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CARTAN SUBALGEBRAS IN FIXED POINT ALGEBRAS OF FINITE GROUP ACTIONS

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Abstract.

We shall consider the following problem: For an outer action α of a finite group G on an approximately finite dimensional factor M, when does the pair $M \supset M^{\alpha}$ contain a common Cartan subalgebra? Here M^{α} denotes the fixed point subalgebra of M. We give a complete answer in terms of the group G and the invariants for the action α.

§1. Introduction.

In [14], V. Jones and S. Popa showed some interesting results on MASA's in the AFD (approximately finite dimensional) type II_1 factor R. Among them, the following result is contained: Let G be a finite group of outer automorphisms of R. Then $R \supset R^G$ contains a common Cartan subalgebra if and only if G is abelian. Here R^G denotes the fixed point subalgebra of R.

Motivated by this result, in this note we shall consider the following problem: For an outer action α of a finite group G on an AFD factor M, when does $M \supset M^{\alpha}$ contain a common Cartan subalgebra? We shall give a necessary and sufficient condition for having a common Cartan subalgebra. Namely, we characterize it in terms of the group G and the invariants for the action α. The essential part of this proof is to construct a model with suitable properties and depends heavily on the classification results obtained in [23], [26] and [16]. The fundamental tools we use here are the model constructed in [10] and the canonical extension of an action in the sense of [8]. The whole idea is led from the index theoretic one ([15], [19], [22]).

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§2. Preliminaries.

2.1 *Results for the case of the* AFD *type* II_1 *factor.*

At first we recall the notion of Cartan subalgebra. A Cartan subalgebra in a von Neumann algebra M is by the definition ([4]), a regular MASA which is the range of a faithful normal conditional expectation from M.

THEOREM (Jones and Popa [14; Corollary 2.3 and Theorem 3.4]). *Let R be the* AFD *type* II_1 *factor and G be a finite group of outer automorphisms of R. Then $R \supset R^G$ contains a common Cartan subalgebra if and only if G is abelian. Here R^G denotes the fixed point subalgebra of R.*

We note that the implication "only if" holds for an arbitrary factor from their proof.

On the other hand, in the crossed product case the pair $R \rtimes G \supset R$ always contains a common Cartan subalgebra. For example, see the proof of [14; Theorem 3.1].

2.2 *Invariants for actions.*

For each type, we recall the complete invariant up to cocycle conjugacy for actions on an AFD factor.

(i) The case of the AFD type II_∞ factor $R_{0,1}$ ([23]).

Let $\alpha: G \to \mathrm{Aut}\, R_{0,1}$ be an action of a discrete amenable group G on $R_{0,1}$. Then the complete invariant is a triple $(N(\alpha), \chi, \mathrm{mod})$, where $N(\alpha) = \alpha^{-1}(\mathrm{Int}(R_{0,1}))$, χ is the characteristic invariant arising from $N(\alpha)$ and mod is the homomorphism from G into the positive numbers $\mathbf{R}_+$ which is defined as the scaling constant of a fixed faithful normal semi-finite trace on $R_{0,1}$.

(ii) The case of AFD type III_λ factors, $\lambda \neq 1$ ([26]).

Let $\alpha: G \to \mathrm{Aut}\, M$ be an action of a discrete amenable group G on an AFD type III_λ factor M ($\lambda \neq 1$). Then the complete invariant is $(N(\alpha), \mathrm{mod}, \chi, \nu)$, where $N(\alpha) = \alpha^{-1}(\mathrm{Cnt}(M))$, mod is the fundamental homomorphism defined by Connes and Takesaki [3], χ is the characteristic invariant arising from $N(\alpha)$ and ν is the modular invariant which is the homomorphism from $N(\alpha)$ into the first cohomology group of the flow of weights.

(iii) The case of the AFD type III_1 factor ([16]).

Let $\alpha: G \to \mathrm{Aut}\, M$ be an action of a finite group or a discrete abelian group G on the AFD type III_1 factor M. Then the complete invariant is $(N(\alpha), \chi, \nu)$.

2.3 *Canonical extensions.*

Let $\alpha: G \to \mathrm{Aut}\, M$ be an action of a discrete group G on a von Neumann algebra M. For a faithful normal semi-finite weight φ on M, the canonical extension $\alpha^\sim$ of α in the sense of Haagerup and Størmer is defined as an action of G on the crossed product algebra $M \rtimes_{\sigma^\varphi} \mathbf{R}$ satisfying

$$\alpha_g^{\sim}(\pi_{\sigma^\varphi}(x)) = \pi_{\sigma^\varphi}(\alpha_g(x)), \; x \in M,$$

$$\alpha_g^{\sim}(\lambda(t)) = \pi_{\sigma^\varphi}((D\varphi \circ \alpha_g^{-1} : D\varphi)_t)\lambda(t), \; t \in \mathsf{R}.$$

Though the extension $\alpha^{\sim}$ is defined after the choice of a weight φ, it is independent of the weight φ up to conjugacy. Therefore it is canonically defined by the action α.

Further, there exists an isomorphism Φ from $(M \rtimes_\alpha G) \rtimes_{\sigma^{\varphi^\sim}} \mathsf{R}$ onto $(M \rtimes_{\sigma^\varphi} \mathsf{R}) \rtimes_{\alpha^\sim} G$ such that

$$\Phi(\pi_{\sigma^{\varphi^\sim}}(\pi_\alpha(x))) = \pi_{\alpha^\sim}(\pi_{\sigma^\varphi}(x)), \; x \in M,$$

$$\Phi(\pi_{\sigma^{\varphi^\sim}}(\lambda(g))) = \lambda^{\sim}(g), \; g \in G,$$

$$\Phi(\Lambda(t)) = \pi_{\alpha^\sim}(\lambda(t), \; t \in \mathsf{R}.$$

Here $\varphi^{\sim}$ is the dual weight of φ on $M \rtimes_\alpha G$ ([6]).

Let us assume that M is an AFD type III factor. If we set $N(\alpha^{\sim}) = (\alpha^{\sim})^{-1}(\mathrm{Int}(M \rtimes_{\sigma^\varphi} \mathsf{R}))$ and consider the characteristic invariant $\chi^{\sim}$ coming from $N(\alpha^{\sim})$, it is known that $N(\alpha) = N(\alpha^{\sim})$ and $\chi = \chi^{\sim}$. For details, see [8], [9] and [16] (see also [25]).

2.4 *Construction of models.*

Let M be an AFD type III factor and $((X, \mu), \{F_t\}_{t \in \mathsf{R}})$ be the flow of weights of M ([3]). We shall consider the following transformations. Let S be an ergodic type III_1 transformation on a measure space (Y, ν). We set

$$\Omega = Y \times X \times \mathsf{R}$$

with the measure $m = \nu \times \mu \times e^u du$, where du means the Lebesgue measure on R. We define the commuting transformations $S^{\sim}$ and $T_t^{\sim}$, $t \in \mathsf{R}$ on Ω by

$$S^{\sim}(y, x, u) = (S(y), x, u - \log \frac{d\nu \circ S}{d\nu}(y)),$$

$$T_t^{\sim}(y, x, u) = (y, F_t(x), u + t - \log \frac{d\mu \circ F_t}{d\mu}(x)).$$

Let G be the countable abelian group generated by $S^{\sim}$ and $T_t^{\sim}$ $(t \in \Gamma)$, where Γ is a countable dense subgroup in R. If we denote the Krieger factor constructed by Feldman and Moore by $W^*(R_G)$, M and $W^*(R_G)$ are isomorphic. This means that any AFD type III factor is expressed as a Krieger factor. For details, we refer to [4] and [10]. (see [2] and [7] for the classification of AFD factors.)

2.5 *Type* III *index theory.*

We recall the basic facts on the type III index theory which are useful for understanding the arguments in this note.

Let M be a (σ-finite) type III factor with a subfactor N and $E: M \to N$ be a faithful normal conditional expectation with finite index in the sense of Kosaki [19]. Let M_1 be the basic extension of $M \supset N$ and $E_M: M_1 \to M$ be the canonical conditional expectation arising from E^{-1}. Fixing a faithful normal state φ on N, we set $\psi = \varphi \circ E \in M_*^+$ and $\chi = \psi \circ E_M \in (M_1)_*^+$. Then we get the following inclusion of type II_∞ von Neumann algebras:

$$M_1^\sim = M_1 \rtimes_{\sigma^\chi} \mathsf{R} \supset M^\sim = M \rtimes_{\sigma^\psi} \mathsf{R} \supset N^\sim = N \rtimes_{\sigma^\varphi} \mathsf{R}.$$

On the other hand, we can consider the basic extension from the inclusion $M^\sim \supset N^\sim$. Then it is exactly $M_1^\sim$. This says that the basic extension is compatible with taking the crossed product with respect to the modular action. See [11], [12] and [18] for details.

Finally, the following result is implicit in [24; Proposition 1.5] or [20; page 164], and is probably a folklore among specialists. However, we remark here explicitly for the convenience to the reader which is not familiar to the type III index theory. It is a key lemma in the proof of main result in this note (see Theorem 2).

LEMMA. *Let $M \supset N$ be a pair of infinite factors with finite index. Let $N \subset M \subset M_1 \subset M_2$ be the factors constructed by iterating the basic extension. Then $M_1 \subset M_2$ is conjugate to $N \subset M$.*

§3. Results.

We deal with type II_∞ case and type III case, separately.

3.1 *The case of the* AFD *type* II_∞ *factor.*

PROPOSITION. *Let $R_{0,1}$ be the* AFD *type* II_∞ *factor. For an outer action α of a finite group G on $R_{0,1}$, $R_{0,1} \supset (R_{0,1})^\alpha$ contains a common Cartan subalgebra if and only if G is abelian.*

PROOF. It is sufficient to show if part. Since G is finite, the module of the action α is trivial. Hence α is cocycle conjugate to the action $\beta \otimes \mathrm{id}$. of G on $R \otimes B(H)$, where β is the unique outer action of G on the AFD type II_1 factor R ([13]) and H is a separable infinite dimensional Hilbert space.

Therefore we get

$$\begin{array}{ccc} R_{0,1} & & R\otimes B(H) \\ \cup & \cong & \cup \\ (R_{0,1})^\alpha & & R^\beta \otimes B(H) \end{array}.$$

From the Jones and Popa theorem, $R_{0,1} \supset (R_{0,1})^\alpha$ possesses a common Cartan subalgebra.

3.2 *The case of* AFD *type* III *factors.*

Before showing main results, we need some preparations.

LEMMA 1. *Let M be a von Neumann algebra and φ be a faithful normal semi-finite weight on M. If M contains a Cartan subalgebra, then so does $M \rtimes_{\sigma^\varphi} \mathbf{R}$.*

PROOF. Suppose that M has a Cartan subalgebra A, that is, A is a regular MASA in M and there exists a faithful normal conditional expectation E from M onto A. There exists a measure space (X, μ) such that A is isomorphic to $L^\infty(X, \mu)$.

Since the isomorphism class of $M \rtimes_{\sigma^\varphi} \mathbf{R}$ is independent of the choice of a weight φ, we may and do assume that $\varphi = \mu \circ E$. We get the inclusion

$$M^\sim = M \rtimes_{\sigma^\varphi} \mathbf{R} \supset A^\sim = A \rtimes_{\sigma^\mu} \mathbf{R}.$$

Since

$$\sigma_t^\mu = \text{id.}, \quad t \in \mathbf{R},$$

that is,

$$a\lambda(t) = \lambda(t)a, \quad a \in A, \quad t \in \mathbf{R},$$

$A^\sim$ is the abelian von Neumann algebra which is isomorphic to $A \otimes L^\infty(\mathbf{R})$. Here $\{\lambda(t)\}_{t \in \mathbf{R}}$ is the usual generator of $A^\sim$ coming from $\mathbf{R}$. As $L^\infty(\mathbf{R})$ is a MASA in $B(L^2(\mathbf{R}))$, it follows that $A^\sim$ is a MASA in $M^\sim$. In order to prove the regularity of $A^\sim$, it is sufficient to show the following: If u belongs to the normalizer of A in M, then u normalizes $A^\sim$. For $t \in \mathbf{R}$ and $a \in A$, we compute

$$\begin{aligned} &u\lambda(t)u^*\lambda(t)^*a \\ &= u\lambda(t)u^*a\lambda(t)^* \\ &= u\lambda(t)u^*auu^*\lambda(t)^* \\ &= uu^*au\lambda(t)u^*\lambda(t)^* \\ &= au\lambda(t)u^*\lambda(t)^*. \end{aligned}$$

Hence we have

$$u\lambda(t)u^*\lambda(t)^* \,(= u\sigma_t^\varphi(u^*)) \in A' \cap M = A.$$

This means that $u\lambda(t)u^* \in A^\sim$. Finally, the existence of a faithful normal conditional expectation from $M^\sim$ onto $A^\sim$ follows from Takesaki's theorem [27].

From now on let M be a type III factor and G be a finite group. We choose and fix a faithful normal semi-finite weight φ on M, and set $M^\sim = M \rtimes_{\sigma^\varphi} \mathbf{R}$. For an action α of G on M, we denote the canonical extension of α by $\alpha^\sim$, and the characteristic invariant of α by $\chi = [\lambda, \mu]$.

LEMMA 2. *With the above notations, the following conditions are equivalent.*

(i) $(M^\sim)' \cap (M^\sim \rtimes_{\alpha^\sim} G) = Z(M^\sim)$, *where* $Z(M^{\sim})$ *means the center of* $M^{\sim}$.

(ii) $N(\alpha) = \{e\}$,

where e is the unit in G.

Proof. It follows from the fact that $(M^{\sim})' \cap (M^{\sim} \rtimes_{\alpha^{\sim}} G)$ is anti-isomorphic to $Z(M^{\sim}) \rtimes_{\mathrm{id},\mu} N(\alpha)$. (See [17] or [25] for details.)

Lemma 3. *Suppose that G is abelian. Let* $(\alpha^{\sim})^{\wedge}$ *be the dual action of* $\alpha^{\sim}$. *Then the following conditions are equivalent.*

(i) $(M^{\sim})' \cap (M^{\sim} \rtimes_{\alpha^{\sim}} G) = Z(M^{\sim} \rtimes_{\alpha^{\sim}} G)$.

(ii) $(M^{\sim} \rtimes_{\alpha^{\sim}} G)' \cap (M^{\sim} \rtimes_{\alpha^{\sim}} G \rtimes_{(\alpha^{\sim})^{\wedge}} G^{\wedge}) = Z(M^{\sim} \rtimes_{\alpha^{\sim}} G)$.

(iii) $\begin{cases} \operatorname{mod} \alpha_g = 1, \ g \in G, \\ \lambda(g, h) = 1, \ g \in G, h \in N(\alpha). \end{cases}$

Proof.

(i) $\leftrightarrow$ (ii): We note that $M \rtimes_{\alpha} G \rtimes_{\alpha^{\wedge}} G^{\wedge}$ is the basic extension of the pair $M \rtimes_{\alpha} G \supset M$, where $\alpha^{\wedge}$ means the dual action of α. Lifting up to the inclusion of type II_{∞} von Neumann algebras by taking the crossed product relative to the modular action, we get

$$M^{\sim} \rtimes_{\alpha^{\sim}} G \rtimes_{(\alpha^{\sim})^{\wedge}} G^{\wedge} \supset M^{\sim} \rtimes_{\alpha^{\sim}} G \supset M^{\sim}.$$

(Hence $M^{\sim} \rtimes_{\alpha^{\sim}} G \rtimes_{(\alpha^{\sim})^{\wedge}} G^{\wedge}$ is the basic extension of $M^{\sim} \rtimes_{\alpha^{\sim}} G \supset M^{\sim}$.) If we denote the modular conjugation arising from $M^{\sim} \rtimes_{\alpha^{\sim}} G$ by $J^{\sim}$, it is known that

$$\begin{aligned} &J^{\sim}((M^{\sim})' \cap (M^{\sim} \rtimes_{\alpha^{\sim}} G))J^{\sim} \\ &= (M^{\sim} \rtimes_{\alpha^{\sim}} G)' \cap (M^{\sim} \rtimes_{\alpha^{\sim}} G \rtimes_{(\alpha^{\sim})^{\wedge}} G^{\wedge}) \end{aligned}$$

and

$$J^{\sim} x^* J^{\sim} = x, \ x \in Z(M^{\sim} \rtimes_{\alpha^{\sim}} G).$$

Here the second equality follows from [1] or [5]. Then the equivalence of (i) and (ii) follows from these.

(i) $\leftrightarrow$ (iii): If we define the action γ of G on $Z(M^{\sim}) \rtimes_{\mathrm{id},\mu} N(\alpha)$ by

$$\gamma_g\left(\sum_{h \in N(\alpha)} c_h z_h\right) = \sum_{h \in N(\alpha)} \lambda(g, h)(\operatorname{mod} \alpha_g)(c_h) z_h,$$

we know

$$Z(M^{\sim} \rtimes_{\alpha^{\sim}} G) \cong (Z(M^{\sim}) \rtimes_{\mathrm{id},\mu} N(\alpha))^{\gamma}$$

(see [17] or [25]). Since the implication (iii) $\to$ (i) is trivial, we shall prove the converse. Assume that the condition (i) holds. This means that the following equality is valid:

$$\sum_{h\in N(\alpha)} \lambda(g,h)(\operatorname{mod}\alpha_g)(c_h)z_h = \sum_{h\in N(\alpha)} c_h z_h$$

for any $g\in G$, $h\in N(\alpha)$ and $c_h\in Z(M^\sim)$. We get

$$\lambda(g,h)(\operatorname{mod}\alpha_g)(c) = c,\ \ c\in Z(M^\sim).$$

Hence we obtain the conclusion.

Now we shall prove our main results in this note.

THEOREM 1. *Let $\alpha: G\to \operatorname{Aut} M$ be an outer action of a finite group G on an* AFD *type* III *factor M. Then the following conditions are equivalent.*

(i) *$M \rtimes_\alpha G \supset M$ contains a common Cartan subalgebra.*

(ii) *$N(\alpha) = \{e\}$.*

PROOF.

(i) $\to$ (ii): Let us assume that $M \rtimes_\alpha G\supset M$ possesses a common Cartan subalgebra. By lemma 1, the canonical pair $M^\sim \rtimes_{\alpha^\sim} G\supset M^\sim$ has a common Cartan subalgebra. We have

$$(M^\sim)'\cap(M^\sim \rtimes_{\alpha^\sim} G) = Z(M^\sim)$$

by the standard argument. Therefore it follows from lemma 2 that $N(\alpha) = \{e\}$.

(ii) $\to$ (i): We shall prove the implication by constructing a model which contains a common Cartan subalgebra and is conjugate to $M \rtimes_\alpha G\supset M$.

Let $((X,\mu), \{F_t\}_{t\in\mathbf{R}})$ be the flow of weights of M. We set

$$Y = \prod_{k=-\infty}^{\infty} Y_k,\ \ Y_k = G,$$

$$\nu = \prod_{k=-\infty}^{\infty} \nu_k,\ \ \nu_k(\{g\}) = \frac{1}{n}\text{ for } g\in G,$$

where $n = |G|$ (the order of G). Let S' be an ergodic type III_1 transformation on (Z,p). We define the measure space (Ω, m) by

$$\Omega = Y\times G\times Z\times X\times \mathbf{R},$$

$$m = \nu\times\nu'\times p\times\mu\times e^u du,$$

where ν' is the measure on G given by $\nu'(\{g\}) = \frac{1}{n}$, $g\in G$. We define the transformations σ, S and T_t, $t \in \mathrm{R}$ on Ω as follows:

$$\sigma(y, g, z, x, u) = (\sigma_0(y), y_0 g, z, x, u),$$

where σ_0 is the shift i.e.,

$$(\sigma_0(y))_k = y_{k+1},\ y = (\cdots, y_{-1}, y_0, y_1, \cdots),$$

and $y_0 g$ means the product of y_0 and g in G,

$$S(y, g, z, x, u) = \left(y, g, S'(z), x, u - \log \frac{dp \circ S'}{dp}(z) \right),$$

$$T_t(y, g, z, x, u) = \left(y, g, z, F_t(x), u + t - \log \frac{d\mu \circ F_t}{d\mu}(x) \right)$$

We simply denote the transformation mod α_g on X which commutes with $\{F_t\}_{t \in \mathrm{R}}$ by $g^\wedge$, and define the G-action β on Ω by

$$\beta_g(y, h, z, x, u) = (y, hg^{-1}, z, g^\wedge(x), u - \log \frac{d\mu \circ (g^\wedge)}{d\mu}(x)),\ g \in G.$$

By H_1 (resp. H_2), we denote the countable discrete amenable group generated by $\sigma, S, T_t (t \in \Gamma)$ and $\beta_g (g \in G)$ (resp. σ, S and $T_t (t \in \Gamma)$), where Γ is a countable dense subgroup in R. Then we get the relation and subrelation arising from $H_1 \supset H_2$, so we have the AFD type III factor and its subfactor which contains a common Cartan subalgebra. Since $\beta_g (g \in G)$ commutes with H_2, β_g belongs to the normalizer of the full group $[H_2]$ of H_2. Hence G induces the action on the subfactor.

We shall consider the invariant of this G-action β. Namely, we consider the skew transformations. We set

$$\Omega^\sim = \Omega \times \mathrm{R}$$

with the measure $m^\sim = m \times e^v dv$ and define the transformations $\sigma^\sim$, $S^\sim$, $T_t^\sim$ $(t \in \Gamma)$ and $\beta_g^\sim$ $(g \in G)$ on $\Omega^\sim$ as follows:

$$\begin{aligned} \sigma^\sim(\omega, v) &= (\sigma(\omega), v), \\ S^\sim(\omega, v) &= (S(\omega), v), \\ T_t^\sim(\omega, v) &= (T_t(\omega), v - t), \\ \beta_g^\sim(\omega, v) &= (\beta_g(\omega), v). \end{aligned}$$

By $H_2^\sim$, we denote the countable abelian group generated by $\sigma^\sim$, $S^\sim$ and $T_t^\sim (t \in \Gamma)$. From the construction, $\beta_g^\sim (g \in G, g \neq e)$ does not belong to the full group $[H_2^\sim]$ of $H_2^\sim$. This means that $N(\beta) = \{e\}$. On the other hand, it follows from the proof in [10] that mod $\beta_g = g^\wedge$, $g \in G$. Hence the invariant of the action β coincides with that of a given action α. Therefore β is cocycle conjugate to α, so that the model is conjugate to $M \rtimes_\alpha G \supset M$.

THEOREM 2. *Let* $\alpha: G \to \operatorname{Aut} M$ *be an outer action of a finite group* G *on an* AFD *type* III *factor* M. *Then the following conditions are equivalent.*

(i) $M \supset M^{\alpha}$ *possesses a common Cartan subalgebra.*

$$\text{(ii)} \begin{cases} G \text{ is abelian,} \\ \operatorname{mod} \alpha_g = 1,\ g \in G, \\ \lambda(g, h) = 1,\ g \in G,\ h \in N(\alpha). \end{cases}$$

PROOF. (i) $\to$ (ii): We assume that $M \supset M^{\alpha}$ contains a common Cartan subalgebra. It is known that G must be abelian. The easy calculation shows that the inclusion

$$M^{\alpha} \subset M \subset M \rtimes_{\alpha} G \subset M \rtimes_{\alpha} G \rtimes_{\alpha^{\wedge}} G^{\wedge}$$

is the tower of the basic extension arising from the first inclusion $M^{\alpha} \subset M$. Therefore $M \rtimes_{\alpha} G \rtimes_{\alpha^{\wedge}} G^{\wedge} \supset M \rtimes_{\alpha} G$ is conjugate to $M \supset M^{\alpha}$. Hence it contains a common Cartan subalgebra, and so does $M^{\sim} \rtimes_{\alpha^{\sim}} G \rtimes_{(\alpha^{\sim})^{\wedge}} G^{\wedge} \supset M^{\sim} \rtimes_{\alpha^{\sim}} G$. We get

$$(M^{\sim} \rtimes_{\alpha^{\sim}} G)' \cap (M^{\sim} \rtimes_{\alpha^{\sim}} G \rtimes_{(\alpha^{\sim})^{\wedge}} G^{\wedge}) = Z(M^{\sim} \rtimes_{\alpha^{\sim}} G).$$

By lemma 3, we obtain the conclusion.

(ii) $\to$ (i): From lemma 3, we have

$$(M^{\sim} \rtimes_{\alpha^{\sim}} G)' \cap (M^{\sim} \rtimes_{\alpha^{\sim}} G \rtimes_{(\alpha^{\sim})^{\wedge}} G^{\wedge}) = Z(M^{\sim} \rtimes_{\alpha^{\sim}} G).$$

Applying lemma 2 to $(M \rtimes_{\alpha} G, \alpha^{\wedge})$, we know

$$N(\alpha^{\wedge}) = \{0\},$$

where 0 means the unit in $G^{\wedge}$. Hence it follows from theorem 1 that $M \rtimes_{\alpha} G \rtimes_{\alpha^{\wedge}} G^{\wedge} \supset M \rtimes_{\alpha} G$ possesses a common Cartan subalgebra. However, it is conjugate to $M \supset M^{\alpha}$.

We concentrate on the case where M is of type III_1 and consider the conditions in the above theorems. It is known that the module is trivial. Since G is finite, the modular invariant is trivial. So $N(\alpha) = \{e\}$ from the outerness. Namely, all the invariants are trivial. This means that a finite group G has the unique outer action α on the AFD type III_1 factor M which arises from the unique outer action β on the AFD type II_1 factor R. More precisely, α is cocycle conjugate to the action $\beta \otimes \mathrm{id.}$ on $R \otimes M$. Hence we get

$$\begin{matrix} M \rtimes_{\alpha} G & & (R \rtimes_{\beta} G) \otimes M \\ \cup & \cong & \cup \\ M & & R \otimes M \end{matrix}$$

and

$$\begin{matrix} M & & R\otimes M \\ \cup & \cong & \cup \\ M^{\alpha} & & R^{\beta}\otimes M \end{matrix} .$$

Summing up the above argument, we obtain

COROLLARY. *Let M be the* AFD *type* II_{∞} *factor or the* AFD *type* III_1 *factor and let G be a finite group of outer automorphisms of M. Then we have*

(i) *$M \rtimes G \supset M$ always contains a common Cartan subalgebra.*

(ii) *$M \supset M^G$ contains a common Cartan subalgebra if and only if G is abelian.*

§4. Connection with the canonical decomposition.

We shall consider the relationship between the existence of a common Cartan subalgebra and the canonical decomposition constructed in [21], and give a typical example to understand the difference between type III_1 case and type III_λ case, $\lambda \neq 1$. The reason why the difference happens is the existence of "type III" action.

At first we recall the basic facts on the canonical decomposition. Let $M \supset N$ be a pair of type III factors with finite index and $M^{\sim} \supset N^{\sim}$ be the canonical inclusion of type II_{∞} von Neumann algebras obtained by taking the crossed product relative to the modular action. Setting

$$\begin{aligned} Z^{\sim} &= Z((N^{\sim})' \cap M^{\sim}), \\ A^{\sim} &= (Z^{\sim})' \cap M^{\sim}, \\ B^{\sim} &= N^{\sim} \vee Z^{\sim}, \end{aligned}$$

it follows that

$$\begin{aligned} & M^{\sim} \supset A^{\sim} \supset B^{\sim} \supset N^{\sim}, \\ & Z(M^{\sim}) \subset Z^{\sim}, \ Z(N^{\sim}) \subset Z^{\sim}, \\ & Z(A^{\sim}) = Z(B^{\sim}) = Z^{\sim}. \end{aligned}$$

Let X (resp. X_M, X_N) be the spectrum of $Z^{\sim}$ (resp. $Z(M^{\sim})$, $Z(N^{\sim})$). Then there exist positive integers m and n such that

$$X = X_M \times \{1, 2, \cdots, m\} = X_N \times \{1, 2, \cdots, n\}.$$

(see [11], [12], [21] and [18] for details.)

Let us assume that $M \supset N$ possesses a common Cartan subalgebra. Then $M^{\sim} \supset N^{\sim}$ also has a natural common Cartan subalgebra, so we get

$$(N^{\sim})' \cap M^{\sim} = Z(N^{\sim}).$$

Therefore it follows that $Z^{\sim} = Z(N^{\sim})$ and $B^{\sim} = N^{\sim}$. This means that the flow space X_M of M is "smaller" than or "equal" to that of N and the lower inclusion disappears.

Let us consider the conditions in the above theorems. Since $M \rtimes_\alpha G \supset M$ and $M \supset M^\alpha$ are completely dual to each other, we consider here only the inclusion $M \rtimes_\alpha G \supset M$. Let us assume for the simplicity, that N is a type III_λ factor, $0 < \lambda < 1$ and G is the cyclic abelian group $\mathbf{Z}_n$ for a fixed positive integer n. There exists a faithful normal state φ on N such that $\sigma_T^\varphi = \mathrm{id.}$, where $T = -2\pi/\log\lambda$. If we set $\alpha = \sigma_{T/n}^\varphi$, α defines the outer action of G on N. We have from the definition, $N(\alpha) = G$. Putting $M = N \rtimes_\alpha G$, the direct computation shows that

$$M^\sim = A^\sim = B^\sim \supset N^\sim$$

and

$$X_M = X_N \times \{1, 2, \cdots, n\}.$$

This says that the lower inclusion actually appears and the flow space X_M of M is "bigger" than that of N. (In this case, M is of type III_{λ^n}.)

On the other hand, in the case where the outer action α of a finite group G on a type III factor N comes from an outer action on a type II_1 factor, it is easy to see that $N(\alpha) = \{e\}$ and

$$M^\sim = A^\sim \supset B^\sim = N^\sim.$$

Therefore the flow space X_M of M is "equal" to that of N.

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