in G : |h - g| < \gamma\} = B_\gamma + g;$$

$$B_l(g; \gamma) = \{h \in G : |-g + h| < \gamma\} = g + B_\gamma.$$

The corresponding closed balls are defined similarly. These sets are not necessarily subgroups.

The pseudo-metric functions $(g, h) \to |h - g|$ and $(g, h) \to |-g + h|$ induce two in general different topologies τ_r and τ_l on G in such a way that the balls $B_r(g; \gamma)$, $\gamma \in \Gamma^*$, and $B_l(g; \gamma)$, $\gamma \in \Gamma^*$, resp., form fundamental systems of neighbourhoods of a point $g \in G$. These topological spaces are homeomorphic by $x \to -x$, and they are Hausdorff if and only if the semi-valuation $|\cdot|$ is a valuation.

Note that the group G with the topology τ_r or τ_l is not necessarily a topological group. Examples can be found in [12]. In case the group G is a topological group with respect to τ_r, the semi-valuation $|\cdot|$ is said to be *topological*. The following result gives criteria for a semi-valuation to be topological; cf. [6], where an analogous result is given in the case of a valued division ring.

2.4. THEOREM. *Let $|\cdot|$ be a semi-valuation. The following conditions are equivalent*:

(1) *The semi-valuation $|\cdot|$ is topological.*
(2) *The group G is a topological group with respect to τ_l.*
(3) *The left translations $g \to a + g$ are all τ_r-continuous.*
(4) *The topology τ_l is finer than the topology τ_r.*
(5) *The topologies τ_l and τ_r are same.*
(6) *Every right open ball contains a left open ball with the same center.*
(7) *Given any $g \in G$ and any $\gamma \in \Gamma^*$ there exists $\delta \in \Gamma^*$ such that the conjugate $g + B_\delta - g$ is contained in B_γ.*

PROOF. We shall show that the following diagram of implications holds true:

$$\begin{array}{ccccccccc} (1) & \Rightarrow & (3) & & & & & & \\ \Uparrow & & \Downarrow & & & & & & \\ (2) & \Leftarrow & (7) & \Leftrightarrow & (6) & \Leftrightarrow & (4) & \Leftrightarrow & (5) \end{array}$$

It is clear that (2) implies (1) because the mapping $g \to -g$ is a homeomorphism between the topologies τ_l and τ_r. The implication $(1) \Rightarrow (3)$ is trivial.

The τ_r-continuity of the left translation $g \to a + g$ in 0 means that for all $\gamma \in \Gamma^*$ we can find $\delta \in \Gamma^*$ such that $a + B_\delta \subset B_\gamma + a$, which is equivalent with (7).

The condition (7) can be written in the form $-B_l(g; \delta) \subset B_l(-g; \gamma)$, which means that the mapping $g \to -g$ is τ_l-continuous in g. Furthermore, if $g, h \in G$ and $\gamma \in \Gamma^*$, then by (7) there exists $\delta \in]0, \gamma]$ such that $-h + B_\delta + h \subset B_\gamma$. Hence

$$B_l(g;\delta) + B_l(h;\delta) = g + h + (-h + B_\delta + h) + B_\delta \subset g + h + B_\gamma + B_\delta = B_l(g + h;\gamma),$$

which means that the mapping $(g, h) \to g + h$ is τ_l-continuous. Thus the condition (7) implies the condition (2).

The equivalencies $(7) \Leftrightarrow (6)$ and $(6) \Leftrightarrow (4)$ are easy to check. To prove the last claim $(4) \Leftrightarrow (5)$ note first that τ_l is finer than τ_r if and only if for every $g \in G$ and

$\gamma \in \Gamma^*$ there exists $\delta \in \Gamma^*$ such that $B_l(-g;\delta) \subset B_r(-g;\gamma)$ or $B_\delta + g \subset g + B_\gamma$. This in turn means that $B_r(g;\delta) \subset B_l(g;\gamma)$, i.e., $\tau_r \supset \tau_l$.

There are several different invariance properties of a semi-valuation which are closely related to the semi-valuation being topological.

2.5. Definition. Let $|\cdot|: G \to \Gamma$ be a semi-valuation. It is said to be

(1) *conjugation invariant* or *commutative*, if $|g + h - g| = |h|$ for all $g, h \in G$;

(2) *conjugation isotonic*, if $|g| \leqq |h|$ implies $|k + g - k| \leqq |k + h - k|$ for all $k \in G$;

(3) *normal*, if $|g| < |h|$ implies $|h + g - h| < |h|$;

(4) *uniformly topological*, if for all $\gamma > 0$ there exists $\delta > 0$ such that $g + B_\delta - g \subset B_\gamma$ for all $g \in G$.

The alternate name "commutative" comes from the fact that a semi-valuation is conjugation invariant if and only if the relation $|g + h| = |h + g|$ holds true for all $g, h \in G$. The name "normal" was chosen for the reason that then (and only then) open balls B_γ are normal in closed balls B'_γ.

2.6. Lemma. *A semi-valuation $|\cdot|$ on G is conjugation isotonic if and only if it is strictly conjugation isotonic*, i.e. *$|g| < |h|$ implies $|k + g - k| < |k + h - k|$ for all $k \in G$.*

Proof. Let $|\cdot|$ be conjugation isotonic, and let $|g| < |h|$. If there were an element $k \in G$ such that $|k + g - k| = |k + h - k|$, then

$$|h| = |-k + (k + h - k) + k| \leqq |-k + (k + g - k) + k| = |g|,$$

which is impossible. Thus $|k + g - k| < |k + h - k|$ for all $k \in G$.

For the converse it is enough to show that $|g| = |h|$ implies $|k + g - k| \leqq |k + h - k|$ for all $k \in G$. Suppose the opposite. Then the same argument as above shows that $|g| > |h|$, contrary to the hypothesis.

The natural valuation of an ordered group is an example of a conjugation isotonic valuation, and hence it is topological; see Theorem 2.8 below. The group valuations of [5] are examples of conjugation invariant valuations; see [5], Corollary 1.3. For the semi-valuation of a Hahn group we have

2.7. Proposition. *Let $(G, \Gamma, |\cdot|) = \mathbf{H}(A; G_\alpha, \Gamma_\alpha, |\cdot|_\alpha)$ be a Hahn group.*

(1) *The semi-valuation $|\cdot|$ is conjugation invariant, conjugation isotonic or normal if and only if every "factor semi-valuation" $|\cdot|_\alpha$ has the same property.*

(2) *If the index set A does not have a least element, the semi-valuation $|\cdot|$ is uniformly topological.*

(3) *If α_0 is a least element of A and the semi-valuation $|\cdot|_{\alpha_0}$ is (uniformly) topological, the semi-valuation $|\cdot|$ is also (uniformly) topological.*

PROOF. (1) It is obvious that in each case the condition is necessary. To prove its sufficiency let $a, b, g \in G$ with $|a| \in \Gamma_\alpha^*$ and $|b| \in \Gamma_\beta^*$ be arbitrary. For every $\gamma > \alpha$ we have

$$(g + a - g)(\gamma) = g(\gamma) + 0 - g(\gamma) = 0,$$

$$(g + a - g)(\alpha) = g(\alpha) + a(\alpha) - g(\alpha) \neq 0,$$

hence

$$|g + a - g| = |g(\alpha) + a(\alpha) - g(\alpha)|_\alpha \in \Gamma_\alpha^*.$$

If $|\cdot|_\alpha$ is conjugation invariant, then $|g + a - g| = |a(\alpha)|_\alpha = |a|$.

Let now $|\cdot|_\alpha$ be conjugation isotonic. Suppose that $|a| < |b|$, so that either $\alpha < \beta$ or $\alpha = \beta$ and $|a(\alpha)|_\alpha < |b(\alpha)|_\alpha$. In the latter case

$$|g + a - g| = |g(\alpha) + a(\alpha) - g(\alpha)|_\alpha < |g(\alpha) + b(\alpha) - g(\alpha)|_\alpha = |g + b - g|.$$

In the case $\alpha < \beta$ the result is obvious, because $\Gamma_\alpha^* < \Gamma_\beta^*$.

The case of a normal semi-valuation is handled similarly.

(2) Let $\gamma \in \Gamma^*$ be arbitrary. Then γ is in Γ_β^* for some $\beta \in A$. Choose $\alpha < \beta$ and $\delta \in \Gamma_\alpha^*$. Then $g + B_\delta - g \subset B_\gamma$ for all $g \in G$, because for all $\beta' \geqq \beta$ and $h \in B_\delta$ we have $\max \operatorname{supp}(h) \leqq \alpha < \beta \leqq \beta'$ and hence

$$(g + h - g)(\beta') = g(\beta') + h(\beta') - g(\beta') = g(\beta') + 0 - g(\beta') = 0.$$

(3) This follows by modifying the previous proof in an obvious way.

2.8. THEOREM. *Let $|\cdot|: G \to \Gamma$ be a semi-valuation. The following diagram of implications holds true:*

$$\begin{array}{ccc} |\cdot| \textit{ is conjugation invariant} & \Rightarrow & |\cdot| \textit{ is uniformly topological} \\ \Downarrow & & \Downarrow \\ |\cdot| \textit{ is conjugation isotonic} & \Rightarrow & |\cdot| \textit{ is topological} \\ \Downarrow & & \\ |\cdot| \textit{ is normal} & & \end{array}$$

PROOF. To prove the implication "$|\cdot|$ is conjugation isotonic $\Rightarrow$ $|\cdot|$ is topological", let $g \in G$ and $\gamma \in \Gamma^*$ be arbitrary. If there is $k \in G$ such that $0 < |k| \leqq \gamma$, define $\delta = |-g + k + g|$. Then $\delta > 0$ and $g + B_\delta - g \subset B_\gamma$ which by Theorem 2.4 means that G is a topological group. If $|k| \leqq \gamma$ implies that $|k| = 0$, then τ_r is discrete, and thus G is a topological group.

In light of Lemma 2.6 and Theorem 2.4 the rest of the proof is obvious.

In the article [12] there are examples which show that a normal semi-valuation is not necessarily topological and that the natural valuation of an ordered group is not necessarily uniformly topological.

3. Valued groups and quotient groups.

Let G be a group, and let $|\cdot|: G \to \Gamma$ be a semi-valuation.

3.1. Definition. A triple $(G, \Gamma, |\cdot|)$, or simply G, where the semi-valuation $|\cdot|: G \to \Gamma$ is topological, is called a *semi-valued group*. If in addition the semi-valuation $|\cdot|$ is a valuation, G is a *valued group*.

In the following we represent some basic properties of semi-valued and valued groups.

3.2. Proposition. *Let G be a semi-valued group. Then the right and left hand side open and closed balls are all both open and closed. Furthermore, the semi-valuation $|\cdot|: G \to \Gamma$ is continuous with respect to the order topology of Γ.*

The proof is obvious.

3.3. Proposition. *A valued group is totally disconnected.*

Proof. Let G be a valued group and let $X \subset G$ have at least two elements g and h. Choose $\gamma = |g - h|$. Then the balls $B_r(g; \gamma)$ and $B_r(h; \gamma)$ are disjoint. In this case we can represent X as a union of the sets $X \cap B_r(g; \gamma)$ and $X \backslash B_r(g; \gamma)$, which are mutually disjoint, non-empty and open in X. Thus X cannot be connected.

If X is a subset of G and $g \in G$, then the *distance* of g to X is

$$|g, X| = \inf\{|g - x| : x \in X\},$$

provided that the infimum in question exists. This happens always, if the ordered set Γ is *conditionally complete*, i.e., every non-empty subset of Γ has the greatest lower bound in Γ. Note that the relation "$|g, X| = 0$" makes sense in any case.

3.4. Proposition. *Let closure be with respect to the τ_r-topology. Then*

$$\bar{X} = \{g \in G : |g, X| = 0\}$$

for all subsets X of G.

Proof. The set $X' = \{g \in G : |g, X| = 0\}$ is τ_r-closed. Indeed, if $g \notin X'$, then the ball $B_r(g; |g, X|)$ does not intersect the set X'. This means that the complement of X' is open.

Furthermore, if a τ_r-closed set Y includes X, it includes also the set X'. Otherwise with an element $g \in X' \backslash Y$ there would exist an element $x \in X$ such that $|g - x| < \gamma$, where $\gamma > 0$ is so chosen that $B_r(g; \gamma) \subset G \backslash Y$; but this is impossible because $X \cap B_r(g; \gamma) = \emptyset$.

As usual we get the following results.

3.5. PROPOSITION. *Let H be a normal subgroup of G and define $|\cdot|^\wedge : G/H \to \Gamma$ so that $|g + H|^\wedge = |g, H|$ for all $g + H \in G/H$. Suppose that the infima in question exist. Then $|\cdot|^\wedge$ is a semi-valuation, so called* quotient semi-valuation. *It is a valuation if and only if H is τ_r-closed. Furthermore, if $(G, \Gamma, |\cdot|)$ is a semi-valued group, the quotient group $(G/H, \Gamma, |\cdot|^\wedge)$ is also a semi-valued group and the corresponding $\tau_r^\wedge$-topology is the same as the quotient topology.*

3.6. COROLLARY. *Let $(G, \Gamma, |\cdot|)$ be a semi-valued group, and define $H = \ker |\cdot|$. Then H is normal in G and $(G/H, \Gamma, |\cdot|^\wedge)$ is a valued group.*

In the previous results it is essential that the valuations are not required to be surjective. If this requirement is made, then a valued group can have a non-topological quotient group. As an example, consider the group $G = H \times K$, where $(H, \Gamma_H, |\cdot|_H)$ is a valued group and $(K, \Gamma_K, |\cdot|_K)$ is a group with a non-topological valuation both valuations being surjective. Define the ordered set Γ to be the ordered sum $\Gamma_H \oplus \Gamma_K^*$, and set

$$|(h, k)| = \begin{cases} |k|_K, & \text{if } k \neq 0, \\ |h|_H, & \text{if } k = 0. \end{cases}$$

Then $(G, \Gamma, |\cdot|)$ is a valued group with a surjective valuation. On the other hand, the quotient group G/H equipped with the surjective quotient valuation $|\cdot|^\wedge : G/H \to \{0_H\} \oplus \Gamma_K^*$, being isomorphic to K, is not topological.

Instead we have the following result, in which it is supposed that the quotient semi-valuation is coinitial. Generally, a semi-valuation $|\cdot| : G \to \Gamma$ is called *coinitial*, if the set $|G| \setminus \{0\}$ is coinitial in Γ^*, i.e., for all $\gamma \in \Gamma^*$ there exists an element $g \in G$ such that $0 < |g| \leqq \gamma$.

3.7. PROPOSITION. *Suppose that $(G, \Gamma, |\cdot|)$ is a semi-valued group with surjective semi-valuation and H is a normal subgroup of G. If the quotient semi-valuation $|\cdot|^\wedge : G/H \to \Gamma$ exists and is coinitial, then $(G/H, |G/H|^\wedge, |\cdot|^\wedge)$, the quotient semi-valuation regarded here as a surjective mapping, is a semi-valued group.*

PROOF. Denote by $B_\gamma^\wedge$ an open ball with center 0 and radius γ in G/H. As the set $\Gamma^{\wedge *} = |G/H|^\wedge \setminus \{0\}$ is coinitial in Γ^*, the fundamental systems of neighbourhoods $\{B_\gamma^\wedge : \gamma \in \Gamma^{\wedge *}\}$ and $\{B_\gamma^\wedge : \gamma \in \Gamma^*\}$ define the same topology. On the other hand, $B_\gamma^\wedge = \pi(B_\gamma)$ for all $\gamma \in \Gamma^*$, where $\pi : G \to G/H$ is the canonical mapping. Thus the topology induced by the balls $B_\gamma^\wedge$ is the quotient topology.

4. Sequences.

In this section we suppose that G is a group and $|\cdot| : G \to \Gamma$ a semi-valuation.

Let X be a non-empty set. We call $(x_\lambda) = (x_\lambda)_{\lambda \in \Lambda}$, where every $x_\lambda \in X$, a *sequence in X*, if the *index set* Λ is a well-ordered set without a last element.

4.1. Definition. A sequence $(g_\lambda)_{\lambda\in\Lambda}$ in G is said to *converge from right* or *r-converge to a point* $g\in G$, if for every $\gamma\in\Gamma^*$ there exists $\lambda_\gamma\in\Lambda$ such that $|g_\lambda - g| < \gamma$ for all $\lambda \geqq \lambda_\gamma$. In this case we write $g = \text{r-lim}_\Lambda g_\lambda$ or simply $g = \text{r-lim}\, g_\lambda$.

A sequence $(g_\lambda)_{\lambda\in\Lambda}$ in G is said to be a *right Cauchy sequence*, if for every $\gamma\in\Gamma^*$ there exists $\lambda_\gamma\in\Lambda$ such that $|g_\lambda - g_\mu| < \gamma$ for all $\lambda, \mu \geqq \lambda_\gamma$.

In an analogous way one can define also *left convergence* and *left Cauchy sequences*. If a sequence is both left and right convergent or Cauchy sequence, we say simply that it is *convergent* or *Cauchy sequence*, resp.

It is obvious that the usual results concerning convergence and Cauchy sequences hold true also in this general one-sided case. In the sequel we list some of those mainly in order to draw attention to possible side dependencies. The results are usually formulated only for the right hand case.

4.2. Proposition.

(1) *Every right convergent sequence is a right Cauchy sequence.*

(2) *A sequence* (g_λ) *is a right Cauchy sequence if and only if the sequence* $(-g_\lambda)$ *is a left Cauchy sequence. Analogously* (g_λ) *has a right limit* g *if and only if* $(-g_\lambda)$ *has a left limit* $-g$.

(3) *In a semi-valued group right convergent and left convergent sequences coincide.*

(4) *If* g *and* g' *are both right limits of a sequence, then* $|g - g'| = 0$.

(5) *Let* X *be a non-empty subset of* G. *Then a point* $x\in G$ *is in the* τ_r*-closure of* X *if and only if some sequence in* X *r-converges to* x.

Proof. As an example, we prove the third and last assertion.

Let $(g_\lambda)_{\lambda\in\Lambda}$ be a sequence in G which r-converges to a point g in G. If $\gamma\in\Gamma^*$ is arbitrary, then by Theorem 2.4 there is $\delta\in\Gamma^*$ such that $-g + B_\delta + g \subset B_\gamma$. Now, if $\lambda_\delta\in\Lambda$ is so chosen that $|g_\lambda - g| < \delta$ for all $\lambda \geqq \lambda_\delta$, then

$$-g + g_\lambda = -g + (g_\lambda - g) + g \in -g + B_\delta + g \subset B_\gamma$$

for all $\lambda \geqq \lambda_\delta$. Thus $(g_\lambda)_{\lambda\in\Lambda}$ converges from left to g.

Let us prove the last assertion. It is obvious that $\text{r-lim}\, x_\lambda \in \bar{X}$ for all r-convergent sequences (x_λ) in X. Suppose now that $x\in\bar{X}$. By Proposition 3.4 this means that for every $\lambda\in\Gamma^*$ there exists a point $x_\lambda\in X$ such that $|x - x_\lambda| < \lambda$.

Let us denote by $\Gamma^>$ the set Γ^* equipped with the dual order of Γ^*. Assume first that the set $\Gamma^>$ does not have a last element. For an index set Λ we choose a well-ordered cofinal subset of $\Gamma^>$. Then the sequence $(x_\lambda)_{\lambda\in\Lambda}$ r-converges to the point x. If there is a last element $\varepsilon\in\Gamma^>$, then it is the smallest element in Γ^*. In this case $|x - x_\varepsilon| = 0$, which means that the constant sequence (x_λ), with $x_\lambda = x_\varepsilon$ for all λ belonging to an arbitrary index set, r-converges to x.

Note that Cauchy sequences do not generally have the symmetry property analogous to the third item in the previous proposition. Even if the semi-valuation is conjugation isotonic, right and left Cauchy sequences do not necessarily coincide; an example may be found in [4], p. 79, where sequences in ordered groups are considered. On the other hand, in a Hahn group this cannot happen:

4.3. PROPOSITION. *If in a Hahn group $G = \mathbf{H}(A; G_\alpha)$ the index set A does not have a least element, then G is Cauchy complete. Furthermore, right and left Cauchy sequences coincide. If there is a least element α_0 in A, then G is Cauchy complete if and only if G_{α_0} is.*

PROOF. Suppose that the index set A does not have a least element. We show that every right Cauchy sequence has a right limit, which is also a left limit. Let (g_λ) be a right Cauchy sequence in G. Thus for all $\gamma = \gamma_\alpha \in \Gamma_\alpha^* \subset \Gamma^*$ there exists λ_γ such that $|g_\lambda - g_\mu| < \gamma$ for all $\lambda, \mu \geqq \lambda_\gamma$. If $\beta \in A$ with $\alpha < \beta$ then necessarily $(g_\lambda - g_\mu)(\beta) = 0$ for all $\lambda, \mu \geqq \lambda_\gamma$, i.e., the sequence $(g_\lambda(\beta))_\Lambda$ is constant in G_β. As the chain A does not have a least element the definition $g(\beta) = \lim_\Lambda g_\lambda(\beta) \in G_\beta$ is meaningful for every $\beta \in A$. The same reason, together with the fact that $(g - g_\lambda)(\beta) = 0$ for all $\lambda \geqq \lambda_\gamma$ and $\beta > \alpha$, imply that g is in the Hahn product $\mathbf{H}(A; G_\alpha)$ and is a right limit of the sequence (g_λ). Since equally $(-g + g_\lambda)(\beta) = 0$ for all $\lambda \geqq \lambda_\gamma$ and $\beta > \alpha$, the element g is also a left limit of the sequence (g_λ).

The case where A has a least element can be handled similarly.

4.4. DEFINITION. A sequence $(y_\delta)_{\delta \in \Delta}$ is a *subsequence* of a sequence $(x_\lambda)_{\lambda \in \Lambda}$, if there exists a convergent mapping $\alpha: \Delta \to \Lambda$ such that $y_\delta = x_{\alpha(\delta)}$ for all $\delta \in \Delta$.

Recall that a mapping $\alpha: \Delta \to \Lambda$ between two index sets is called *convergent*, if for all $\lambda \in \Lambda$ there is $\delta_\lambda \in \Delta$ such that $\alpha(\delta) \geqq \lambda$ for all $\delta \geqq \delta_\lambda$. A mapping $\alpha: \Delta \to \Lambda$ is said to be *cofinal*, if the range $\alpha(\Delta)$ is *cofinal* in Λ, i.e., for an arbitrary $\lambda \in \Lambda$ there exists $\delta \in \Delta$ with $\alpha(\delta) \geqq \lambda$.

For instance, if $(x_\lambda)_{\lambda \in \Lambda}$ is a sequence, Δ an index set and $\alpha: \Delta \to \Lambda$ an isotonic and cofinal mapping, then $(x_{\alpha(\delta)})_{\delta \in \Delta}$ is a subsequence of the sequence $(x_\lambda)_{\lambda \in \Lambda}$.

In order to characterize the existence of a subsequence we state in the following some important facts about the cofinality type of index sets. These results are also needed in the next section, but they might be of independent interest.

The *cofinality type* $\mathrm{cf}(\Delta)$ of an ordered set Δ is the smallest order type $\mathrm{OT}(\Delta')$ of well-ordered, cofinal subsets Δ' of Δ.

4.5. LEMMA. *Let Δ and Λ be index sets and let $\alpha: \Lambda \to \Delta$ be an isotonic and surjective mapping. Then* $\mathrm{OT}(\Delta) \leqq \mathrm{OT}(\Lambda)$.

Proof. Using the axiom of choice one can find a mapping $\beta: \Delta \to \Lambda$ such that $\alpha\beta$ is the identity mapping. Then β is injective and isotonic, which implies the result.

4.6. **Lemma.** *An index set and its cofinal subsets have the same cofinality type.*

Proof. Let Λ be an index set, and let $\Delta \subset \Lambda$ cofinal. Since the cofinality is a transitive relation, it is clear that $\mathrm{cf}(\Lambda) \leqq \mathrm{cf}(\Delta)$. For the converse, choose a cofinal subset Λ' of Λ such that $\mathrm{OT}(\Lambda') = \mathrm{cf}(\Lambda)$, and define a cofinal mapping $\alpha: \Lambda' \to \Delta$ by $\alpha(\lambda) = \min\{\delta \in \Delta : \delta \geqq \lambda\}$. Using this mapping with Lemma 4.5 we get

$$\mathrm{cf}(\Lambda) = \mathrm{OT}(\Lambda') \geqq \mathrm{OT}(\alpha(\Lambda')) \geqq \mathrm{cf}(\Delta).$$

4.7. **Lemma.** *Let Δ and Λ be index sets and let $\alpha: \Lambda \to \Delta$ be an isotonic mapping. Then there exists a cofinal subset Λ' in Λ such that the restriction $\alpha|_{\Lambda'}: \Lambda' \to \alpha(\Lambda)$ is bijective.*

Proof. A direct computation shows that the definition

$$\Lambda' = \{\lambda \in \Lambda : \lambda' < \lambda \Rightarrow \alpha(\lambda') < \alpha(\lambda)\}$$

meets the requirements.

4.8. **Theorem.** *Let Λ and Δ be two index sets. Then the following facts are equivalent*:

(1) $\mathrm{cf}(\Delta) = \mathrm{cf}(\Lambda)$;
(2) *there exists a convergent mapping* $\alpha: \Delta \to \Lambda$;
(3) *there exists an isotonic and cofinal mapping* $\beta: \Lambda \to \Delta$.

Proof. Assume first that $\mathrm{cf}(\Delta) = \mathrm{cf}(\Lambda)$. Choose cofinal subsets $\Delta' \subset \Delta$ and $\Lambda' \subset \Lambda$ so that $\mathrm{OT}(\Delta') = \mathrm{cf}(\Delta)$ and $\mathrm{OT}(\Lambda') = \mathrm{cf}(\Lambda)$. Then there exists an order isomorphism $\alpha_0: \Delta' \to \Lambda'$. Define a mapping $\alpha: \Delta \to \Lambda$ by the formula

$$\alpha(\delta) = \min\{\alpha_0(\delta') : \delta' \in \Delta' \;\&\; \delta' \geqq \delta\}.$$

This mapping is easily seen to be convergent.

Suppose now that a convergent mapping $\alpha: \Delta \to \Lambda$ exists. Then the sets $\Delta_\lambda = \{\delta_\lambda \in \Delta : \delta \geqq \delta_\lambda \Rightarrow \alpha(\delta) \geqq \lambda\}$ are non-empty and thus the mapping $\beta: \Lambda \to \Delta$ such that $\beta(\lambda) = \min \Delta_\lambda$ is well-defined. One can show that the mapping β is isotonic and cofinal.

Assume finally that there is an isotonic and cofinal mapping $\beta: \Lambda \to \Delta$. By Lemma 4.7 there exists a cofinal subset Λ' of Λ such that the mapping $\beta|_{\Lambda'}: \Lambda' \to \beta(\Lambda)$ is an order isomorphism. Thus we have the following chain of equalities

$$\mathrm{cf}(\Lambda) = \mathrm{cf}(\Lambda') = \mathrm{cf}(\beta(\Lambda)) = \mathrm{cf}(\Delta).$$

We note that it is possible to extend the above list of equivalent statements by the condition that there is a so-called *Tukey function* $\beta: \Lambda \to \Delta$ that maps cofinal subsets to cofinal subsets. See e.g. [16], where this condition is shown to be equivalent to the existence of a convergent mapping $\alpha: \Delta \to \Lambda$ even for directed sets.

4.9. COROLLARY. *Let $(x_\lambda)_{\lambda\in\Lambda}$ be a sequence and Δ an index set. There exists a subsequence of (x_λ) with index set Δ if and only if* $\mathrm{cf}(\Delta) = \mathrm{cf}(\Lambda)$.

A right Cauchy sequence $(x_\lambda)_{\lambda\in\Lambda}$ is called *proper*, if there is no index $\lambda' \in \Lambda$ such that $|x_\lambda - x_{\lambda'}| = 0$ for all $\lambda \geqq \lambda'$. Recall that $\Gamma^>$ denotes the set Γ^* equipped with the dual order of Γ^*.

4.10. THEOREM. *If a right Cauchy sequence $(x_\lambda)_{\lambda\in\Lambda}$ is proper, then* $\mathrm{cf}(\Lambda) = \mathrm{cf}(\Gamma^>)$.

PROOF. Choose a well-ordered and cofinal subset Γ' of $\Gamma^>$, and define

$$\Lambda_\gamma = \{\lambda \in \Lambda: \mu \geqq \lambda \Rightarrow |x_\mu - x_\lambda| < \gamma\}$$

for all $\gamma \in \Gamma'$. Since (x_λ) is a right Cauchy sequence, these sets are all non-empty. The mapping $\alpha: \Gamma' \to \Lambda$ such that $\alpha(\gamma) = \min \Lambda_\gamma$ is clearly isotonic.

As the sequence (x_λ) is proper, the set $\bigcap_{\gamma\in\Gamma'} \Lambda_\gamma$ is empty. Thus for every $\lambda \in \Lambda$ there is $\gamma \in \Gamma'$ with $\lambda \notin \Lambda_\gamma$. If $\alpha(\gamma) < \lambda$, then for all $\mu \geqq \lambda$ would $|x_\mu - x_{\alpha(\gamma)}| < \gamma$ hold, because $\alpha(\gamma)$ is in Λ_γ. By the ultrametric inequality this would imply that λ is in Λ_γ, which is impossible. Thus $\alpha(\gamma) \geqq \lambda$, which means that the range of α is cofinal. Theorem 4.8 implies now the result.

5. Pseudo-convergent sequences.

In this section we suppose that G is a group and $|\cdot|: G \to \Gamma$ is a semi-valuation.

As in the case of usual valuation we can define the notion of pseudo-convergence:

5.1. DEFINITION. A sequence $(g_\lambda)_{\lambda\in\Lambda}$ in G is said to *pseudo-converge from right to a point* $g \in G$, if there exists $\lambda' \in \Lambda$ such that $|g_\mu - g| < |g_\lambda - g|$ for all $\mu > \lambda \geqq \lambda'$. In this case we say that g is an *rp-limit* of the sequence $(g_\lambda)_{\lambda\in\Lambda}$ and we write $g = \text{rp-lim}_\Lambda g_\lambda = \text{rp-lim}\, g_\lambda$.

A sequence $(g_\lambda)_{\lambda\in\Lambda}$ in G is said to be a *right pseudo-Cauchy sequence*, if there exists $\lambda' \in \Lambda$ such that $|g_\nu - g_\mu| < |g_\mu - g_\lambda|$ for all $\nu > \mu > \lambda \geqq \lambda'$.

In an analogous way one can define also *left pseudo-convergence*, and *left pseudo-Cauchy sequences*. If a sequence is both left and right pseudo-convergent or a pseudo-Cauchy sequence, we say that it is *pseudo-convergent* or a *pseudo-Cauchy sequence*, resp.

As in the previous section we usually consider only the right hand side properties of these notions, which are of course very similar to the usual situation; see e.g. [10] or [15]. For the proofs of the following results, see the references just mentioned or [9].

5.2. PROPOSITION. *A right pseudo-convergent sequence is a right pseudo-Cauchy sequence.*

5.3. PROPOSITION. *Let* $(g_\lambda)_{\lambda \in \Lambda}$ *be a right pseudo-Cauchy sequence in G. Then*

(1) *the sequence* $(|g_\lambda|)_{\lambda \geqq \lambda'}$ *in* Γ *is either strictly decreasing or constant;*

(2) $\pi_\lambda = |g_\mu - g_\lambda|$ *is well defined for* $\mu > \lambda \geqq \lambda'$, *and the sequence* $(\pi_\lambda)_{\lambda \geqq \lambda'}$ *is strictly decreasing;*

(3) $g = \text{rp-lim}\, g_\lambda$ *if and only if there exists* $\lambda'' \geqq \lambda'$ *such that* $|g_\lambda - g| = \pi_\lambda$ *for all* $\lambda \geqq \lambda''$.

5.4. DEFINITION. The *right breadth* of a right pseudo-Cauchy sequence $(g_\lambda)_{\lambda \in \Lambda}$ is the subgroup $B = \{g \in G: |g| < \pi_\lambda \ \forall \lambda \geqq \lambda'\}$.

5.5. LEMMA. *Let* $g = \text{rp-lim}\, g_\lambda$. *Then* $h = \text{rp-lim}\, g_\lambda$ *if and only if* $h \in B + g$.

5.6. PROPOSITION. *If B is a right breadth of a right pseudo-Cauchy sequence in G, then the set* $\Sigma\,\{\gamma \in \Gamma: \gamma < \pi_\lambda \ \forall \lambda \geqq \lambda'\}$ *has the following properties:*

(1) Σ *is a lower segment of* Γ, i.e., $\sigma \in \Sigma$, $\gamma \in \Gamma$ *and* $\gamma \leqq \sigma$ *imply* $\gamma \in \Sigma$;

(2) *the set* $|G| \setminus \Sigma$ *is not empty;*

(3) *the set* $|G| \setminus \Sigma$ *does not have a smallest element.*

Conversely, if a set Σ *in* Γ *has these three properties, then the set* $B = \{g \in G: |g| \in \Sigma\}$ *is a breadth of some right pseudo-Cauchy sequence in G.*

It is known that pseudo-Cauchy sequences are not necessarily Cauchy sequences, and conversely, Cauchy sequences are not necessarily pseudo-Cauchy sequences; see e.g. [8], p. 32. The following two results clarify further relationships between these two notions.

5.7. THEOREM. *A proper right Cauchy sequence has a right pseudo-Cauchy subsequence.*

PROOF. Let $(g_\lambda)_{\lambda \in \Lambda}$ be a proper right Cauchy sequence. The set of those indices, for which the corresponding element of the sequence is not a right limit of the sequence, is cofinal in Λ. Furthermore, the corresponding subsequence is also a proper right Cauchy sequence. Thus we can suppose that no element of the original sequence equals to a possible right limit of the sequence.

Choose a well-ordered cofinal subset $\Gamma_0^>$ of the set $\Gamma^>$ so that $\text{OT}(\Gamma_0^>) = \text{cf}(\Gamma^>)$. As the right Cauchy sequence (g_λ) is proper, $\text{cf}(\Gamma^>) = \text{cf}(\Lambda)$ by Theorem 4.10. Thus we can find an isotonic cofinal mapping $\beta: \Gamma_0^> \to \Lambda$.

Define for every $\gamma \in \Gamma_0^>$ and $\lambda \in \Lambda$

$$\Lambda_\gamma = \{\lambda \in \Lambda : \lambda \geqq \beta(\gamma) \;\&\; \exists \lambda' > \lambda \text{ s.t. } g_\mu \in B_r(g_\lambda; \gamma)\ \forall \mu \geqq \lambda'\},$$
$$\Gamma_\lambda = \{\gamma \in \Gamma_0^{>} : \forall \mu \in \Lambda\ \exists \mu' \geqq \mu \text{ s.t. } g_{\mu'} \notin B_r(g_\lambda; \gamma)\}.$$

These sets are non-empty; the former because the sequence (g_λ) is a right Cauchy sequence and the latter because the sequence does not include its possible right limit point.

Define further the mappings

$$\eta: \Gamma_0^{>} \to \Lambda \text{ s.t. } \eta(\gamma) = \min \Lambda_\gamma,$$
$$\varphi: \Lambda \to \Gamma_0^{>} \text{ s.t. } \varphi(\lambda) = \min \Gamma_\lambda.$$

Both of these have cofinal ranges. In addition, the mapping η is isotonic and $\eta\varphi(\lambda) > \lambda$ for all $\lambda \in \Lambda$. With the help of these mappings and a transfinite induction we shall find a cofinal subset $\Delta \subset \Lambda$ such that the subsequences $(g_\lambda)_{\lambda \in \Delta}$ is a right pseudo-Cauchy sequence. The elements of the index set Δ are chosen using a kind of ping-pong-principle with the sets Λ and Γ_0 as sides and with the mappings φ and η as players.

For the first elements of Δ we choose

$$\delta_0 = \min \Lambda, \qquad \delta_1 = \eta\varphi(\delta_0), \qquad \delta_2 = \eta\varphi(\delta_1) = (\eta\varphi)^2(\delta_0).$$

Let $\lambda \in \Lambda$, $\lambda > \delta_2$ be arbitrary. We make the following hypothesis for the transfinite induction:

(Δ) for all $\mu \in \Lambda$ with $\mu < \lambda$ the choice between "μ belongs to Δ" and "μ does not belong to Δ" has been made in such a way that for all $\mu \in \Delta$ we have:

a) $\eta\varphi(\mu) < \lambda$ implies $\eta\varphi(\mu) \in \Delta$,

b) $\mu < \nu < \eta\varphi(\mu)$ and $\nu < \lambda$ imply $\nu \notin \Delta$;

(A) for all $\lambda_1, \lambda_2 \in \Delta$ with $\lambda_1 < \lambda_2 < \lambda$ we have $g_{\lambda_2} \notin B_r(g_{\lambda_1}; \varphi(\lambda_1))$;

(B) for all $\lambda_0, \lambda_1, \lambda_2 \in \Delta$ with $\lambda_0 < \lambda_1 < \lambda_2 < \lambda$ we have $g_{\lambda_2} \in B_r(g_{\lambda_1}; \varphi(\lambda_0))$.

In making the induction step there are the following possibilities:

(1) there exists $\mu \in \Delta$ such that $\mu < \lambda$ and $\eta\varphi(\mu) > \lambda$;

(2) for all $\mu \in \Delta$ with $\mu < \lambda$ we have $\eta\varphi(\mu) \leqq \lambda$, but $\eta\varphi(\mu_0) = \lambda$ for some $\mu_0 \in \Delta$ with $\mu_0 < \lambda$;

(3) for all $\mu \in \Delta$ with $\mu < \lambda$ we have $\eta\varphi(\mu) < \lambda$, but

$$g_\lambda \notin \bigcap_{\substack{\mu_0 < \mu_1 < \lambda \\ \mu_0, \mu_1 \in \Delta}} B_r(g_{\mu_1}; \varphi(\mu_0));$$

(4) for all $\mu \in \Delta$ with $\mu < \lambda$ we have $\eta\varphi(\mu) < \lambda$ and

$$g_\lambda \in \bigcap_{\substack{\mu_0 < \mu_1 < \lambda \\ \mu_0, \mu_1 \in \Delta}} B_r(g_{\mu_1}; \varphi(\mu_0)).$$

In the cases (1) and (3) we don't select the index λ to the index set Δ. Then the hypotheses (Δ), (A) and (B) are obviously valid up to the index λ. In the cases (2) and (4) we choose λ to the index set Δ. In these cases the hypothesis (Δ) is clearly valid for λ. The verifications of the hypotheses (A) and (B) require long but straightforward calculations, which will be omitted.

Using the cofinality properties of the mappings η and φ one can show that the achieved index set Δ is cofinal in Λ. Furthermore, the construction of Δ guarantees that the subsequence $(g_\mu)_{\mu\in\Delta}$ is indeed a right pseudo-Cauchy sequence.

Note that in [9], Lemma 11, p. 44, a similar result is stated in the case of a valued Abelian group, but the proof there seems to be incomplete: Using the notation of the previous proof, the process used in [9] to form the desired subsequence is picking different elements from the set sequence $(\{g_\lambda : \lambda \in \Lambda_\gamma\})_{\gamma\in\Gamma_0}$. This method does guarantee that the achieved sequence is a pseudo-Cauchy sequence, but it does not guarantee that this sequence is a *subsequence* of the original sequence. As one can see from the previous proof the main difficulty is to get both properties at the same time. That is why we had to replace the simple picking process by the more complicate ping-pong process. Note also that our proof uses essentially the results of Section 4 concerning cofinality types of index sets.

5.8. Proposition. *Let $(g_\lambda)_{\lambda\in\Lambda}$ be a right pseudo-Cauchy sequence and B its right breadth. If (g_λ) is a right Cauchy sequence, then $B = \ker|\cdot|$. Conversely, if $B = \ker|\cdot|$, then (g_λ) is a right Cauchy sequence, provided that the semi-valuation is coinitial.*

Proof. It is obvious that $\ker|\cdot| \subset B$ always. If (g_λ) is a right Cauchy sequence and $g \in B$, then for every $\gamma \in \Gamma^*$ we have $|g| < \pi_\lambda < \gamma$ for sufficiently large $\lambda \in \Lambda$. This means that $g \in \ker|\cdot|$.

Conversely, suppose that the semi-valuation is coinitial and $B = \ker|\cdot|$. Let $\gamma \in \Gamma^*$ be arbitrary. The coinitiality of the semi-valuation implies that $\pi_{\lambda''} < \gamma$ for some $\lambda'' \in \Lambda$ with $\lambda'' \geqq \lambda'$. An application of Proposition 5.3 yields

$$|g_\mu - g_\lambda| \leqq \pi_{\min\{\lambda,\mu\}} \leqq \pi_{\lambda''} < \gamma$$

for all $\lambda, \mu > \lambda''$, i.e., (g_λ) is a right Cauchy sequence.

Modifying the proofs presented in [9], pp. 45–46, one can prove the following results:

5.9. Proposition. *Let B be a right breadth, which is a normal subgroup of G. Equip the quotient group G/B with the valuation $|\cdot|_B : G/B \to \Gamma'$,*

$$|g + B|_B = \begin{cases} |g|, & \text{if } g \notin B, \\ 0, & \text{if } g \in B; \end{cases}$$

here Γ' denotes the set $\{0\} \cup (\Gamma \setminus \Sigma)$ and Σ is as in Proposition 5.6.

(1) *If $(g_\lambda)_{\lambda \in \Lambda}$ is a right pseudo-Cauchy sequence in G with right breadth B, then $(g_\lambda + B)_{\lambda \in \Lambda}$ is a right Cauchy sequence in G/B.*

(2) *If $(g_\lambda + B)_{\lambda \in \Lambda}$ is a proper right Cauchy sequence in G/B, then the sequence $(g_\lambda)_{\lambda \in \Lambda}$ in G has a right pseudo-Cauchy subsequence with breadth B.*

(3) *An element $g \in G$ is an* rp*-limit of a right pseudo-Cauchy sequence $(g_\lambda)_{\lambda \in \Lambda}$ with breadth B if and only if $g + B$ is a right limit of the sequence $(g_\lambda + B)_{\lambda \in \Lambda}$.*

5.10. COROLLARY. *Suppose that in G all right breadths are normal subgroups. Then every right pseudo-Cauchy sequence has an* rp*-limit in G if and only if for every right breadth B in G the quotient group G/B is Cauchy complete with respect to the pseudometric given by the valuation $|\cdot|_B$.*

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