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ON A THEOREM OF GUNDERSEN AND LAINE

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Abstract.

In this paper, we give conditions under which an algebraic differential equation cannot possess any algebroid solution. These conditions generalize a theorem by Gundersen and Laine [3; Theorem 2.2].

1. Preliminaries And Main Results.

We shall apply the usual notations and basic results of Nevanlinna's theory [5] of value distribution. Let $w(z)$ be a v-valued algebroid function in the complex plane defined by an irreducible equation

$$(*) \qquad a_v(z)\,w^v + a_{v-1}(z)\,w^{v-1} + \ldots + a_0(z) = 0,$$

where $a_j(z)$ $(j = 0, 1, \ldots, v)$ are holomorphic functions in the complex plane without any common zeros.

We consider the following algebraic differential equation

$$(1.1) \qquad \Omega(z, w) = \sum_{k=0}^{n} A_k(z)\,w^k,$$

where $A_k(z)$ $(k = 0, 1, \ldots, n)$ are meromorphic functions with $A_n(z) \not\equiv 0$, and $\Omega(z, w)$ is a differential polynomial in w and its derivatives with meromorphic coefficients. Write

$$(1.2) \qquad \Omega(z, w) = \sum_{\lambda \in I} B_\lambda(z)\,(w)^{i_0}(w')^{i_1}\ldots(w^{(n)})^{i_n},$$

where $\lambda = (i_0, i_1, \ldots, i_n)$ denotes the multi-index; each $B_\lambda(z)$ is meromorphic, and I is a finite index set.

Gundersen and Laine [3] discussed the existence of meromorphic solutions of algebraic differential equations and gave conditions under which an algebraic differential equation (1.1) cannot possess a meromorphic solution. In this paper,

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we shall show that such equation cannot have any solution even in the larger class of algebroid functions. To prove this, we shall make use of the following special form of Malmquist's Theorem.

THEOREM A ([2], [4]). *If the equation* (1.1) *has at least one admissible v-valued algebroid solution* $w(z)$, *that is,* $w(z)$ *satisfies the condition:*

$$\sum_{k=0}^{n} T(r, A_k) + \sum_{\lambda \in I} T(r, B_\lambda) = S(r, w),$$

where $S(r, w) = o\{T(r, w)\}$, *then we have*

$$n \leqq \Delta + 2\sigma(v - 1), \tag{1.3}$$

where

$$\Delta = \operatorname{Max}_{\lambda \in I} \left\{ \sum_{\alpha=0}^{n} (\alpha + 1) i_\alpha \right\}; \quad \sigma = \operatorname{Max}_{\alpha=1} \left\{ \sum_{\alpha=1}^{n} (2\alpha - 1) i_\alpha \right\}. \tag{1.4}$$

In [3], it was asked how many distinct meromorphic solutions can a "non-Malmquist" equation of the form (1.1) possess. For this reason, we need the following definition.

DEFINITION ([3]). Consider the algebraic differential equation (1.1) and suppose that q is a fixed integer such that $0 \leqq q \leqq n$. We say that the A_q-hypothesis holds, if there exist $v \not\equiv 0$ meromorphic and h nonconstant entire function such that

$$A_q(z) = v(z) e^{h(z)}, \tag{1.5}$$

where

$$T(r, v) = S(r, e^h), \tag{1.6}$$

and for $k \neq q$, $0 \leqq k \leqq n$, $\lambda \in I$ we have

$$T(r, A_k) = S(r, A_q); \quad T(r, B_\lambda) = S(r, A_q). \tag{1.7}$$

Our main result is Theorem 1 which improves the following Theorem due to Gundersen and Laine [3].

THEOREM B ([3]). *Consider an equation* (1.1) *such that*

$$n > \Delta, \tag{1.8}$$

and assume that the A_q*-hypothesis holds for some* q *that satisfies* $\Delta \leqq q \leqq n - 1$. *Suppose also that* $A_0(z) \not\equiv 0$, $B_{(0,\dots,0)}(z) \equiv 0$. *Then the equation* (1.1) *does not possess a meromorphic solution.*

THEOREM 1. *Consider an equation* (1.1) *such that*

$$n > \Delta + 2\sigma(\nu - 1), \tag{1.9}$$

where Δ and σ are the quantities defined by (1.4), *and assume that the A_q-hypothesis holds for some q that satisfies $\hat{\Delta} \leqq q \leqq n - 1$ with $\hat{\Delta} = \Delta + 2\sigma(\nu - 1)$. Suppose also that $A_0(z) \not\equiv 0$, $B_{(0,\dots,0)}(z) \equiv 0$. Then the equation* (1.1) *does not possess a ν-valued algebroid solution $w(z)$ with $N_\chi(r, w) \leqq 2(\nu - 1)N(r, w)$, where $N_\chi(r, w)$ is the counting function of branch points.*

COROLLARY 1. *Consider* (1.1) *with $n > \Delta$ and assume that the A_q-hypothesis holds for some q that satisfies $\Delta \leqq q \leqq n - 1$. Suppose also that $A_0(z) \not\equiv 0$, $B_{(0,\dots,0)}(z) \equiv 0$. Then the equation* (1.1) *does not possess a ν-valued algebroid solution $w(z)$ with $N_\chi(r, w) = o\{N(r, w)\}$.*

THEOREM 2. *Consider*

$$\frac{dw}{dz} = \frac{\sum_{k=0}^{n} A_k(z)\, w^k}{\sum_{j=0}^{m} B_j(z)\, w^j}, \tag{1.10}$$

such that $n \geqq m + 3 + 2(\nu - 1)(2m - 1)$, and assume that the A_q-hypothesis holds for some q that satisfies $m + 2 + 2(\nu - 1)(2m - 1) \leqq q \leqq n - 1$. Suppose also that $A_0(z) \not\equiv 0$. Then the equation (1.10) *does not possess a ν-valued algebroid solution $w(z)$ with $N_\chi(r, w) \leqq 2(\nu - 1)\, N(r, w)$.*

In order to prove our Theorems, we need

THEOREM C ([1]). *Let $\Omega(z, w)$ be a differential polynomial and let*

$$P(z, w) = \sum_{k=0}^{n} A_k(z)\, w^k; \quad Q(z, w) = \sum_{j=0}^{m} B_j(z)\, w^j,$$

where $w(z)$ is a ν-valued algebroid function satisfying

$$Q(z, w(z))\, \Omega(z, w(z)) = P(z, w(z)).$$

Then, whenever $m \geqq n$, we have $m(r, \Omega(z, w(z))) = S(r, w)$.

2. Proofs.

PROOF OF THEOREM 1. Firstly, the equation (1.1) can be rewritten as

$$A_q(z) = \frac{\Omega(z, w) - \hat{P}(z, w)}{w^q}, \tag{2.1}$$

where

$$\hat{P}(z,w) = A_0(z) + \ldots + A_{q-1}(z)\,w^{q-1} + A_{q+1}(z)\,w^{q+1} + \ldots + A_n(z)\,w^n.$$

By [4; p. 217],

$$(2.2)\quad T(r,\Omega(z,w)) \leqq \{\Delta + 2\sigma(\nu-1)\}\,T(r,w) + O\left\{\Sigma T(r,B_\lambda) + \Sigma m\left(r,\frac{w^{(j)}}{w}\right)\right\}$$

and by [2] or [4; p. 209], we have

$$(2.3)\qquad T(r,\hat{P}(z,w)) = n\,T(r,w) + O\left\{\sum_{k\neq q} T(r,A_k)\right\}.$$

By the A_q-hypothesis (1.7), we can rewrite (2.2) and (2.3) as follow:

$$(2.2)'\qquad T(r,\Omega(z,w)) \leqq \{\Delta + 2\sigma(\nu-1)\}\,T(r,w) + S(r,w) + S(r,A_q)$$

and

$$(2.3)'\qquad T(r,\hat{P}(z,w)) = n\,T(r,w) + S(r,A_q).$$

By (2.1), we obtain

$$\begin{aligned} T(r,A_q) &\leqq q\,T(r,w) + T(r,\Omega(z,w)) + T(r,\hat{P}(z,w)) + O(1)\\ &\leqq D\,T(r,w) + S(r,w) + S(r,A_q), \end{aligned}$$

where $D = q + \Delta + 2\sigma(\nu-1) + n$. Note that $S(r,A_q) = o\{T(r,A_q)\}$. Whenever $r \geqq r_0$, the above inequality can be written as

$$(2.4)\qquad T(r,A_q) \leqq D\,T(r,w) + S(r,w).$$

Hence, we get

$$(2.5)\qquad \sum_{k\neq q} T(r,A_k) + \sum_{\lambda\in I} T(r,B_\lambda) = S(r,A_q) \leqq S(r,w).$$

By $n > \hat{\Delta}$ and from (1.1), any pole of $w(z)$ must be either a zero of $A_n(z)$, or a pole of some $A_k(z)$ $(k \neq n)$, or a pole of some B_λ. By taking multiplicities into account, we have

$$N(r,w) \leqq n\left\{N\left(r,\frac{1}{A_n}\right) + \sum_{k=0}^{n-1} N(r,A_k) + \sum_{\lambda\in I} N(r,B_\lambda)\right\}.$$

By (1.5), (1.6), (1.7) and (2.4), we obtain

$$N(r,A_q) = S(r,A_q) = S(r,w)$$

and

$$(2.6)\qquad N(r,w) = S(r,w).$$

Taking logarithmic derivative on both sides of (2.1), we get

$$\frac{A_q'(z)}{A_q(z)} = -q\frac{w'(z)}{w(z)} + \frac{\Omega'(z,w(z)) - \hat{P}'(z,w(z))}{\Omega(z,w(z)) - \hat{P}(z,w(z))}.$$

This reduces to

$$(2.7)\qquad w^n\left[(n-q)A_n(z)\,w' + \left(A_n'(z) - A_n(z)\frac{A_q'(z)}{A_q(z)}\right)w\right] = Q_n(z,w),$$

where $Q_n(z,w)$ is a differential polynomial in w and its derivatives with total degree $\leq n$; whose coefficients $c_\mu(z)$ are one of the following forms: $A_k(z)$, $A_k'(z)$ $(k \neq q)$, $A_q'(z)/A_q(z)$, $B_\lambda(z)$ and $B_\lambda'(z)$, $\lambda \in I$. Thus, we have

$$T(r,c_\mu) = m(r,c_\mu) + N(r,c_\mu) = S(r,w), (\mu \in J);$$

where J is a finite index set. Let

$$H(z) \equiv (n-q)A_n(z)\,w' + \left(A_n'(z) - A_n(z)\frac{A_q'(z)}{A_q(z)}\right)w.$$

Then, (2.7) becomes

$$(2.8)\qquad w^n H(z) = Q_n(z,w).$$

We divide into 2 cases for discussion:

(i) If $H(z) \equiv 0$, i.e.,

$$(n-q)\frac{w'}{w} = \frac{A_q'(z)}{A_q(z)} - \frac{A_n'(z)}{A_n(z)}.$$

Since $n \neq q$, integrating the above equation, we obtain

$$(2.9)\qquad A_q(z)\,w^q = c\,A_n(z)\,w^n,$$

where c is a constant. Substituting (2.9) into (1.1), we obtain

$$\Omega(z,w) = A_0(z) + \ldots + A_{q-1}(z)\,w^{q-1} + A_{q+1}(z)\,w^{q+1} + \ldots + (c+1)A_n(z)\,w^n.$$

By (2.2)′, (2.3)′ and (2.4), we get

$$n\,T(r,w) \leqq \{\Delta + 2\sigma(v-1)\}\;T(r,w) + S(r,w)$$

which leads to $n \leqq \Delta + 2\sigma(v-1)$. This contradicts our hypothesis. It means that $H(z) \equiv 0$ cannot happen.

(ii) If $H(z) \not\equiv 0$, then by spplying Theorem C to (2.8), we get $m(r,H) = S(r,w)$. On the other hand, because

$$n(r,H) \leqq n(r,A_n') + n(r,w') + \bar{n}(r,A_q) + \bar{n}\left(r,\frac{1}{A_q}\right),$$

where $\bar{n}(r, A_q)$ and $\bar{n}(r, 1/A_q)$ is the number of poles and zeros of A_q and each pole or zero being counted only once, we know in [4]

$$n(r, w') \leqq n(r, w) + \bar{n}(r, w) + n_\chi(r, w),$$

where $n_\chi(r, w)$ is the number of branch points of $w(z)$. By hypothesis of Theorem 1, $N_\chi(r, w) \leqq 2(v-1)\,N(r, w)$. Thus, we have

$$\begin{aligned} N(r, H) &\leqq N(r, w) + \bar{N}(r, w) + 2(v-1)\,N(r, w) + N(r, A_n) \\ &\quad + \bar{N}(r, A_n) + \bar{N}(r, A_q) + \bar{N}\left(r, \frac{1}{A_q}\right) \\ &= S(r, w). \end{aligned}$$

So, $T(r, H) = S(r, w)$. By definition of $H(z)$, we obtain

$$w' = c_{10}(z) + c_{11}(z) w, \tag{2.10}$$

where

$$c_{10}(z) = \frac{H(z)}{(n-q)\,A_n(z)} \quad \text{and} \quad c_{11} = \left[A_n(z)\frac{A_q'(z)}{A_q(z)} - A_n'(z)\right]\frac{1}{(n-q)\,A_n(z)}.$$

Hence,

$$\begin{aligned} &T(r, c_{10}) \leqq T(r, H) + T(r, A_n) + O(1) = S(r, w); \\ &T(r, c_{11}) \leqq T\left(r, \frac{A_q'}{A_q}\right) + T\left(r, \frac{A_n'}{A_n}\right) + O(1) \leqq S(r, w). \end{aligned} \tag{2.11}$$

Differentiating (2.10), we get

$$w'' = c_{10}'(z) + c_{11}'(z)\,w + c_{11}(z)\,w' = c_{20}(z) + c_{21}(z)\,w,$$

where $c_{20}(z) = c_{10}'(z) + c_{10}(z)c_{11}(z)$ and $c_{21}(z) = c_{11}'(z) + c_{11}^2(z)$. By (2.11), we have $T(r, c_{20}) + T(r, c_{21}) = S(r, w)$. In the same fashion, we can prove that $w^{(p)} = c_{p0}(z) + c_{p1}(z)\,w$ and $T(r, c_{p0}) + T(r, c_{p1}) = S(r, w)$. Above all, (1.1) becomes

$$\sum_{i=0}^{|\lambda|} D_i(z)\,w^i = \sum_{k=0}^{n} A_k(z)\,w^k, \text{ where } |\lambda| = \max_{\lambda \in I}\{i_0 + i_1 + \ldots + i_n\}.$$

Hence,

$$\sum_{k=0}^{n} c_k(z)\,w^k = 0, \tag{2.12}$$

where $c_k(z) = A_k(z) - D_k(z)$. Because $|\lambda| < \Delta \leqq \hat{\Delta}$, so when $k \geqq q \geqq \hat{\Delta}$, we have $c_k(z) = A_k(z)$.

Denote the set of zeros of $w(z)$ by $E = \cup E_j\,(j = 1, 2, 3, 4)$. E_1 (or E_2) is the set of zeros of $w(z)$ with multiple order $\tau \leqq \nu$, but is not (or is) the zeros and poles of $A_k(z)$ and $D_k(z)$. However, E_3 (or E_4) is the set of zeros of $w(z)$ with multiple order $\tau \geqq \nu + 1$, but is not (or is) the zeros and poles of $A_k(z)$ and $D_k(z)$. Let $N_{\nu)}(r, 1/w)$ denote the counting functions corresponding to the set of zeros of $w(z)$ in E_1. We claim

$$T(r, w) = N_{\nu)}\left(r, \frac{1}{w}\right) + S(r, w). \tag{2.13}$$

In fact, by definition of $H(z)$, we know that

$$m\left(r, \frac{1}{w}\right) \leqq m\left(r, \frac{1}{H}\right) + m(r, A_n) + m\left(r, \frac{w'}{w}\right) + m\left(\frac{A_n'}{A_n}\right) + m\left(r, \frac{A_q'}{A_q}\right) + 0(1) = S(r, w).$$

Thus, we have $T(r, w) = N(r, 1/w) + S(r, w)$.

In the following, we shall prove that the contribution of counting functions corresponding to the points of E_2, E_3 and E_4 in $N(r, 1/w)$ are equal to $S(r, w)$. Let $N_{(j)}(r, 1/w)$ denote the counting functions corresponding to the points in $E_j (j = 2, 3, 4)$.

Firstly, it is easy to see that

$$N_{(2)}\left(r, \frac{1}{w}\right) \leqq \nu\left\{\sum\left[N(r, A_k) + N\left(r, \frac{1}{A_k}\right)\right] + \sum\left[N(r, D_k) + N\left(r, \frac{1}{D_k}\right)\right]\right\} = S(r, w).$$

For $z_0 \in E_3$, we have $\tau \geqq \nu + 1$. So z_0 is the zero of $w'(z)$ of order $\tau - \nu (\geqq 1)$. As

$$\frac{H(z)}{A_n(z)} = (n - q)\, w' + \left(\frac{A_n'(z)}{A_n(z)} - \frac{A_q'(z)}{A_q(z)}\right) w(z), \tag{2.14}$$

z_0 must be a zero of $H(z)/A_n(z)$ of order $\tau - \nu$. Since $\tau \geqq \nu + 1$, we have $(\tau - \nu)(\nu + 1) \geqq \tau$. Hence,

$$N_{(3)}\left(r, \frac{1}{w}\right) \leqq (\nu + 1)\, N\left(r, \frac{H(z)}{A_n(z)}\right) = S(r, w).$$

Lastly, if z_0 is the zero of $w(z)$ of order $\tau (\geqq \nu + 1)$, and also z_0 is the zero and pole of $A_n(z)$ and $A_q(z)$, then z_0 is a simple pole of $A_n'(z)/A_n(z)$ and $A_q'(z)/A_q(z)$. Therefore, if z_0 is the zero of $w(z)$ of order $\tau (\geqq \nu + 1)$, then it must be the zeros of $A_n'(z)\, w(z)/A_n(z)$ and $A_q'(z)\, w(z)/A_q(z)$ of order $\tau - 1$ $(\geqq \tau - \nu)$. From (2.14), we know that z_0 must be the zero of $H(z)/A_n(z)$ of order $\tau - \nu$. So, we also have

$$N_{(4)}\left(r, \frac{1}{w}\right) \leqq (\nu + 1)\, N\left(r, \frac{H(z)}{A_n(z)}\right) = S(r, w).$$

This proves (2.13).

For $z_0 \in E_1$, by (2.12), we get $c_0(z_0) = 0$, $z_0 \in E_1$. Since $T(r, c_0) = S(r, w)$ and $N(r, c_k) = S(r, w)$ $(k = 1, 2, \ldots, n)$, if $c_0(z) \not\equiv 0$, then

$$N_{\nu)}\left(r, \frac{1}{w}\right) \leqq \nu N\left(r, \frac{1}{c_0}\right) + S(r, w) = S(r, w).$$

This is impossible. So, we must have $c_0(z) \equiv 0$. Thus, (2.12) becomes

$$\sum_{k=0}^{n-1} c_{k+1}(z)\, w^k = 0.$$

For the same reason, we obtain

$$\sum_{k=0}^{n-q} c_{k+q}(z)\, w^k = 0,$$

that is,

$$\sum_{k=0}^{n-q} A_{k+q}(z)\, w^k = 0.$$

In particular, if $z_0 \in E_1 \cup E_2$, then the above equation still holds. In such case, there may appear either

(i) if $A_k(z_0) \neq \infty$, for all k with $k \neq q$, then necessarily $A_q(z_0) = 0$ or
(ii) if $A_q(z_0) \neq 0$, then it must have some k such that $A_k(z_0) = \infty$.

In both cases, we must have

$$N_{\nu)}\left(r, \frac{1}{w}\right) \leqq \nu \left\{ N\left(r, \frac{1}{A_q}\right) + \sum_{k=q+1}^{n} N\left(r, \frac{1}{A_k}\right) \right\} = S(r, w).$$

This contradiction to (2.13) proves Theorem 1.

Proof of Corollary 1. In fact, by the given condition, we have

$$T(r, \Omega(z, w)) \leqq \Delta T(r, w) + S(r, w).$$

Similar proof in Theorem 1 gives our conclusion.

Proof of Theorem 2.
Rearrange (1.10) into the form:

$$\sum_{j=0}^{m} B_j(z)\, w^j\, w' = \sum_{k=0}^{n} A_k(z)\, w^k.$$

This becomes the special case of (1.1). Here, we take $\Delta = m + 2$, $\sigma = 2m - 1$. So, by applying Theorem 1, the result of Theorem 2 is achieved.

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