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ON THE λ-DIMENSION OF CARTESIAN SQUARES

SUSANA SCRIVANTI

By a ring, we always mean a commutative ring with identity. Consider a commutative square of rings and ring homomorphisms

$$\begin{array}{ccc} A & \xrightarrow{i_1} & A_1 \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ A_2 & \xrightarrow{j_2} & A_0 \end{array} \tag{1}$$

Then (1) is called a cartesian square (or a pullback, or a fiber product) if given $a_1 \in A_1$, $a_2 \in A_2$ with $j_1(a_1) = j_2(a_2)$, there exists a unique element $a \in A$ such that $i_1(a) = a_1$ and $i_2(a) = a_2$. We shall use the equivalent definition that diagram (1) is a pullback if the restriction of i_2 to ker i_1 is an isomorphism onto ker j_2, and j_1 induces an injection of coker i_1 into coker j_2.

The following definitions and the proofs of the several assertions can be found in [GS]. Let M be an A-module, a finite n-presentation of M is an exact sequence

$$F_n \to F_{n-1} \to \ldots \to F_0 \to M \to 0$$

with F_i finitely and free A-modules.

If M is a finitely generated A-module, we denote by:

$$\lambda_A(M) = \sup\{n \mid \text{there is a finite } n\text{-presentation of } M\}$$

If M is not finitely generated we put $\lambda(M) = -1$.

Let

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

be an exact sequence of A-modules then:

(2) $\lambda(M_2) \geqq \inf\{\lambda(M_1), \lambda(M_3)\}$

(3) $\lambda(M_3) \geqq \inf\{\lambda(M_2), \lambda(M_1) + 1\}$.

(4) $\lambda(M_1) \geqq \inf\{\lambda(M_2), \lambda(M_3) - 1\}$.

The λ-dimension of A, denoted by $\lambda \dim A$ is the least integer such that

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$\lambda_A(M) \geqq n$ implies $\lambda_A(M) = \infty$ for all A-modules M. If no such n exists we set $\lambda \dim A = \infty$.

It follows that:

$\lambda \dim A \leqq 0$ iff A is noetherian.

$\lambda \dim A \leqq 1$ iff A is coherent.

This paper is motivated by the results in [O] and [G]. In [O], Ogoma studies necessary and sufficient conditions for the fiber product of noetherian rings to be noetherian, while Greenberg [G] studies coherence over a special case of a pull-back ring, in terms of coherence of the components.

We will prove the result (cf. Theorem 4 and its corollaries below) that generalizes their results, and which in a certain sense is best possible.

In [O, Theorem 2.1] Ogoma showed that if (1) is a cartesian square with A_1 and A_2 noetherian, then the fiber product A is noetherian if only if

(i) $C = j_1(A_1) \cap j_2(A_2)$ is noetherian and

(ii) a_1/a_1^2 and a_2/a_2^2 are finite C-modulus where $a_i = \ker j_i$ $(i = 1,2)$.

On the other hand, Greenberg showed in [G, Theorem 2.4] that if (1) is a cartesian square in which i_2 is a flat epimorphism, a_2 is a flat ideal of A (also of A_2) and $A_1 \simeq A/a_2$, the A is coherent in each of the following cases:

(i) A_1 and A_2 are coherent and $a_2 \in MAX(A_2)$.

(ii) A_1 is coherent and reduced and A_2 is hereditary.

(iii) A_1 is notherian and A_2 is coherent.

We give a sufficient condition for the fiber product of rings with λdim $\leqq n$ to be a ring with λdim $\leqq n$.

The following proposition is used several times throughout.

Proposition 1. *Suppose given a pullback diagram* (1) *with i_1 surjective, and an A-module M. The following then hold.*

(a) *M is projective iff $A_i \otimes_A M$ is a projective A_i-module* $(i = 1,2)$.

(b) *M is flat iff $A_i \otimes_A M$ is a flat A_i-module* $(i = 1,2)$.

(c) *M is finitely generated iff $A_i \otimes_A M$ is a finitely generated A_i-module* $(i = 1,2)$.

Proof. The assertions concerning projectivity and flatness are proved in [W] and [FV] respectively. We will prove statement c). The only if direction is clear.

Let $A_i \otimes_A M$ be generated as an A_i-module by $a_{i_k} \otimes m_{i_k}$, $m_{i_k} \in M$, $a_{i_k} \in A_i$, for $i = 1,2$ and $1 \leqq 1_k \leqq n_1$, $1 \leqq 2_k \leqq n_2$. We then have exact sequences.

$$A_i \otimes_A A^{n_i} \simeq A_i^{n_i} \to A_i \otimes_A M \to 0$$

$$1 \otimes (0, \ldots 1, \ldots, 0) \mapsto a_{i_k} \otimes m_{i_k}$$

We can obtain a map φ

$$A^{n_1+n_2} \xrightarrow{\varphi} M \to \operatorname{coker} \varphi \to 0$$

$$(0, \ldots, 1, \ldots, 0) \mapsto m_k$$

with $m_s = m_{1_s}$, when $1 \leqq s \leqq n_1$, and $m_s = m_{2_{s-n_1}}$ when $n_1 < s \leqq n_2$. Then $A_i \otimes_A \operatorname{coker} \varphi = 0$ $(i = 1, 2)$. Indeed

$$A_i \otimes_A A^{n_1+n_2} \xrightarrow{f_i} A_i \otimes_A M \to A_i \otimes_A \operatorname{coker} \varphi \to 0$$
$$1 \otimes (0, \ldots, 1, \ldots, 0) \mapsto 1 \otimes m_k$$

is exact and f_i is surjective.

Consider the exact sequences of A-modules

$$0 \to A \xrightarrow{\{i_1, i_2\}} A_1 \oplus A_2 \xrightarrow{j_1 - j_2} j_2(A_2) \to 0$$

By a) $\operatorname{coker} \varphi$ is a projective A-module. So tensoring the sequence with $\operatorname{coker} \varphi$, we obtain an exact sequence:

$$0 \to \operatorname{coker} \varphi \to (A_1 \otimes_A \operatorname{coker} \varphi) \oplus (A_2 \otimes_A \operatorname{coker} \varphi) \to j_2(A_2) \otimes_A \operatorname{coker} \varphi \to 0$$

Since $A_i \otimes_A \operatorname{coker} \varphi = 0$ we obtain $\operatorname{coker} \varphi = 0$. We conclude that the map $A^{n_1+n_2} \to M$ is surjective and M is a finitely generated A-module.

The case $n = 1$ of the following proposition generalizes proposition 2.1 of [G].

PROPOSITION 2. *Suppose given a pullback diagram* (1), *with i_1 surjective, M an A-module and $\lambda_{A_i}(\operatorname{Tor}_j^A(A_i, M)) \geqq n - j$ for $1 \leqq j \leqq n$ (empty condition if $n = 0$) and $i = 1, 2$. Then*

$$\lambda_A(M) \geqq n \textit{ iff } \lambda_{A_i}(A_i \otimes_A M) \geqq n \textit{ for } i = 1, 2$$

PROOF. The proof is by intduction on n.

For $n = 0$ the proposition follows from proposition 1.c).

Assume that $\lambda_{A_i}(\operatorname{Tor}_j^A(A_i, M)) \geqq n - j$, $1 \leqq j \leqq n$, $i = 1, 2$.

Let $\lambda_A(M) \geqq n$. We then have an exact sequence of A-modules

$$0 \to K \to A^k \xrightarrow{f} M \to 0 \tag{5}$$

and $\lambda_A(M) \geqq n$ iff $\lambda_A(K) \geqq n - 1$.

Since $\operatorname{Tor}_j^A(A_i, K) \simeq \operatorname{Tor}_{j+1}^A(A_i, M)$ for $j \geqq 1$, by the induction hypothesis $\lambda_{A_i}(A_i \otimes_A K) \geqq n - 1$ $(i = 1, 2)$. In addition, tensoring (5) with A_i over A, and put $K_i = \ker(1_{A_i} \otimes f)$, we obtain two exact sequences

$$0 \to K_i \to A_i^k \xrightarrow{1_{A_i} \otimes f} A_i \otimes_A M \to 0 \quad \& \quad 0 \to \operatorname{Tor}_1^A(A_i, M) \to A_i \otimes_A K \to K_i \to 0$$

By (3) $\lambda_{A_i}(K_i) \geqq n - 1$, hence $\lambda_{A_i}(A_i \otimes_A M) \geqq n (i = 1, 2)$ as desired.

If on the other hand $\lambda_{A_i}(A_i \otimes_A M) \geqq n$, then $\lambda_{A_i}(K_i) \geqq n - 1$ and by (2) $\lambda_{A_i}(A_i \otimes_A K) \geqq n - 1$. $(i = 1, 2)$ Hence we conclude that $\lambda_A(M) \geqq$ n iff $\lambda_{A_i}(A_i \otimes_A M) \geqq n (i = 1, 2)$.

THEOREM 3. *Let diagram* (1) *be a pullback in which i_1 is surjective and A_1, A_2 are flat A-modules. Then*

$$\lambda\dim A \leqq \max_{i=1,2}\{\lambda\dim A_i\}$$

PROOF. Since A_1, A_2 are flat A-modules, it is easy to see that for every A-module N we have that $\lambda_A(N) \geqq k$ iff $\lambda_{A_i}(N \otimes A_i) \geqq k$ for $k \geqq -1$.

Let $n = \max_{i=1,2}\{\lambda\dim A_i\}$ and M be an A-module with $\lambda_A(M) \geqq n$. Then we have that $\lambda_{A_i}(M \otimes A_i) \geqq n (i = 1,2)$. Because $\lambda\dim A_i \leqq n$ we get $\lambda_{A_i}(M \otimes A_i) \geqq n + 1$ and we can conclude that $\lambda_A(M) \geqq n + 1$ for $i = 1,2$.

THEOREM 4. *Suppose given a pullback diagram* (1) *with* i_1 *surjective. Suppose that for all* A*-modules* M *with* $\lambda_A(M) \geqq n$ *we have* $\lambda_{A_i}(\mathrm{Tor}_j^A(A_i, M)) \geqq n + 1 - j$ *for* $1 \leqq j \leqq n + 1$ *and* $i = 1,2$. *Then*

$$\lambda\dim A_i \leqq n (i = 1,2) \Rightarrow \lambda\dim A \leqq n.$$

PROOF. Let M be an A-module with $\lambda_A(M) \geqq n$ and $\lambda_{A_i}(\mathrm{Tor}_j^A(A_i, M)) \geqq n + 1 - j (i = 1,2)$. It follows from proposition 2 that $\lambda_{A_i}(A_i \otimes_A M) \geqq n (i = 1,2)$.

But $\lambda\dim A_i \leqq n (i = 1,2)$ and hence $\lambda_{A_i}(A_i \otimes_A M) \geqq n + 1 (i = 1,2)$. Using proposition 2 we have that $\lambda_{A_i}(A_i \otimes_A M) \geqq n + 1 (i = 1,2)$ iff $\lambda_A(M) \geqq n + 1$.

We have shown that $\lambda_A(M) \geqq n$ implies $\lambda_A(M) \geqq n + 1$. Hence $\lambda\dim A \leqq n$.

COROLLARY 5. *Let diagram* (1) *be a pullback with* i_1 *surjective and* A_1, A_2 *noetherian. Then* A *is noetherian iff* $\mathrm{Tor}_1^A(A_i, A/a)$ *is a finitely generated* A_i*-module for* $i = 1,2$, *and all ideals* a *of* A.

PROOF. The only if assertion follows from theorem 4. We will prove the converse.

Let a be an ideal of A. Since A is noetherian, the ideal is finitely generated and hence $A_i \otimes_A a$ is a finitely generated A_i-module for $(i = 1,2)$.

We tensor the exact sequence of A-modules

(6) $$0 \to a \to A \xrightarrow{\pi} A/a \to 0$$

with $A_i (i = 1,2)$ over A, put $H_i = \ker(1_{A_i} \otimes \pi)$, and we obtain two exact sequences

(7) $$0 \to H_i \to A_i \xrightarrow{1_{A_i} \otimes \pi} A_i \otimes_A A/a \to 0$$

and

(8) $$0 \to \mathrm{Tor}_1^A(A_i, A/a) \to A_i \otimes_A a \to H_i \to 0$$

Since $\lambda_{A_i}(A_i \otimes_A A/a) = \infty$, $\lambda_{A_i}(H_i) = \infty (i = 1,2)$. From $\lambda_{A_i}(A_i \otimes_A a) = \infty$ and $\lambda_{A_i}(H) = \infty$ it follows that $\mathrm{Tor}_i^A(A_i, A/a)$ is a finitely generated A_i-module $(i = 1,2)$.

From corollary 5 and proposition 2.1 of [O] we have the following:

REMARK. *Suppose given a pullback* (1) *with* A_1, A_2 *noetherian. Then* $\mathrm{Tor}_i^A(A_i, A/a)$ *is a finitely generated* A_i*-module for* $(i = 1,2)$ *iff* a_2/a_2^2 *is a finitely generated* $j_1(A_1)$*-module* (*where* $a_2 = \ker j_2$).

LEMMA 6. *Let* (1) *be a pullback with* i_1 *surjective. Suppose that* a_2/a_2^2 *is a finitely generated* $j_1(A_1)$*-module and that for all ideals* b *of* A_2, *there exists an ideal* a *of* A *such that* $A_2 \otimes_A a \simeq b$. *Then* A *is noetherian iff* A_1 *and* A_2 *are noetherian.*

PROOF. The only if assertion follows from corollary 5. In showing the converse, note that since i_1 is surjective, A_1 is noetherian.

Let b be an ideal of A_2. Since $A_2 \otimes_A a \simeq b$ and a is a ideal of A, a is a finitely generated. It follows that b is a finitely generated ideal of A_2.

COROLLARY 7. *Let the diagram* (1) *be a pullback with* i_1 *surjective and* A_1, A_2 *coherent. Then* A *is coherent iff* $\mathrm{Tor}_1^A(A_i, A/a)$ *is a finitely presented* A_i*-module and* $\mathrm{Tor}_2^A(A_i, A/a)$ *is a finitely generated* A_i*-module, for all finitely geneated ideals* a *of* A, $(i = 1,2)$.

PROOF. The only if assertion follows from theorem 4, we will prove the converse.

From the sequences (6), (7) and (8) we have $\lambda_{A_i}(\mathrm{Tor}_1^A(A_i, A/a)) \geqq 1$, i.e. $\mathrm{Tor}_1^A(A_i, A/a)$ is a finitely presented A_i-module $(i = 1,2)$.

Since a is a finitely presented ideal of A, we have an exact sequence

$$0 \to P \to A^r \xrightarrow{g} a \to 0 \tag{9}$$

where P is a finitely generated A-module. If we apply the functor $\mathrm{Tor}_*^A(A_i, -)$ to (9) and put $P_i = \ker(1_{A_i} \otimes g)$ $(i = 1,2)$ we obtain two exact sequences.

$$0 \to P_i \to A_i^r \xrightarrow{1_{A_i} \otimes g} A_i \otimes_A a \to 0 \quad \& \quad 0 \to \mathrm{Tor}_i^A(A_i, a) \to A_i \otimes_A P \to P_i \to 0.$$

It follows from (4) and $\lambda_{A_i}(A_i \otimes_A a) = \infty$, that $\lambda_{A_i}(P_i) = \infty$, and from $\lambda_{A_i}(A_i \otimes_A P) \geqq 0$, $\lambda_{A_i}(P_i) = \infty$ and (4) that $\lambda_{A_i}(\mathrm{Tor}_i^A(A_i, a)) \geqq 0$ $(i = 1,2)$.

LEMMA 8. *Let* (1) *be a pullback with* i_1 *surjective. Suppose that for all finitely generated ideals* a *of* A, $\mathrm{Tor}_i^A(A_i, A/a)$ *is a finitely presented* A_i*-module and that* $\mathrm{Tor}_2^A(A_i, A/a)$ *is a finitely geneated* A_i*-module* $(i = 1,2)$. *Supose, moreover, that for all finitely generated ideals* b_i *of* A_i, *there exist finitely generated ideals* a_i *of* A *such that* $A_i \otimes_A a_i \simeq b_i$ $(i = 1,2)$. *Then* A *is coherent iff* A_1 *and* A_2 *are coherent.*

PROOF. The only if assertion follows from corollary 7. We will prove the converse.

Let b_i be a finitely generated ideal of A_i $(i = 1,2)$. By hypothesis $b_i \simeq A_i \otimes_A a_i$, where a_i is a finitely generated ideal of A. Then a_i is finitely presented and we conclude that b_i is a finitely presented A_i-module $(i = 1,2)$.

REMARK. Corollary 7 is a generalization of proposition 2.4 of [G].

To see this, note that Greenberg has studied the case where A_2 and A/a_2 are coherent rings, A is a subring of A_2, which is a flat epimorphic image of A, and a_2 is a flat ideal of A. That is, we have a diagram

$$\begin{array}{ccc} A & \xrightarrow{i_1} & A/a_2 \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ A_2 & \xrightarrow{j_2} & A_2/a_2 \end{array}$$

Since A_2 is a flat A-module, we have that $\operatorname{Tor}_i^A(A_2, -) = p$ and since a_2 is a flat ideal of A, $\operatorname{Tor}_2^A(A/a_2, -) = 0$. Then by corollary 7, A is coherent iff $\operatorname{Tor}_1^A(A/a_2, A/a)$ is finitely presented A/a_2-module for all finitely generated ideals a of A.

The three cases studied by Greenberg in theorem 2.4 give that $A/a_2 \otimes_A a$ are finitely presented for all finitely generated ideals a of A. Equivalently, that $\operatorname{Tor}_1^A(A/a_2, A/a)$ is finitely presented, because if we consider the sequences (6), (7) and (8), we see that $\lambda_{A/a_2}(\operatorname{Tor}_1^A(A/a_2, A/a)) \geqq 1$ as $\lambda_{A/a_2}(A/a_2 \otimes_A a) \geqq 1$ and $\lambda_{A/a_2}(H) = \infty$.

COROLLARY 9. *Let* (1) *be a pullback with* i_1 *surjective*, A_1, A_2 *coherent and* $\mathrm{fd}_A(A_i) \leqq 1$. *Then*

$$\lambda\dim A \leqq 2$$

PROOF. Since $\operatorname{Tor}_j^A(A_i, -) = 0$ for $j \geqq 2$, and $\lambda_{A_i}(N) \geqq 1$ implies that $\lambda_{A_i}(N) = \infty$ for all A_i-modules N, it follows from theorem 4, that we only need to show that $\operatorname{Tor}_1^A(A_i, M)$ is a finitely presented A_i-module for all A-modules M with $\lambda_A(M) \geqq 2\,(i = 1,2)$.

Let M be an A-module with $\lambda_A(M \geqq 2$. Then we have an exact sequence of A-modules

$$(5) \qquad 0 \to K \to A^k \xrightarrow{f} M \to 0$$

with $\lambda_A(K) \geqq 1$. Since $\lambda_A(K) \geqq 1$ implies $\lambda_{A_i}(A_i \otimes_A K) \geqq 1$, we obtain $\lambda_{A_i}(A_i \otimes_A K) = \infty\,(i = 1,2)$.

If we apply the functor $\operatorname{Tor}^A(A_i, -)$ to (5), and put $K_i = \ker(1_{A_i} \otimes f)\,(i = 1,2)$, we obtain two exact sequences

$$0 \to K_i \to A_i^k \xrightarrow{1_{A_i} \otimes f} A_i \otimes_A M \to 0 \quad \& \quad 0 \to \operatorname{Tor}_1^A(A_i, M) \to A_i \otimes_A K \to K_i \to 0$$

Since A_i are coherent and $\lambda_{A_i}(A_i \otimes_A M) \geqq 1$ we see that $\lambda_{A_i}(K) = \infty$, and it follows from (4) that $\lambda_{A_i}(\operatorname{Tor}_1^A(A_i, M)) = \infty\,(i = 1,2)$.

Given a pullback (1) with i_1 and i_2 surjective, it is well-known that A is noetherian iff A_1 and A_2 are noetherian. We now show that a similar result holds if we replace noetherian by coherent.

PROPOSITION 10. *Let (1) be a pullback with i_1 and i_2 surjective, and suppose that* $\ker j_1$ *and* $\ker j_2$ *are finitely generated A-modules. Then A is coherent iff A_1 and A_2 are coherent.*

PROOF. Recall that A is coherent iff $\prod_I A$ is flat for all I. By proposition 1,b) A is coherent if $A_i \otimes_A \prod_I A$ is flat for all I $(i = 1, 2)$. Since A_i are finitely presented A-modules, $A_i \otimes_A \prod_I A \simeq \prod_I A_i$. Thus A coherent if $\prod_I A_i$ is A_i-flat for all I $(i = 1.2.$

We conclude with the following example where corollary 7 and corollary 9 can be applied.

EXAMPLE. Let T be a ring consisting of those power series with positive rational exponents increasing towards ∞ in an indeterminate x over a field k, denoted by $T = k[[\mathbb{Q}^+]]$ (for more details cf. [S]).

Thus T is a valuation ring with a non principal ideal $m = \sum_{i=1}^{\infty} x^{1/i}T$. Moreover, gldim $T = 2$, and T is coherent (since every f. g. ideal is principal thus free).

Consider the cartesian square, where T_1 and T_2 are two copies of T

$$\begin{array}{ccc} R & \xrightarrow{i_1} & T_1 \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ T_2 & \xrightarrow{j_2} & T/m \end{array}$$

where the maps onto T/m are the natural ones.

Then $R = \{(a, b) \in T \times T | a - b \in m\}$. Hence R is a local ring with zero divisors and maximal ideal $J = m \times m$.

In [OB, theorem 2.37] B. Osofsky showed that gldim $R = 3$ and wd $R = 2$

We now address the question of determining the λ-dimesion of R.

First we determine $\mathrm{fd}_R T_k$ $(k = 1, 2)$. But exactly as in [S, example 1 (ii)] one shows that $\mathrm{fd}_R T_k) \leqq 1$.

Using corollary 7 we can show that R is not a coherent ring.

To see this, let I be a finitely generated ideal of R. From [OB, theorem 2.37] we obtain that $I = (\mathrm{a}, \mathrm{b})R$ where $\mathrm{a} \neq 0$ and $\mathrm{b} \neq 0$ (which is a projective ideal of R), or $I = (a, 0)R \oplus (0, \mathrm{b})R$.

We may assume that $I = (\mathrm{a}, 0)R$. We then have an exact sequence

$$(10) \qquad 0 \to I \to R \to R/I \to 0.$$

Tensoring (10) with T_2, we get

$$0 \to \mathrm{Tor}_1^R(T_2, R/I) \to T_2 \otimes_R I \xrightarrow{g_2} T_2 \to T_2 \otimes_R R/I \to 0$$

But $g_2 = 0$, because $g_2(\mathrm{t} \otimes (\mathrm{a}, 0)) = \mathrm{t}(\mathrm{a}, 0) = 0$, and hence we have that

$\mathrm{Tor}_1^R(T_2, R/I) \simeq T_2 \otimes_R I$. Consider the exact sequence

$$0 \to (0, m) \to R \to I \to 0. \tag{11}$$

Tensoring (11) with T_2 we obtain

$$0 \to T_2 \otimes_R (0, m) \to T_2 \to T_2 \otimes_R I \to 0.$$

Since $T_2 \otimes_R (0, m) \simeq m$, and m is not a finitely generated T_2-module, it follows that $T_2 \otimes_R I$ is not a finitely presented T_2-module, and we conclude from corollary 7 that R is not a coherent ring.

We shall now show that $\lambda\dim R = 2$:

PROOF. Since $\mathrm{fd}_R(T_k) \leqq 1$ $(k = 1, 2)$ and T is coherent, by corllary 9 we obtain that $\lambda\dim R \leqq 2$. But since we know that R is not coherent, we conclude that $\lambda\dim R = 2$.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF STOCKHOLM
SWEDEN

CURRENT ADDRES
VÄSTERBY BACKE 16, 4TR
163 72 SPÅNGA
SWEDEN