MATH. SCAND. 70 (1992), 311–319

ON THE IRREDUCIBILITY OF FLENSTED-JENSEN'S FUNDAMENTAL SERIES REPRESENTATIONS FOR SEMISIMPLE SYMMETRIC SPACES

H. THORLEIFSSON

Abstract.

In [5] T. Oshima and T. Matsuki gave a construction of the discrete series representations of a symmetric space G/H, satisfying the equal rank condition rank G/H = rank $K/K \cap H$, and proved their irreducibility for regular infinitesimal character. In [11] D. Vogan proved the irreducibility of these representation also for singular infinitesimal character. In [3] Flensted-Jensen used the construction of T. Oshima and T. Matsuki to construct fundamental series representations $\mathscr{V}(\mathfrak{a}, \Sigma^+, \lambda)$ of functions on G/H, when the space G/H satisfies a certain condition, and proves their irreducibility for regular infinitesimal character. In this paper we characterize the spaces handled by Flensted-Jensen, and give a proof that for those spaces $\mathscr{V}(\mathfrak{a}, \Sigma^+, \lambda)^c$ is cohomological induced from a one-dimensional representation. As a consequence we get some irreducibility results for these representations.

1. Introduction.

Throughout this paper let G be a connected real semisimple linear group and H an open subgroup of the fixed point subgroup of an involution σ of G. Then $X = G/H$ is a semisimple symmetric space. Let $\mathscr{E}(X)$ be the space of smooth function on X. We are interested in certain G-submodules of $\mathscr{E}(X)$. Before we can formulate the statement we are going to look at we have to put together some notations.

Let $\mathfrak{g}_0$ be the Lie algebra of G. We will use the same letter σ to denote the involution on $\mathfrak{g}_0$. Let

$$\mathfrak{g}_0 = \mathfrak{h}_0 \oplus \mathfrak{s}_0$$

be the decomposition of $\mathfrak{g}_0$ into the $+1$ and the -1 eigenspaces of σ. Choose a Cartan involution θ of $\mathfrak{g}_0$, commuting with σ, and let

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$$

be the corresponding Cartan decomposition. Let $\mathfrak{a}_0 \subset \mathfrak{s}_0$ be a fundamental Cartan subspace for $X = G/H$, so $\mathfrak{t}_0 := \mathfrak{a}_0 \cap \mathfrak{k}_0$ is maximal abelian in $\mathfrak{k}_0 \cap \mathfrak{s}_0$. Let

Received April 30, 1991.

$\mathfrak{c}_0$ be a fundamental Cartan subalgebra of $\mathfrak{g}_0$ containing $\mathfrak{t}_0$. Let $\Sigma(\mathfrak{g},\mathfrak{a})$ (resp. $\Delta(\mathfrak{g},\mathfrak{c})$) be the set of roots of $\mathfrak{a}$ (resp. $\mathfrak{c}$) in $\mathfrak{g}$ (we use gothic letters with suffix 0 to denote real Lie algebras and their subspaces, and the same letter without the suffix 0 to denote the complexification), and let $W = W(\mathfrak{g},\mathfrak{a})$ be the corresponding Weyl group.

Let $\Sigma^+(\mathfrak{g},\mathfrak{a})$ be a positive system of restricted roots, and $\Delta^+(\mathfrak{g},\mathfrak{c})$ a positive system of roots compatible with $\Sigma^+(\mathfrak{g},\mathfrak{a})$. Suppose further that $\Sigma^+ = \Sigma^+(\mathfrak{g},\mathfrak{a})$ is θ-compatible, that is $\alpha\in\Sigma^+(\mathfrak{g},\mathfrak{a})$, $\alpha\,|\,\mathfrak{t} \neq 0$ implies $\theta\alpha\in\Sigma^+(\mathfrak{g},\mathfrak{a})$. Put

$$(1)\qquad \mathfrak{l} := \mathfrak{g}^{\mathfrak{t}} = \sum_{\substack{\alpha\in\Sigma(\mathfrak{g},\mathfrak{a})\\ \alpha|\mathfrak{t}=0}} \mathfrak{g}_\alpha \oplus \mathfrak{g}^{\mathfrak{a}},\quad \mathfrak{u} := \sum_{\substack{\alpha\in\Sigma^+(\mathfrak{g},\mathfrak{a})\\ \alpha|\mathfrak{t}\neq 0}} \mathfrak{g}_\alpha$$

($\mathfrak{g}_\alpha$ is the root space of α and $\mathfrak{g}^{\mathfrak{t}}$ is the centralizer of $\mathfrak{t}$ in $\mathfrak{g}$) and $\mathfrak{q} := \mathfrak{l}\oplus\mathfrak{u}$. Then $\mathfrak{q}$ is a θ-stable parabolic subalgebra of $\mathfrak{g}$.

We are mainly interested in symmetric spaces $X = G/H$ satisfying the following condition, which is condition (22) from [3] §VI.3.

$$(2)\qquad \alpha\,|\,\mathfrak{t} \neq 0 \text{ for each } \alpha\in\Sigma(\mathfrak{g},\mathfrak{a}).$$

Further let $\mathbf{D}(X)$ be the algebra of invariant differential operators on X, and $\mathfrak{S}(\mathfrak{a})^W$ the subspace of the symmetric algebra $\mathfrak{S}(\mathfrak{a})$ of elements invariant under W. For each $\lambda\in\mathfrak{a}^*$ (the algebraic dual of $\mathfrak{a}$) there is a homomorphism $\chi_\lambda\colon \mathbf{D}(X)\to\mathbf{C}$ defined through the canonical isomorphism $\mathbf{D}(X)\to\mathfrak{S}(\mathfrak{a})^W$, and evaluation at λ (see for example [3] Theorem II.2). Put $\mathscr{E}_\lambda(X) = \{f\in\mathscr{E}(X)\,|\,Df = \chi_\lambda(D)f$ for all $D\in\mathbf{D}(X)\}$.

Define $\rho(\mathfrak{u})\in\mathfrak{l}^*$ through $\rho(\mathfrak{u})(Y) := \frac{1}{2}\,\mathrm{Trace}(\mathrm{ad}(Y)\,|\,\mathfrak{u})$, $Y\in\mathfrak{l}$, and $\rho(\mathfrak{u}\cap\mathfrak{k})\in(\mathfrak{l}\cap\mathfrak{k})^*$ through $\rho(\mathfrak{u}\cap\mathfrak{k})(Y) := \frac{1}{2}\,\mathrm{Trace}(\mathrm{ad}(Y)\,|\,\mathfrak{u}\cap\mathfrak{k})$, $Y\in\mathfrak{l}\cap\mathfrak{k}$. Let $\langle\,,\rangle$ denote the Cartan-Killing from on $\mathfrak{g}$. We use $\langle\,,\rangle$ also for the corresponding form on $\mathfrak{g}^*$. Using $\langle\,,\rangle$ we look at $\mathfrak{t}^*$, $\mathfrak{a}^*$, $\mathfrak{c}^*$ and $\mathfrak{l}^*$ as subspaces of $\mathfrak{g}^*$. The decompositions $\mathfrak{g} \triangleq \mathfrak{u}\oplus\mathfrak{l}\oplus\sigma\mathfrak{u}$ and $\mathfrak{k} = (\mathfrak{u}\cap\mathfrak{k})\oplus(\mathfrak{l}\cap\mathfrak{k})\oplus\sigma(\mathfrak{u}\cap\mathfrak{k})$ imply $\rho(\mathfrak{u})\,|\,\mathfrak{l}\cap\mathfrak{h} = 0$ and $\rho(\mathfrak{u}\cap\mathfrak{k})\,|\,\mathfrak{l}\cap\mathfrak{k}\cap\mathfrak{h} = 0$. The θ-invariance of $\mathfrak{u}$ implies $\rho(\mathfrak{u})\,|\,\mathfrak{l}\cap\mathfrak{p} = 0$. Since $\mathfrak{t}$ is maximal abelian in $\mathfrak{k}\cap\mathfrak{s}$, we get $\mathfrak{l}\cap\mathfrak{k}\cap\mathfrak{s} = \mathfrak{t}$. All this implies $\rho(\mathfrak{u})$, $\rho(\mathfrak{u}\cap\mathfrak{k})\in\mathfrak{t}^*$. Put $\Delta^+(\mathfrak{l},\mathfrak{c}) := \Delta^+(\mathfrak{g},\mathfrak{c})\cap\Delta(\mathfrak{l},\mathfrak{c})$, and let $\rho(\mathfrak{l})\in\mathfrak{c}^*$ be half the sum of the roots in $\Delta^+(\mathfrak{l},\mathfrak{c})$. For $\lambda\in\mathfrak{a}^*$ put

$$\mu_\lambda := \lambda\,|\,\mathfrak{t} + \rho(\mathfrak{u}) - 2\rho(\mathfrak{u}\cap\mathfrak{k})\in\mathfrak{t}^*,$$

and let $\mathscr{V}(\mathfrak{a},\Sigma^+,\lambda)\subset\mathscr{E}_\lambda(X)$ be the $(\mathfrak{g},K)$-module constructed by Flensted-Jensen in [3] §V.3. We will recall the construction in the next section. The following theorem is a part of [3] Theorem VI.8.

THEOREM 1.1. *Suppose G/H is simply connected and condition* (2) *is satisfied. Then $\mathscr{V}(\mathfrak{a},\Sigma^+,\lambda)$ is a finitely-generated* $(\mathfrak{g},K)$*-module, with infinitesimal character* $-(\lambda+\rho(\mathfrak{l}))$, *and the following holds:*

A. *If* $\mathscr{V}(\mathfrak{a}, \Sigma^+, \lambda) \neq 0$, *then*
(i) $\langle \mu_\lambda, \alpha \rangle / \langle \alpha, \alpha \rangle \in \mathbf{Z}$ *for each* $\alpha \in \Sigma^+(\mathfrak{g}, \mathfrak{a})$.
(ii) *Any K-type occurring in* $\mathscr{V}(\mathfrak{a}, \Sigma^+, \lambda)$ *has lowest weight* $-\mu$, *where*

$$\mu = \mu_\lambda + \sum_{\beta \in \Delta(\mathfrak{u} \cap \mathfrak{p}, \mathfrak{t} \cap \mathfrak{k})} n_\beta \beta$$

for some $n_\beta \in \mathbf{N} \cup \{0\}$.

B. *Assume* $\langle \mu_\lambda, \alpha \rangle / \langle \alpha, \alpha \rangle \in \mathbf{N} \cup \{0\}$ *for each* $\alpha \in \Sigma(\mathfrak{u} \cap \mathfrak{k}, \mathfrak{t})$. *If*

$$\operatorname{Re}\langle \lambda + \rho(\mathfrak{l}), \alpha \rangle \geqq 0 \text{ for each } \alpha \in \Sigma^+(\mathfrak{g}, \mathfrak{a}), \tag{3}$$

then $\mathscr{V}(\mathfrak{a}, \Sigma^+, \lambda)$ *is irreducible.*

If G/H is not simply connected the statements remain true, if one adds an integrability condition on μ_λ in part B (μ_λ must be the differential of a weight of a finite dimensional K-module).

In Corollary 2.3 these modules are written as modules cohomologically induced from one dimensional representations of a two-fold (metaplectic) cover of L, extending the same result for the discrete series for G/H ([3] Theorem VIII.2). In Proposition 2.4 a description of the symmetric spaces G/H satisfying condition (2), and in Therem 2.6 some irreducibility results are given.

Finally I want to thank G. Ólafsson and H. Schlichtkrull for helpfull discussions while working on this article.

2. The fundamental series as cohomologically induced modules.

We keep the notations from the introduction. In [3] M. Flensted-Jensen generalized a construction of the discrete series given by T. Oshima and T. Matsuki in [5] to construct fundamental series representations. Let me recall this construction (see [3] § V.3 for more details).

Let G be a real form of the complex Lie group $G_{\mathbf{C}}$ and $(\mathfrak{g}_0^d, \mathfrak{h}_0^d, \mathfrak{k}_0^d)$ denote the dual symmetric triple corresponding to $(\mathfrak{g}_0, \mathfrak{h}_0, \mathfrak{k}_0)$ ([3] § 1.4), defined by

$$\mathfrak{g}_0^d = (\mathfrak{h}_0 \cap \mathfrak{k}_0) \oplus i(\mathfrak{h}_0 \cap \mathfrak{p}_0) \oplus i(\mathfrak{s}_0 \cap \mathfrak{k}_0) \oplus (\mathfrak{s}_0 \cap \mathfrak{p}_0),$$

$$\mathfrak{h}_0^d = (\mathfrak{h}_0 \cap \mathfrak{k}_0) \oplus i(\mathfrak{s}_0 \cap \mathfrak{k}_0),$$

$$\mathfrak{k}_0^d = (\mathfrak{h}_0 \cap \mathfrak{k}_0) \oplus i(\mathfrak{h}_0 \cap \mathfrak{p}_0).$$

Further let G^d, K^d and H^d be the analytic subgroups of $G_{\mathbf{C}}$ corresponding to $\mathfrak{g}_0^d$, $\mathfrak{k}_0^d$ and $\mathfrak{h}_0^d$. K^d is a maximal compact subgroup of G^d. Put

$$\mathfrak{a}_0^d = \mathfrak{a} \cap \mathfrak{g}_0^d = i(\mathfrak{a}_0 \cap \mathfrak{k}_0) \oplus (\mathfrak{a}_0 \cap \mathfrak{p}_0),$$

and let A^d be the corresponding analytic subgroup of G^d.

Let $\hat{K}$ (resp. $\hat{H}^d(K)$) be the set of equivalence classes of irreducible finite dimensional representations of K (resp. H^d that extend to a holomorphic representation of $K_{\mathbf{C}}$). Thus $\hat{K}$ and $\hat{H}^d(K)$ are in one-to-one correspondence via holomorphic representations of $K_{\mathbf{R}}$. For a H^d-module V (resp. K-module) let V_{H^d} (resp. V_K) be the space of sums of $v \in V$, that behave under H^d (resp. K) corresponding to some representation in $\hat{H}^d(K)$ (resp. $\hat{K}$). In [2] (Theorem 2.3) Flensted-Jensen uses analytic continuation on $G_{\mathbf{C}}/H_{\mathbf{C}}$ to give an isomorphism

$$\eta\colon \mathscr{E}(G^d/K^d)_{H^d} \to \mathscr{E}(G/H)_K,$$

that commutes with the left $\mathfrak{U}(\mathfrak{g})$-action (we are using the formulation given in [5] Proposition 1).

Let $\Sigma^+ = \Sigma^+(\mathfrak{g}, \mathfrak{a})$ be as in the introduction, and let P^d be the corresponding parabolic subgroup of G^d, with Langlands decomposition $P^d = M^d A^d N^d$, and $\rho \in \mathfrak{a}^*$ equal to half the sum of roots in Σ^+. For $\lambda \in \mathfrak{a}^*$ let $\mathbf{C}_\lambda$ be the one-dimensional P^d-module on which $M^d N^d$ operates trivially and $\mathfrak{a}$ with weight λ, and let $P^d \to \mathbf{C}$, $p \mapsto p^\lambda$ be the corresponding one-dimensional character of P^d.

Let (ξ, V) be a smooth Fréchet space representation of P, and $\mathscr{D}(G^d/P^d; V)$ the space of smooth functions on G with values in V, satisfying $f(xp) = p^{-\rho}\xi(p)^{-1} f(x)$, for all $x \in G^d$, $p \in P^d$, given the topology described in [1] Definition 4; 1 (note that G^d/P^d is compact). Let $\mathscr{D}'(G^d/P^d; V)$ the space of distributions T on G with value in V, satisfying $T(xp) = p^{-\rho}\xi(p)^{-1} T(x)$, for all $x \in G^d$, $p \in P^d$. There is a canonical isomorphism $\mathscr{D}'(G^d; V') \cong \mathscr{D}(G^d; V)'$ ([1] Proposition 1; 1, see also [1] for the topology used). Using [1] Proposition 4; 1 and [1] Corollaire 3; 3 one gets an isomorphism

(4) $$\mathscr{D}'(G^d/P^d; V') \cong \mathscr{D}(G^d/P^d; V)'.$$

(Here we only need this for finite dimensional V.)

The Poisson transform $\mathscr{P}_\lambda$ defined in [5] (1.3) (see also [3] §IV.1) defines, when restricted to $\mathscr{D}'(G^d/P^d; \mathbf{C}_{-\lambda})_{H^d}$, a $(\mathfrak{g}, H^d)$-morphism

(5) $$\mathscr{P}_\lambda\colon \mathscr{D}'(G^d/P^d; \mathbf{C}_{-\lambda})_{H^d} \to \mathscr{E}_\lambda(G^d/K^d)_{H^d}.$$

If $\lambda \in \mathfrak{a}^*$ satisfies

(6) $$\operatorname{Re}\langle \lambda, \alpha \rangle \geqq 0 \text{ for all } \alpha \in \Sigma^+(\mathfrak{g}, \mathfrak{a}),$$

then (5) is an isomorphism (this follows from [3] Theorem IV.2 and Corollary IV.10).

Now put $\mathscr{V}(\mathfrak{a}, \Sigma^+, \lambda) = \eta^{-1} \circ \mathscr{P}_\lambda(\{T \in \mathscr{D}'(G^d/P^d, \mathbf{C}_{-\lambda})_{H^d} \mid \operatorname{supp} T \subset H^d P^d\})$, where $\operatorname{supp} T$ denotes the support of T. If $\lambda \in \mathfrak{a}^*$ satisfies (6) then $\mathscr{V}(\mathfrak{a}, \Sigma^+, \lambda)$ is actually isomorphic to the space of $T \in \mathscr{D}'(G^d/P^d; \mathbf{C}_{-\lambda})_{H^d}$, with $\operatorname{supp} T \subset H^d P^d$, as $\mathfrak{U}(\mathfrak{g})$-module.

For $\mu \in \mathfrak{c}^*$, the extremal weight of the finite-diemsional G-module $F(\mu)$, we let $\psi_{-(\lambda+\rho(\mathfrak{l}))}^{-(\lambda+\rho(\mathfrak{l})+\mu)}$ be the Jantzen-Zuckerman translation functor ([9] Definition 4.5.7).

LEMMA 2.1. *Let $\lambda \in \mathfrak{a}^*$ satisfy* (6) *and $\mu \in \mathfrak{a}^*$ be the highest weight of an $H_{\mathbf{C}}$-spherical representation of $G_{\mathbf{C}}$. Then*

$$(7) \qquad \psi_{-(\lambda+\rho(\mathfrak{l}))}^{-(\lambda+\rho(\mathfrak{l})+\mu)}\mathscr{V}(\mathfrak{a},\Sigma^+,\lambda+\mu) \cong \mathscr{V}(\mathfrak{a},\Sigma^+,\lambda).$$

PROOF. Put $V = \mathbf{C}_{\lambda+\mu}$. Let (π, E) be an irreducible finite dimensional G^d-module, and (π', E') the contragredient representation. The map $\mathscr{D}(G^d; V \otimes E') \to \mathscr{D}(G^d; V \otimes E') \cong \mathscr{D}(G^d; V) \otimes E'$, $f \mapsto \tilde{f}$, with $\tilde{f}(x) := (I \otimes \pi'(x))f(x)$, $x \in G^d$, is an isomorphism. Restriction gives an isomorphism $\mathscr{D}(G^d/P^d; V \otimes E') \to \mathscr{D}(G^d/P^d; V) \otimes E'$. Now (4) and dualizing gives an isomorphism Q: $\mathscr{D}'(G^d/P^d; V') \otimes E \to \mathscr{D}'(G^d/P^d; V' \otimes E)$, in such a way, that the support of $Q(T \otimes v)$ is equal to the support of T, if $v \in V$ is non-zero, and $T \in \mathscr{D}'(G^d/P^d; V')$.

$F \mapsto \mathscr{D}'(G^d/H^d; F)$ is an exact functor from the category of finite dimensional (continuous) P-modules into the category of G-module. The left exactness is obvious, and the right exactness follows from (4) using the left exactness of $\mathscr{D}(G^d/P^d; F)$. This means that there is a G-module filtration of $\mathscr{D}'(G^d/P^d; V' \otimes F(\mu))$ with subquotients of the form $\mathscr{D}'(G^d/P^d; V' \otimes F)$, where F is an irreducible subquotient of $F(\mu) \mid MAN$.

Now suppose F has lowest weight ν. Then $\mathfrak{Z}(\mathfrak{g})$ operates on $\mathscr{D}'(G^d/P^d; V' \otimes F)$ with infinitesimal character $-(\lambda + \rho(\mathfrak{l}) + \mu) + \nu$. If this subquotient occurs in $\psi_{-(\lambda+\rho(\mathfrak{l}))}^{-(\lambda+\rho(\mathfrak{l})+\mu)}\mathscr{D}'(G^d/P^d; V')$, then $-(\lambda + \rho(\mathfrak{l}) + \mu) + \nu \in -W(\mathfrak{g},\mathfrak{a})(\lambda + \rho(\mathfrak{l}))$. Now we can apply Lemma 4.8 of [11] (with $\mathfrak{q}$ of that paper equal to the opposite of our $\mathfrak{q}$). We get $\nu = \mu$ and that F must be the one-dimensional MAN-submodule of $F(\mu)$ of weight μ. This implies $\mathscr{D}'(G^d/P^d; \mathbf{C}_{-\lambda}) \cong \psi_{-(\lambda+\rho(\mathfrak{l}))}^{-(\lambda+\rho(\mathfrak{l})+\mu)}\mathscr{D}'(G^d/P^d; \mathbf{C}_{-(\lambda+\mu)})$, the isomorphism being given by the embedding $\mathbf{C}_{-\lambda} \to \mathbf{C}_{-(\lambda+\mu)} \otimes F(\mu)$ and the inverse of the isomorphism Q above.

Since $Q(T \otimes v)$ has same support as T, for non-zero $v \in F(\mu)$, we also get an isomorphism when we restrict to the $\mathfrak{U}(\mathfrak{g})$-submodules of distributions with support contained in H^dP^d. Now the above definition of $\mathscr{V}(\mathfrak{a},\Sigma^+,\lambda)$ implies the lemma.

Let L be the normalizer of $\mathfrak{q}$ in G, and $L^\sim$ the metaplectic (two-fold) cover of L ([10] Definition 5.7). Write ζ for the non-trivial element of the kernel of the covering map. A metaplectic representation of $L^\sim$ is one that is -1 on ζ ([11] Definition 5.7). Further let $(L \cap K)^\sim$ be the preimage of $L \cap K$ under this covering map. $2\rho(\mathfrak{u})$ is the differential of a one dimensional character of L and, by definition of $L^\sim$, $\rho(\mathfrak{u})$ is the differential of a one dimensional metaplectic representation of $L^\sim$.

The proof of the following lemma follows the lines of the proof of [7] Lemma 5.5. In case condition (2) is satisfied this lemma shows, with Theorem 1.1, that for the irreducibility or vanishing of $\mathscr{V}(\mathfrak{a},\Sigma^+,\lambda)$ it is enough to look at $\lambda \in \mathfrak{a}^*$, which are the differential of one-dimensional metaplectic representations of $L^\sim$.

LEMMA 2.2. *Suppose condition* (2) *is satisfied. Then* $\lambda \in \mathfrak{a}^*$ *is the differential of a one-dimensional metaplectic* $\tilde{L}$*-module if and only if* μ_λ *is the differential of a weight of a finite dimensional K-module.*

PROOF. Since $\mathfrak{a}$ is contained in the center of $\mathfrak{l}$ we get $\langle \lambda + \rho(\mathfrak{u}), \alpha \rangle = 0$, for all $\alpha \in \Delta(\mathfrak{l}, \mathfrak{c})$, using [9] Lemma 3.2.4a. By the definition of $\tilde{L}$, λ is the differential of a one dimensional metaplectic representation of $\tilde{L}$, if and only if $\lambda + \rho(\mathfrak{u})$ is the differential of a one-dimensional representation of L. Since $\lambda + \rho(\mathfrak{u})$ is orthogonal to all $\alpha \in \Delta(\mathfrak{l}, \mathfrak{c})$ this is equivalent to $\lambda + \rho(\mathfrak{u})$ being the differential of a one-dimensional representation of the Cartan subgroup C of L corresponding to $\mathfrak{c}$. Again this is equivalent to μ_λ being the differential of a one dimensional character of $C \cap K$. Since $C \cap K$ is a Cartan subgroup of K, this proves the lemma.

Recall the Zuckerman functors $\mathscr{R}^j_{\mathfrak{q}}$, which are covariant functors from the category of meaplectic $(\mathfrak{l}, (L \cap K)^{\sim})$-modules to the category of $(\mathfrak{g}, K)$-modules. ([10] Definition 6.20). Let $\Sigma^+(\mathfrak{l}, \mathfrak{a}) = \Sigma^+(\mathfrak{g}, \mathfrak{a}) \cap \Sigma(\mathfrak{l}, \mathfrak{a})$. The next proposition describes $\mathscr{V}(\mathfrak{a}, \Sigma^+, \lambda)^c$ as cohomologically induced module. The argument is the same as given by D. Vogan in [11] (at the end of section 4) for the discrete series of G/H, using the Langlands parameter of $\mathscr{V}(\mathfrak{a}, \Sigma^+, \lambda)$ and the Jantzen-Zuckerman translation functor.

PROPOSITION 2.3. *Suppose condition* (2) *is satisfied. Let* $\lambda \in \mathfrak{a}^*$ *be the differential of a one-dimensional metaplectic representation* $\mathbf{C}_\lambda$ *of* $\tilde{L}$, *and put* $S = \dim(\mathfrak{u} \cap \mathfrak{k})$. *If* λ *satisfies* (6), *then*

$$\mathscr{V}(\mathfrak{a}, \Sigma^+, \lambda) \cong \mathscr{R}^S_{\mathfrak{q}}(\mathbf{C}_\lambda)^c.$$

PROOF. In [6] the Langlands parameters of $\mathscr{V}(\mathfrak{a}, \Sigma^+, \lambda)^c$ are calculated for those λ satisfying condition (3). If condition (3) is satisfied, then by [6] Satz 4.4

$$\mathscr{V}(\mathfrak{a}, \Sigma^+, \lambda)^c \cong X_G(\mathfrak{q}, \mathscr{V}(\mathfrak{a}, \Sigma^+(\mathfrak{l}, \mathfrak{a}), \lambda - \rho(\mathfrak{u}))^c, \mu_\lambda).$$

Here $X_G(*)$ is the "holomorphic induction" from [8] (see section 4 of that paper).

Suppose condition (2) is satisfied. Part A.ii of Theorem 1.1 shows that the L-module $\mathscr{V}(\mathfrak{a}, \Sigma^+(\mathfrak{l}, \mathfrak{a}), \lambda - \rho(\mathfrak{u}))$ is one-dimensional with differential $-(\lambda - \rho(\mathfrak{u}))$, so we get $\mathscr{V}(\mathfrak{a}, \Sigma^+, \lambda)^c \cong X_G(\mathfrak{q}, \mathbf{C}_{\lambda - \rho(\mathfrak{u})}, \mu_\lambda)$.

But [8] Proposition 5.18 (see Theorem 4.23 for parameters) and [9] Theorem 8.2.4 (Independence of polarization) shows that $\mathscr{R}^S_{\mathfrak{q}}(\mathbf{C}_\lambda) \cong X_G(\mathfrak{q}, \mathbf{C}_{\lambda - \rho(\mathfrak{u})}, \mu_\lambda)$, for λ satisfying (3), giving the statement for those λ. Now [11] Proposition 4.7 and (7) give the statement for other λ.

We now want to look closer at condition (2), and give a characterization in the next proposition. But first we need some preparation.

LEMMA 2.4. *With notations as above the following conditions are equivalent:*

(a) $\mathfrak{s}^{\mathfrak{t}} = \mathfrak{a}$.

(b) $\mathfrak{g}^{\mathfrak{a}} = \mathfrak{g}^{\mathfrak{t}}$.

(c) $\alpha \mid \mathfrak{t} \neq 0$ *for all* $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a})$.

PROOF. (b) $\Leftrightarrow$ (c): This is trivial.

(b) $\Rightarrow$ (a): $\mathfrak{s}^{\mathfrak{t}} = \mathfrak{g}^{\mathfrak{t}} \cap \mathfrak{s} = \mathfrak{g}^{\mathfrak{a}} \cap \mathfrak{s} = \mathfrak{s}^{\mathfrak{a}} = \mathfrak{a}$.

(a) $\Rightarrow$ (c): Suppose there exists an $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a})$ with $\alpha \mid \mathfrak{t} = 0$. Let $X \in \mathfrak{g}_\alpha$ be non-zero. Then $X - \sigma X \in \mathfrak{s}^{\mathfrak{t}} = \mathfrak{a}$. Since $\sigma X \in \mathfrak{g}_{-\alpha}$ this is a contradiction.

PROPOSITION 2.5. *Suppose* $\mathfrak{g}_0$ *is simple. Then condition* (2) *is satisfied if and only if one of the following conditions* (a)–(d) *is satisfied.*

(a) $\mathfrak{g}_0$ *has a complex structure and* σ *is complex linear.*

(b) $\mathfrak{g}_0$ *is complex and* $\mathfrak{h}_0$ *is a quasi-split real form of* $\mathfrak{g}_0$.

(c) $\operatorname{rank} G/H = \operatorname{rank} K/K \cap H$.

(d) $\mathfrak{g}_0^d \cong \mathfrak{sl}(n, \mathbf{R}), \mathfrak{so}(n,n), \mathfrak{e}_{6(6)}, \mathfrak{su}^*(2n)$ *or* $\mathfrak{e}_{6(-26)}$, *and* $\theta \mid \mathfrak{a}$ *is an involutive outer automorphism of* $\Sigma(\mathfrak{g}, \mathfrak{a})$, *leaving invariant some positive system of restricted roots* $\Sigma^+(\mathfrak{g}, \mathfrak{a})$.

Suppose we are in case (d). *If* $\mathfrak{g}_0^d$ *is one of* $\mathfrak{sl}(n, \mathbf{R}), \mathfrak{so}(n,n)$ *or* $\mathfrak{e}_{6(6)}$, *then* $\mathfrak{a}_0$ *is a non-compact Cartan subalgebra of* $\mathfrak{g}_0$ *is* $\mathfrak{su}^*(2n)$, *then* $\mathfrak{l}$ *is isomorphic to* $\mathfrak{gl}(2, \mathbf{C})^{n-1}$.

PROOF. If $\mathfrak{g}_0$ is complex, then either σ is complex linear or $\mathfrak{h}_0$ is a real form of $\mathfrak{g}_0$. We can therefore split the proof into the following cases corresponding to the cases in the proposition.

(a) Suppose $\mathfrak{g}_0$ is complex and σ complex linear. Then $\mathfrak{k}_0$ is a compact real form of $\mathfrak{g}_0$, and $\mathfrak{a}_0 = (\mathfrak{a}_0 \cap \mathfrak{k}_0) \oplus i(\mathfrak{a}_0 \cap \mathfrak{k}_0)$. Since every $\alpha \in \Sigma(\mathfrak{g}_0, \mathfrak{a}_0)$ is C-linear, we get $\alpha \mid \mathfrak{t}_0 \neq 0$, for all $\alpha \in \Sigma(\mathfrak{g}_0, \mathfrak{a}_0)$. Thus condition (2) is always satisfied.

(b) Suppose $\mathfrak{g}_0$ is complex and $\mathfrak{h}_0$ a real form of $\mathfrak{g}_0$. Let $\mathfrak{a}_0 \subset \mathfrak{h}_0$ be a maximally split Cartan subalgebra of $\mathfrak{h}_0$. Then $i\mathfrak{a}_0$ is a fundamental Cartan subspace of $\mathfrak{s}_0 = i\mathfrak{h}_0$ and by Lemma 2.4a, condition (2) is satisfied if and only if the centralizer of $\mathfrak{k}_0 \cap i\mathfrak{a}_0 = i(\mathfrak{a}_0 \cap \mathfrak{p}_0)$ in $\mathfrak{s}_0$ is equal to $i\mathfrak{a}_0$, which is equivalent to $\mathfrak{h}_0$ being quasi-split.

(c) Suppose $\operatorname{rank} G/H = \operatorname{rank} K/K \cap H$. In this case $\mathfrak{a} \subset \mathfrak{k} \cap \mathfrak{s}$, so condition (2) is always satisfied.

(d) Suppose $\mathfrak{g}_0$ is simple without a complex structure and the equal rank condition (condition in part (c)) is not satisfied.

Suppose condition (2) is satisfied. Let $\mathfrak{a}_0 \subset \mathfrak{s}_0$ be a fundamental Cartan subspace and $\Sigma^+(\mathfrak{g}, \mathfrak{a})$ be a positive system of restricted roots invariant under θ. Since $\mathfrak{a}_0$ is non-compact, θ defines a non-trivial involutive automorphism of the Dynkin diagram corresponding to $\Sigma^+(\mathfrak{g}, \mathfrak{a})$. Such an automorphism must be an outer automorphism and only exists if the Dynkin diagram is of type A_l ($l \geqq 2$), D_l

($l \geqq 4$) or E_6 ([4] Theorem 3.29). Looking at Table VI in [4] one sees that this can only happen for $(\mathfrak{g}_0^d, \mathfrak{k}_0^d)$ of type AI, DI oe EI with $\mathfrak{a}_0$ a non-compact Cartan subalgebra of $\mathfrak{g}_0$, or of type AII or EIV. The sructure of $\mathfrak{g}_0^d$ can be read off from [4] Table V (in case DI, $\mathfrak{a}_0^d$ is a Cartan subalgebra of $\mathfrak{g}_0^d$, so we only get $\mathfrak{so}(n, n)$), giving also the last statement.

Now suppose $\mathfrak{g}_0^d$ is one of the Lie algebras in part (d) and θ satisfies the condition in part (d). For $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a})$ we get $\alpha \mid \mathfrak{t} = \frac{1}{2}(\alpha + \theta a) \neq 0$, since $\theta\alpha \neq -\alpha$. This gives condition (2).

An example of a symmetric space satisfying the conditions of part (d) in the proposition above is given by $SU^*(2n)/SO^*(2n)$. The corresponding dual symmetric triple in $(\mathfrak{sl}(2n, \mathbb{R}), \mathfrak{sp}(n, \mathbb{R}), \mathfrak{so}(2n, \mathbb{R}))$.

We can now give irreducibility statements for $\mathscr{V}(\mathfrak{a}, \Sigma^+, \lambda)$ that follow from the last proposition and known irreducibility results for $\mathscr{R}_{\mathfrak{q}}^S(\mathbb{C}_\lambda)$. Call (G, H) complex if G has a complex structure with σ holomorphic.

THEOREM 2.6. *Suppose $\mathfrak{g}_0$ is simple, (G, H) not complex and $\mathfrak{g}_0^d$ not isomorphic to $\mathfrak{e}_{6(-26)}$. If condition* (2) *is satisfied and $\lambda \in \mathfrak{a}^*$ satisfies*

$$\text{(8)} \qquad \operatorname{Re}(\lambda, \alpha) > 0, \textit{ for all } \alpha \in \Sigma^+(\mathfrak{g}, \mathfrak{a}),$$

then $\mathscr{V}(\mathfrak{a}, \Sigma^+, \lambda)$ is irreducible or zero.

PROOF. We look at the cases given in Proposition 2.4. (a) has been excluded, so we can start with

(b) Here $\mathfrak{l} = \mathfrak{g}^a$ is abelian, so we get a fundamental series representation of G which is irreducible by [11] Theorem 2.6(a).

(c) This is the discrete series case, which has been handled in [11] Theorem 2.10.

(d) By Proposition 2.5, either $\mathfrak{l}$ is abelian or isomorphic to $\mathfrak{gl}(2, \mathbb{C})^{n-1}$. In case $\mathfrak{l}$ is abelian the statement follows from [11] Theorem 2.6(a). Suppose $\mathfrak{l} \cong \mathfrak{gl}(2, \mathbb{C})^{n-1}$. Then for simple $\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{c})$ one gets $\operatorname{Re}\langle\rho(\mathfrak{l}), \alpha^\vee\rangle \in \{\pm 1\}$ ($\alpha^\vee$ the α-coroot). If (8) is satisfied, then $\langle\lambda + \rho(\mathfrak{l}), \alpha^\vee\rangle$ is not a negative integer. By [11] Theorem 2.6(b), $\mathscr{R}_{\mathfrak{q}}^S(\mathbb{C}_\lambda)$ is irreducible or zero.

For (G, H) complex we do not get such strong restrictions on the structure of $\mathfrak{l}$. For $\mathfrak{g}_0^d \cong \mathfrak{e}_{6(-26)}$ one can use the τ-invariant to prove irreducibility, with the exception of one λ. Possibly this case could be handled by looking closer at the coherent continuation of $\mathscr{V}(\mathfrak{a}, \Sigma^+, \lambda)$, starting with the Langlands parameters of $\mathscr{V}(\mathfrak{a}, \Sigma^+, \lambda)$, for λ satisfying (3).

REFERENCES

1. F. Bruhat, *Sur les représentations induits des groupes de Lie*, Bull. Soc. Math. France 84 (1956), 97–205.
2. M. Flensted-Jensen, *Discrete series for semisimple symmetric spaces*, Ann. of Math. 111 (1980), 253–311.
3. M. Flensted-Jensen, *Analysis on Non-riemannian Symmetric Spaces*, CBMS 61 AMS, 1986.
4. S. Helgason, *Differential Geometry, Lie groups and Symmetric Spaces*, Pure and Appl. Math., Vol. 80, Academic Press, New York-London, 1978.
5. T. Oshima, T. Matsuki, *Discrete Series for Symmetric Spaces*, Adv. Stud. Pure Math. 4 (1984), 331–390.
6. G. Ólafsson, *Die Langlands-Parameter für die Flensted-Jensensche fundamentale Reihe*, Math. Scand. 55 (1984).
7. H. Schlichtkrull, *On some series of representations related to symmetric spaces*, Mém. Soc. Math. France (N.S.) 15 (1984), 277–289.
8. B. Speh, D. Vogan, *Reducibility of generalized principal series representations*, Acta Math. 145 (1980), 227–299.
9. D. Vogan, *Representations of real reductive Lie groups*, Progr. in Math. 15 (1981), Birrkäuser Verlag.
10. D. Vogan, *Unitary Representations of Reductive Lie Groups*, Princeton University Press, 1987.
11. D. Vogan, *Irreducibility of discrete series of representations for semisimple symmetric spaces*, Adv. Stud. Pure Math. 14 (1988), 191–221.

MATHEMATISCHES INSTITUT
UNIVERSITÄT GÖTTINGEN
BUNSENSTRASSE 3–5
W-3400 GÖTTINGEN
GERMANY