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INDECOMPOSABLE MODULES OVER MULTICOIL ALGEBRAS

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§0. Introduction.

Let k be an algebraically closed field, and A be a finite dimensional k-algebra (associative, with an identity). We are interested in the description of the category mod A of finitely generated right A-modules, thus of the indecomposable objects in mod A. All the recent investigations about tame algebras [13] point out to the importance of those indecomposable modules which lie in tubes [21]. For instance, it is shown in [12] that, if A is tame, then all but finitely many non-isomorphic indecomposable A-modules which have the same dimension as k-vector spaces lie in homogeneous tubes.

In the study of the simply connected algebras of polynomial growth [22] appeared a natural generalisation of the notion of tube, called a coil ([4], observe that the use of the term "coil" in the present paper and in [4] deviates from its use in an earlier publication [3]). A multicoil consists, roughly speaking, of a finite number of coils glued together by some directed parts, and a multicoil algebra is an algebra having the property that each cycle of non-zero non-isomorphisms lies in one standard coil of a multicoil. In particular, multicoil algebras are cycle-finite [3] and hence tame. They generalise the coil algebras as defined in [3] and contain all the best understood examples of algebras of polynomial growth and finite global dimension. We shall prove here that, if a multicoil algebra has a sincere indecomposable module lying in a stable tube, then this algebra is either tame concealed or tubular [21].

THEOREM (A). *Let A be a finite dimensional, basic and connected algebra over an algebraically closed field. If A is a multicoil algebra, the following conditions are equivalent:*

(i) *A is either tame concealed or tubular,*

(ii) *There exists a sincere indecomposable A-module lying in a stable tube of the Auslander-Reiten quiver of A,*

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(iii) *There exist infinitely many non-isomorphic sincere modules of the same dimension lying in homogeneous tubes of the Auslander-Reiten quiver of A.*

If follows from this theorem and the results of [4] that indecomposable modules lying in a stable tube of a multicoil algebra have as their support a tame concealed or a tubular full convex subcategory of the algebra. The structure of such indecomposables is completely described in [21]. Also, we recall that tame concealed algebras and tubular algebras are of polynomial growth. In fact, we shall show:

THEOREM (B). *Let A be a finite dimensional, basic and connected multicoil algebra over an algebraically closed field. Then A is of polynomial growth.*

As a consequence, a multicoil algebra is domestic [20] if and only if it does not contain a tubular algebra as a full convex subcategory.

The paper is organised as follows. After a preliminary section (1), in which we recall those facts about tame algebras that will be needed in this paper, section (2) is devoted to a description of coils, multicoil algebras and components of their Auslander-Reiten quivers. In section (3), we study enlargements of cycle-finite algebras by successive one-point extensions and coextensions. Finally, we prove our main results in section (4).

1. Preliminaries on tame algebras.

1.1. *Notation.* Throughout this paper, k will denote a fixed algebraically closed field. By an algebra A is meant an associative finite dimensional k-algebra with an identity, which we shall moreover assume to be basic and connected. In this case, it is well-known that there exists a connected bound quiver (Q_A, I) and an isomorphism $A \xrightarrow{\sim} kQ_A/I$. Also, $A = kQ_A/I$ can equivalently be considered as a k-category, of which the object class A_0 is the set $(Q_A)_0$ of points of Q_A, and the set of morphisms $A(x, y)$ from x to y is the quotient of the k-vector space $kQ_A(x, y)$ having as basis the set of paths in Q_A from x to y by the subspace $I(x, y) = I \cap kQ_A(x, y)$, see [11].

By an A-module is meant throughout a finitely generated right A-module. We shall denote by mod A the category of A-modules, by $\mathrm{rad}^\infty(\mathrm{mod}\, A)$ the infinite power of the radical of mod A and by ind A a full subcategory of mod A consisting of a set of representatives of isomorphism classes of indecomposable objects in mod A. We shall use freely properties of the Auslander-Reiten translations $\tau = D\,\mathrm{Tr}$ and $\tau^{-1} = \mathrm{Tr}\, D$ and the Auslander-Reiten quiver Γ_A of A, for which we refer to [6, 21]. We shall agree to identify the points in Γ_A with the corresponding indecomposable A-modules. A component Γ of Γ_A is called *standard* if the full subcategory of mod A which consists of the modules which belong to Γ is equivalent to the mesh category $k(\Gamma)$ of Γ, see [11, 21]. A path $x_0 \to x_1 \to \ldots \to x_m$

in Γ_A is called *sectional* if $x_{i-1} \neq \tau x_{i+1}$ for $0 < i < m$, and a connected subquiver Σ of Γ_A is a *subsection* if each path in Σ is sectional. Finally, a subsection is *maximal* if it is not properly contained in another subsection [7]. Thus, if x is a point of a component of Γ_A without oriented cycles, then x determines a unique maximal subsection, by taking all the sectional paths with source x. It follows from [9] that the composition of morphisms lying on a sectional path is non-zero.

For each point i of Q_A, we denote by $S(i)$ the corresponding simple A-module and by $P(i)$ (respectively, $I(i)$) the projective cover (respectively, the injective envelope) of $S(i)$. The *dimension-vector* of a module M_A is the vector $\underline{\dim}\, M = (\dim_k \operatorname{Hom}_A(P(i), M))_{i\in(Q_A)_0}$ in the Grothendieck group $K_0(A)$ of A. The *support* $\operatorname{Supp}(\underline{d})$ of a vector $\underline{d} = (d_i)_{i\in(Q_A)_0}$ in $K_0(A)$ is the full subcategory of A with object class $\{i \in (Q_A)_0 \mid d_i \neq 0\}$. The support $\operatorname{Supp}(M)$ of a module M is the support of its dimension-vector $\underline{d} = \underline{\dim}\, M$. A module M, or a vector $\underline{d}$ in $K_0(A)$, is called *sincere* if its support is equal to A. The support $\operatorname{Supp}(\Gamma)$ of a component Γ of Γ_A is defined to be the full subcategory of A with object class $\{i \in (Q_A)_0 \mid \underline{\dim}\, M)_i \neq 0$ for some $M \in \Gamma_0\}$. Let C be a full subcategory of A, the restriction to C of an A-module M will be denoted by $M|_C$. The subcategory C is called *convex* (in A) if any path in A with source and target in C lies entirely in C. It is called *triangular* if Q_C contains no oriented cycle.

A *path* in mod A is a sequence of non-zero non-isomorphisms $M_0 \to M_1 \to \ldots \to M_t$, where the M_i are indecomposable. Such a path is called a *cycle* if $M_0 \simeq M_t$. A module M is called *directing* if it lies on no cycle in mod A.

1.2. *Tame algebras.* Following [13], we say that an algebra A is *tame* if, for any $\underline{d} \in K_0(A)$, there exists a finite family of functors F_i: $\operatorname{mod} k[X] \to \operatorname{mod} A$, $1 \leqq i \leqq n(\underline{d})$, where $k[X]$ is the polynomial algebra in one variable, satisfying the following conditions:

(i) For each i, $F_i = -\otimes_{k[X]} M_i$ is a $k[X]$-A-bimodule, finitely generated and free as a $k[X]$-module.

(ii) All but at most finitely many isomorphism classes of indecomposable A-modules of dimension-vector $\underline{d}$ are of the form $F_i(S)$, for some $1 \leqq i \leqq n(\underline{d})$, and some simple $k[X]$-module S.

For a tame algebra A, and a vector $\underline{d} \in K_0(A)$, we denote by $\mu_A(\underline{d})$ the least number of functors F_i satisfying (i) and (ii). The algebra A is said to be of *polynomial growth* [22] if there exists $m \in \mathbb{N}$ such that, for every vector $\underline{d} \in K_0(A)$ with non-negative coordinates, we have

$$\mu_A(\underline{d}) \leqq \left(\sum_{i\in(Q_A)_0} d_i \right)^m$$

It is said to be *domestic* [20] if there exist finitely many functors F_i: $\operatorname{mod} k[X] \to \operatorname{mod} A$, $1 \leqq i \leqq n$, satisfying (i) above and:

(ii′) For every $\underline{d} \in K_0(A)$, all but at most finitely many isomorphism classes of indecomposable modules of dimension-vector $\underline{d}$ are of the form $F_i(M)$, for some $1 \leqq i \leqq n$, and some indecomposable $k[X]$-module M.

Domestic algebras are of polynomial growth [23] (2.1).

A translation quiver Γ is called a *tube* [21] if it contains a cyclical path and its topological realiation $|\Gamma| = S^1 \times \mathsf{R}^+$ (when S^1 is the unit circle, and R^+ the set of non-negative real numbers). Tubes containing no projective or injective modules are called *stable*, and tubes consisting of τ-invariant modules are called *homogeneous*. In this paper, all tubes are assumed coherent with length functions [14]. It is shown in [12] that, for a tame algebra A and $\underline{d} \in K_0(A)$ such that $\mu_A(\underline{d}) > 0$, all but finitely many isomorphism classes of indecomposable A-modules of dimension-vector $\underline{d}$ lie in homogeneous tubes. We shall say that *A has sincere tubes* if there exists a sincere vector $\underline{d} \in K_0(A)$ such that $\mu_A(\underline{d}) > 0$.

1.3. *Tilted algebras and tubular algebras.* Let Δ be a finite connected quiver without oriented cycles. An algebra C is called a *tilted algebra* of type Δ if there exists a tilting module T over the path algebra $k\Delta$ such that $C = \operatorname{End}(T_{k\Delta})$, see [15]. If Δ is an euclidean (respectively, wild) quiver, then C is said to be of *euclidean* (respectively, *wild*) *type*. If Δ is a euclidean quiver, and T is a preprojective (or preinjective) $k\Delta$-module, then $C = \operatorname{End}(T_{k\Delta})$ is called a *tame concealed* algebra [16, 21]. It follows from [21] (4.2) that tilted algebras are characterised by complete slices in the module category.

We shall need the notions of truncated branch extensions and coextensions of a tame concealed algebra (tubular extensions and coextensions in the terminology of [21] (4.7)). In particular, a truncated branch extension (respectively, coextension) A of a tame concealed algebra is tame if and only if its tubular type is domestic or tubular [21] (4.9) (5.2), [5] (2.3), [18] (2.1). In the first case A is a tilted algebra of euclidean type having a complete slice in its preinjective (respectively, preprojective) component, and, conversely, every representation-infinite tilted algebra of euclidean type is of one of these two forms [21] (4.9). In the second case, A is a tubular algebra (respectively, cotubular). By [21] (5.2), any tubular algebra is cotubular and conversely.

To describe the module category over a truncated branch extension or coextension, we shall use the following notation inspired from [21]. Let B be a truncated branch extension of a tame concealed algebra C, then

$$\operatorname{ind} B = \mathscr{P}_0^B \vee \mathscr{T}_0^B \vee \mathscr{Q}_0^B$$

where $\mathscr{P}_0^B$ denotes the preprojective component of Γ_C, $\mathscr{T}_0^B$ is a $\mathsf{P}_1(k)$-family of tubes obtained from the corresponding tubes in Γ_C by successive ray insertions, and $\mathcal{Q}_0^B$ denotes the remaining components of Γ_B. The ordering from the left to the right indicates that there are non-zero morphisms only from any of these classes to itself and to the classes on its right. All projective indecomposable B-modules belong to $\mathcal{P}_0^B \vee \mathcal{T}_0^B$. Similarly, if B is a truncated branch coextension of a tame concealed algebra C, then

$$\operatorname{ind} B = \mathcal{P}_\infty^B \vee \mathcal{T}_\infty^B \vee \mathcal{Q}_\infty^B$$

when $\mathcal{Q}_\infty^B$ denotes the preinjective component of Γ_C, $\mathcal{T}_0^B$ is a $\mathsf{P}_1(k)$-family of tubes obtained from the corresponding tubes in Γ_C by successive coray insertions, and $\mathcal{P}_0^B$ denotes the remaining components of Γ_B. All injective indecomposable B-modules belong to $\mathcal{T}_\infty^B \vee \mathcal{Q}_\infty^B$.

If B is a domestic truncated branch extension (respectively, coextension) of C, then $\mathcal{Q}_0^B$ (respectively, $\mathcal{P}_\infty^B$) is the preinjective (respectively, preprojective), component of Γ_B and contains a complete slice [21] (4.9).

If B is a tubular truncated branch extension of C then B is also a tubular truncated branch coextension of a tame concealed algebra C' and

$$\mathcal{Q}_0^B = \left(\bigvee_{q \in \mathbb{Q}^+} \mathcal{T}_q^B\right) \vee \mathcal{T}_\infty^B \vee \mathcal{Q}_\infty^B$$

$$\mathcal{P}_\infty^B = \mathcal{P}_0^B \vee \mathcal{T}_0^B \vee \left(\bigvee_{q \in \mathbb{Q}^+} \mathcal{T}_q^B\right)$$

where each $\mathcal{T}_q^B$ is a $\mathsf{P}_1(k)$-family of stable tubes [21] (5.2).

1.4. LEMMA. *Let B be a truncated branch extension (respectively coextension) of a tame concealed algebra C. If M is an indecomposable B-module lying on the mouth of a tube $\mathcal{T}$ in Γ_B, then $B[M]$ (respectively, $[M]B$) is a truncated branch extension (respectively, coextension) of C.*

PROOF. Let Σ denote the maximal sectional path in $\mathcal{T}$ of target M. We shall consider two cases:

(1) Σ does not contain projective B-modules. Then $\operatorname{Hom}_B(P, M) = 0$ for any indecomposable projective B-module P which is not a C-module. Therefore, M is a C-module. Since M lies on the mouth of $\mathcal{T}$, then the Auslander-Reiten sequence in mod B starting with M has an indecomposable middle term, and so M is not a radical summand of a projective B-module. Also, M_C is a simple regular C-module. Therefore $B[M]$ is a truncated branch extension of C.

(2) Σ contains a projective B-module. Let $P(c)$ be the indecomposable projective B-module lying on Σ such that there exists a sectional subpath $P(c) \to \ldots \to M$ of Σ, and no other projective B-module on this subpath. So, if a denotes the extension point of B in $B[M]$, we have an arrow $a \to c$ in $B[M]$. We must show that, for any arrow $c \to b$ such that $P(b) \notin \Sigma_0$, the composition

$a \to c \to b$ is zero, or, equivalently, that $\mathrm{Hom}_B(P(b), M) = 0$. If $M = P(c)$, there is nothing to show. Otherwise, it follows from the hypothesis that $P(b)$ is not injective. Since M lies on the mouth, there exists a non- projective B-module N_B on the segment of Σ from $P(c)$ to M such that the Auslander-Reiten sequence ending with N has an indecomposable middle term. This implies that $\mathrm{Hom}_B(P(b), N) = 0$ and consequently $\mathrm{Hom}_B(P(b), M) = 0$. Therefore $B[M]$ is a truncated branch extension of C.

2. Coils, multicoils and multicoil algebras.

2.1. We shall give an inductive definition of a coil. Let (Γ, τ) be a connected translation quiver without multiple arrows, and assume there exists $x \in \Gamma_0$ and an infinite sectional path

$$\Sigma: x = x_0 \to x_1 \to x_2 \to \dots$$

such that the support of $k(\Gamma)(x, -)$ is of one of the following types:

(i) $\mathrm{Supp}\, k(\Gamma)(x, -) = \Sigma$

(ii) $\mathrm{Supp}\, k(\Gamma)(x, -)$ is of the form

$$y_t \leftarrow \dots \leftarrow y_2 \leftarrow y_1 \leftarrow x = x_0 \to x_1 \to x_2 \to \dots$$

(in particular, x_0 is injective).

(iii) $\mathrm{Supp}\, k(\Gamma)(x, -)$ is of the form

$$\begin{array}{ccccccccc} y_1 & & \to y_2 & \to \dots \to & y_t & & & & \\ \uparrow & & \uparrow & & \uparrow & & & & \\ x = x_0 & \to & x_1 & \to \dots \to & x_{t-1} & \to & x_t & \to & \dots \end{array}$$

(in particular, x_{t-1} is injective).

In each case, we shall define an operation enlarging (Γ, τ) to a new translation quiver (Γ', τ'). Each of these operations and their duals will be called *admissible*. The point x will be called the *pivot* of the operation.

(i) Let $t \in \mathbb{N}$ be arbitrary, and Γ_t denote the following translation quiver, isomorphic to the Auslander-Reiten quiver of the full $t \times t$ lower triangular matrix ring

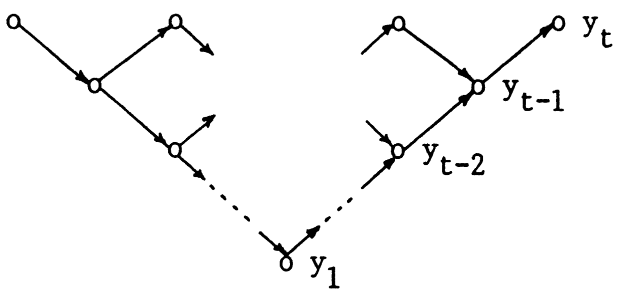
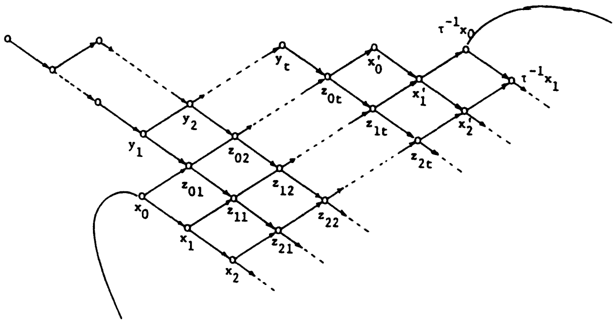


Fig. 1.

We let Γ' be the translation quiver having as points those of Γ, those of Γ_t, additional points z_{ij} and x'_j (where $i \geqq 0, j \geqq 1$), and having arrows as in the figure shown below

Fig. 2



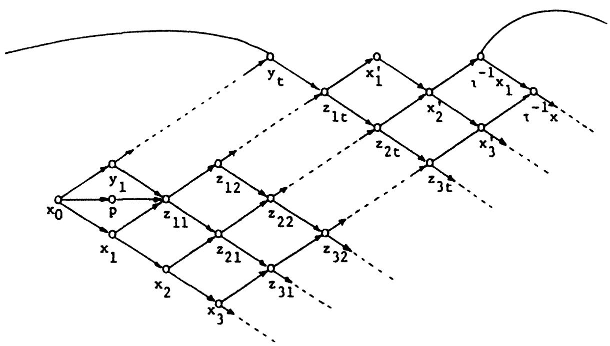
The translation τ' is defined as follows: $\tau' z_{ij} = z_{i-1,j-1}$ if $i \geqq 2, j \geqq 2$, $\tau' z_{i1} = x_{i-1}$ if $i \geqq 1$, $\tau'(\tau^{-1}x_i) = x'_i$ provided x_i is not injective in Γ, otherwise x'_i is injective in Γ'. For the remaining points of Γ (respectively, Γ_t) τ' coincides with the translation τ of Γ (respectively, Γ_t).

(ii) Γ' is the translation quiver having as points those of Γ and additional points denoted by p, z_{ij} and x'_i (where $i \geqq 1, j \geqq 1$), and having arrows as in the figure shown below (see fig. 3).

The translation τ' is defined as follows: p is projective-injective, $\tau' z_{ij} = z_{i-1,j-1}$ if $i \geqq 2$, $j \geqq 2$, $\tau' z_{i1} = x_{i-1}$ if $i \geqq 1$, $\tau' z_{1j} = y_{j-1}$ if $j \geqq 2$, $\tau' x'_j = z_{i-1,t}$ if $i \geqq 2$, $\tau' x'_1 = y_t$, $\tau'(\tau^{-1}x_i) = x'_i$ provided x_i is not injective in Γ, otherwise x'_i is injective in Γ'. For the remaining points of Γ', τ' coincides with the translation τ of Γ.

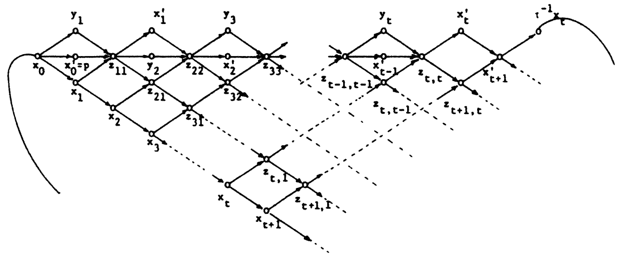
(iii) Γ' is the translation quiver having as points those of Γ and additional points denoted by $x'_0 = p$, z_{ij} and x'_j (where $i \geqq 1, 1 \leqq j \leqq i$), and having arrows as in the figure shown below.

Fig. 3.



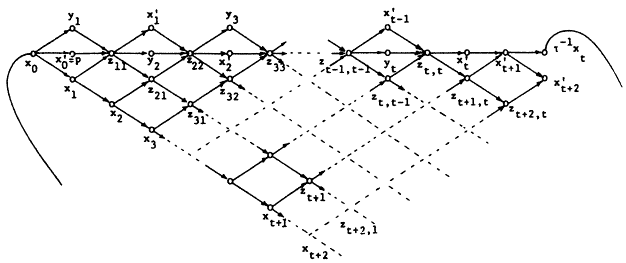
If t is odd

Fig.4



If t is even

Fig. 5.

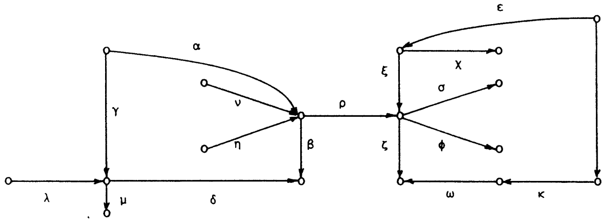


The translation τ' is defined as follows: p is projective, $\tau' z_{ij} = z_{i-1,j-1}$ if $i \geqq 2$, $2 \leqq j \leqq i$, $\tau' z_{i1} = x_{i-1}$ if $i \geqq 1$, $\tau' x'_i = y_i$ if $1 \leqq i \leqq t$, $\tau' x'_i = z_{i-1,t}$ if $i > t$, $\tau'(\tau^{-1} x_i) = x'_i$ if $i \geqq t$ provided x_i is not injective in Γ, otherwise x'_i is injective in Γ'. Observe that x'_{t-1} is always injective. For the remaining points of Γ', τ' coincides with the translation τ of Γ.

DEFINITION. A translation quiver is called a *coil* if there exists a sequence of translation quivers $\Gamma_0, \Gamma_1, \ldots, \Gamma_m = \Gamma$ such that Γ_0 is a stable tube and, for each $O \leqq 1 < m$, Γ_{i+1} is obtained from Γ_i by an admissible operation.

Observe that this use of the term coil deviates from its use in [3]. The present notion of coil is clearly a natural generalisation of the notion of tube. Coils occur frequently as Auslander-Reiten components of algebras of polynomial growth. The following is an example of a simply connected algebra of polynomial growth having a coil as Auslander-Reiten component. Let A be given by the quiver

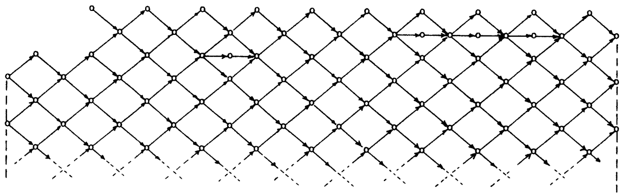
Fig. 6.



bound by $\alpha\beta = \gamma\delta, \lambda\mu = 0, \nu\beta = 0, \eta\beta = 0, \alpha\rho = 0, \rho\zeta = 0, \xi\sigma = 0, \xi\phi = 0\, \varepsilon\chi = 0$, and $\varepsilon\xi\zeta = \psi\kappa\omega$. Indeed, the unique component Γ_A containing all the indecomposables projectives which are not preprojective is the following coil (where we identify along the vertical dotted lines, see fig. 7)

Remark. It is not hard to prove, using vector space categories methods [21], that if Γ is a standard component of the Auslander-Reiten quiver of the algebra A, then to each admissible operation transforming Γ into Γ' corresponds an oper-

Fig. 7.



ation on A which yields a new algebra A' such that Γ' is a standard component of the Auslander-Reiten quiver of A', see [4].

2.2. Definition. Let A be an algebra, a component Γ of Γ_A is said to be a *multicoil* if Γ contains a full subquiver Γ' such that:

(i) Γ' is a finite disjoint union of standard coils.

(ii) every point $\Gamma \setminus \Gamma'$ is directing in mod A.

Accordingly, we can define

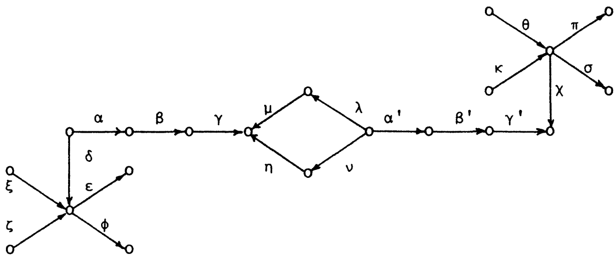
Definition. An algebra A is said to be a *multicoil algebra* if, for any cycle

$$M = M_0 \to M_1 \to \ldots \to M_t = M$$

in mod A, the indecomposable modules M_i all lie in one multicoil of Γ_A.

This clearly implies that the M_i actually lie in one (standard) coil of the multicoil to which they belong. The following is an example of a multicoil algebra. Let A be given by the quiver:

Fig. 8.



bound by $\alpha\beta\gamma = 0$, $\alpha'\beta'\gamma' = 0$, $\lambda\mu = \nu\eta$, $\delta\varepsilon = 0$, $\delta\phi = 0$, $\theta\chi = 0$ and $\kappa\chi = 0$.

A notion of coil algebra, using the earlier definition of coil, was introduced in [3]. It is easily seen that a coil algebra in the sense of [3] is in particular a multicoil algebra in our present sense. Thus, all examples of coil algebras given in [3] are also examples of multicoil algebras. In particular, iterated tilted algebras of euclidean type, tame tilted algebras of wild type, algebras tilting-cotilting equivalent to tubular algebras are multicoil algebras. A representation-finite algebra is a multicoil algebra if and only if it has a directed module category.

2.3. We shall need the following results from [4].

(a) Let A be a multicoil algebra, and B be a full convex subcategory of A. Then B is a multicoil algebra.

(b) Let A be a multicoil algebra, and $\mathscr{T}$ be a stable tube of Γ_A. Then the support algebra of $\mathscr{T}$ is convex in A.

(c) Let A be a multicoil algebra. Then A is triangular.

(d) Let A be a representation-infinite multicoil algebra. Then A contains a tame concealed full convex subcategory.

2.4. In many situations, we shall construct cycles at least one of whose morphisms lies in the infinite power of the radical of the module category.

DEFINITION [3]. An algebra A is said to be *cycle-finite* if, for any cycle

$$M = M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \to \dots \xrightarrow{f_t} M_t = M$$

in mod A, we have $f_i \notin \operatorname{rad}^\infty(\operatorname{mod} A)$ for all $1 \leqq i \leqq t$.

All multicoil algebras are cycle-finite, whereas there are cycle-finite algebras which are not multicoil algebras: indeed, any representation-finite algebra is cycle-finite. By [3] (1.4), any cycle-finite algebra (and in particular any multicoil algebra) is tame.

2.5. Clearly, any quotient of a cycle-finite algebra is also cycle-finite. We shall now show that any full subcategory of a cycle-finite algebra is cycle-finite.

LEMMA. *Let A be a cycle-finite algebra, and B be a full subcategory of A. Then B is cycle-finite.*

PROOF. Assume that B is not cycle-finite and let

$$M = M_0 \xrightarrow{f_1} M_1 \to \ldots \to M_{t-1} \xrightarrow{f_t} M_t = M$$

be a cycle in mod B such that $f_1 \in \mathrm{rad}^\infty(\mathrm{mod}\, B)$. Denote by $E_\bullet$: mod $A \to$ mod B the restriction functor associated to the full embedding $E\colon B \to A$, and by E_λ: mod $B \to$ mod A a left adjoint to $E_\bullet$ such that $E_\bullet E_\lambda \xrightarrow{\sim} 1_{\mathrm{mod}\, B}$. Then, for each i, $E_\lambda(M_i)$ is an indecomposable A-module and

$$E_\lambda(M) = E_\lambda(M_0) \xrightarrow{E_\lambda(f_1)} E_\lambda(M_1) \to \ldots \to E_\lambda(M_{t-1}) \xrightarrow{E_\lambda(f_t)} E_\lambda(M_t) = E_\lambda(M)$$

is a cycle in mod A. We claim that $E_\lambda(f_1) \in \mathrm{rad}^\infty(\mathrm{mod}\, A)$. Indeed, since $f_1 \in \mathrm{rad}^\infty(\mathrm{mod}\, B)$, then, for each $t > 0$, f_1 can be written as a linear combination of compositions of t non-isomorphisms in mod B. Since E_λ is a k-linear functor, $E_\lambda(f_1)$ can be written as a linear combination of compositions of t non-isomorphisms in mod A. This shows our claim, which contradicts the cycle-finiteness of A.

2.6 LEMMA. *Let A be a cycle-finite algebra, and X be an indecomposable sincere A-module lying in a stable tube of Γ_A. Then*

(i) $\overline{\mathrm{Hom}}_A(M, X) = \mathrm{Hom}_A(M, X)$ *for any A-module M,*

(ii) $\underline{\mathrm{Hom}}_A(X, M) = \mathrm{Hom}_A(X, M)$ *for any A-module M.*

PROOF of (i). Suppose that $\bar{f} = 0$ for some non-zero morphism $f\colon M \to X$. Then f factors through an injective A-module, and there exists an indecomposable injective A-module I such that $\mathrm{Hom}_A(I, X) \neq 0$. Thus we have a cycle of non-zero non-isomorphisms $I \to X \to I$ in mod A. Then, by our assumptions, I and X must belong to the same component of Γ_A. This however contradicts the fact that X belongs to a stable tube.

2.7. LEMMA *Let A be a cycle-finite algebra and X be an indecomposable module lying in a stable tube of Γ_A such that, for all $t \geqq 0$, $\tau^{-t}X$ is sincere. Then*

(i) *If P is an indecomposable projective A-module then, for any $t \geqq 0$,* $\mathrm{Hom}_A(\tau^{-t}P, \tau^{-t}X) \neq 0$

(ii) *If I is an indecomposable injective A-module then, for any $s \geqq 0$,* $\mathrm{Hom}_A(\tau^s X, \tau^s I) \neq 0$.

PROOF OF (i). This is done by induction on t. By (2.6) and the sincerity of X, we have

$$\overline{\mathrm{Hom}}_A(P, X) = \mathrm{Hom}_A(P, X) \neq 0$$

which shows the statement for $t = 0$. Let $t = 1$. It follows from the argument used in the proof of (2.6) and the fact that A is cycle-finite and has a sincere indecomposable lying in a stable tube of Γ_A, that A has no projective-injectives. Hence P is not injective and $\tau^{-1}P \neq 0$. By the Auslander-Reiten formula, we have

$$\underline{\mathrm{Hom}}_A(\tau^{-1}P, \tau^{-1}X) \simeq \overline{\mathrm{Hom}}_A(P, X) \neq 0$$

which implies that $\mathrm{Hom}_A(\tau^{-1}P, \tau^{-1}X) \neq 0$. Assume now that the statement holds for t. In particular, $\tau^{-t}P \neq 0$. We claim that $\tau^{-t}P$ is not injective. Indeed, by (2.6) and the sincerity of $\tau^{-t}X$, we have

$$\overline{\mathrm{Hom}}_A(\tau^{-t}P, \tau^{-t}X) \simeq \mathrm{Hom}_A(\tau^{-t}P, \tau^{-t}X) \neq 0$$

thus $\tau^{-t-1}P \neq 0$. Therefore, by the induction hypothesis and the Auslander-Reiten formula, we obtain

$$\underline{\mathrm{Hom}}_A(\tau^{-t-1}P, \tau^{-t-1}X) \simeq \overline{\mathrm{Hom}}_A(\tau^{-t}P, \tau^{-t}X) \neq 0$$

which implies that $\mathrm{Hom}_A(\tau^{-t-1}P, \tau^{-t-1}X) \neq 0$.

REMARK. If a stable tube $\mathscr{T}$ of Γ_A contains an indecomposable sincere module Y, then there exists $X \in \mathscr{T}$ such that, for all $t \geqq 0$, $\tau^{-t}X$ is sincere [21] (3.1).

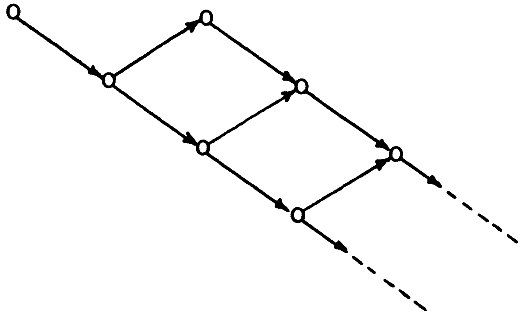
2.8. LEMMA. *Let Γ be a multicoil in an algebra A such that there is no path of irreducible morphisms in Γ of the form*

$$I = M_0 \to M_1 \to \ldots \to M_t = P$$

with I injective and P projective. Then Γ is a standard tube which may contain projectives or injectives but not both.

PROOF. Let Γ be a full subquiver of Γ which is a finite disjoint union of standard coils such that $\Gamma \backslash \Gamma$ consists of directing modules. Let also Γ'' e a standard coil in Γ'. Then Γ'' is obtained from a stable tube by a sequence of admissible operations. Assume that in this sequence we have used either operation (ii) or operation (iii) or their duals. Then there results a path of irreducible morphisms in Γ'' from an injective indecomposable to a projective indecomposable. Moreover, if $X = I$ is the new injective (respectively, $X = P$ is the new projective) created in this process, then the support of $\mathrm{Hom}(X, -)$ in its component contains a full subcategory of the form shown in fig. 9.
Thus X cannot be the pivot of the next admissible operation and so Γ'' contains a path of irreducible morphisms from an injective indecomposable to a projective indecomposable. This shows that Γ'' is obtained from a stable tube by a sequence of applications of (i) and its dual. However, if in this sequence, we have used both (i) and its dual, we again obtain a path of irreducible morphisms from an indecomposable injective to an indecomposable projective. Thus Γ'' is obtained from a stable tube by a sequence of applications of either (i) or its dual. That is, Γ''

Fig. 9.



is a tube which is either stable, or contains injectives or projectives, but not both. Assume that Γ'' is a stable tube. Then $\Gamma = \Gamma''$ because, if this is not the case, then there exists a projective $P \in (\Gamma \backslash \Gamma'')_0$ (respectively, an injective $I \in (\Gamma \backslash \Gamma'')_0$) which has a summand of its radical (respectively, of its socle factor) in Γ'' and therefore is not directing, a contradiction. Finally, assume that Γ'' is a tube containing only projectives (the other case being dual). It follows from the hypothesis that no injective $I \in \Gamma_0$ has a summand of its socle factor in Γ''. But then, if $\Gamma \neq \Gamma''$, then $\Gamma \backslash \Gamma'$ must contain a projective with radical summand in Γ'' which is not directing. This also yields a contradiction, because such a projective should then belong to $\Gamma'' \subset \Gamma'$. Consequently, $\Gamma = \Gamma''$.

2.9. Corollary. *Let A be a multicoil algebra having a sincere indecomposable lying in a stable tube, then all multicoils in Γ_A are standard tubes which may contain projectives or injectives but not both.*

Proof. Let X_A be a sincere indecomposable A-module lying in a stable tube and suppose Γ is a multicoil of Γ_A. If Γ does not satisfy our claim, then by (2.8), there exists in Γ a path of irreducible morphisms

$$I = M_0 \to M_1 \to \dots \to M_t = P$$

with I injective and P projective. Since X is sincere, $\mathrm{Hom}_A(X, I) \neq 0$ and $\mathrm{Hom}_A(P, X) \neq 0$. Consequently we have a cycle in mod A

$$X \to I = M_0 \to M_1 \to \dots \to M_t = P \to X.$$

Since X lies in a stable tube (thus not in Γ), we obtain a contradiction to the fact that the multicoil algebra A is cycle-finite.

2.10. Corollary. *Let A be a multicoil algebra having a sincere indecomposable module lying in a stable tube, then any component of Γ_A may contain projectives or injectives, but not both.*

PROOF. Let Γ be a component of Γ_A containing both a projective module P and an injective module I. Let $\mathcal{T}$ be a stable tube of Γ_A containing an indecomposable sincere module X. By (2.7), Remark, we may assume that all modules in the τ-orbit of X are sincere. In particular, since $\mathcal{T}$ is stable, then $\mathcal{T} \neq \Gamma$. By (2.9), Γ is not a multicoil. By definition of a multicoil algebra, this implies that Γ does not contain τ-periodic modules since Γ contains a projective and, by (2.7), $\tau^S I \neq 0$ for all $s \geqq 0$, there exists a path $\tau^S I \to \ldots \to P'$ with P' projective. Applying (2.7), we have a cycle

$$X \to \ldots \to \tau^S X \to \tau^S I \to \ldots \to P' \to X$$

again a contradiction.

3. Enlargements of cycle-finite algebras.

3.1. LEMMA. *Let C be a tame concealed algebra, and M be a C-module such that $C[M]$ (respectively, $[M]C$) is tame. Then M has no non-zero preprojective (respectively, preinjective) direct summand.*

PROOF. Suppose that M_C has an indecomposable preprojective direct summand N_C. We claim that there exists an indecomposable preinjective C-module X such that $\dim_k \operatorname{Hom}_C(N, X) = m \geqq 3$. This is well-known if C is hereditary. If not, there exist a hereditary algebra H, and a preprojective tilting module T_H such that $C \simeq \operatorname{End} T_H$. Since $N_C \simeq \operatorname{Hom}_H(T, N')$ for some indecomposable preprojective module N'_H, and there exists an indecomposable preinjective module X'_H such that $\dim_k \operatorname{Hom}_k(N', X') = m \geqq 3$, our claim follows from the fact that if $X = \operatorname{Hom}_H(T, X')$, then $\operatorname{Hom}_C(N, X) \simeq \operatorname{Hom}_H(N', X')$.

Since the vector space category $\operatorname{Hom}_C(M, \operatorname{mod} C)$ contains $\operatorname{Hom}_C(N, \operatorname{mod} C)$ then $\operatorname{mod} C[M]$ contains, by [20] (2.4), a full subcategory of the form $\operatorname{mod} H$, where H is a wild hereditary algebra given by m parallel arrows, with $m \geqq 3$, and this contradicts the tameness of $C[M]$.

3.2. LEMMA. *Let B a tubular algebra and*

$$\operatorname{ind} B = \mathcal{P}_0^B \vee \mathcal{T}_0^B \vee \left(\bigvee_{q \in Q^+} \mathcal{T}_q^B \right) \vee \mathcal{T}_\infty^B \vee \mathcal{Q}_\infty^B$$

be the standard decomposition of $\operatorname{ind} B$. *Let M be a B-module such that $B[M]$ (respectively, $[M]B$) is tame. Then all indecomposable summands of M* belong to $\mathcal{T}_\infty^B \vee \mathcal{Q}_\infty^B$ (respectively, $\mathcal{P}_0^B \vee \mathcal{T}_0^B$).

PROOF. Suppose that N is an indecomposable direct summand of M which does not lie in $\mathcal{T}_\infty^B \vee \mathcal{Q}_\infty^B$ Let $p \in Q^+$ be such that $N \in \mathcal{P}_0^B \vee \mathcal{T}_0^B \vee \left(\bigvee_{q < p} \mathcal{T}_q^B \right)$. Take a non-zero morphism $f\colon N \to I$, with I indecomposable injective. Then, by [21] (5.2), $I \in \mathcal{T}_\infty^B \vee \mathcal{Q}_\infty^B$. Moreover, $\mathcal{T}_p^B$ is a family $(\mathcal{T}_p^B(\lambda))_{\lambda \in \mathbb{P}_1(k)}$ of pairwise orthogonal tubes separating $\mathcal{P}_0^B \vee \mathcal{T}_0^B \vee \left(\bigvee_{q<p} \mathcal{T}_q^B\right)$ from $\left(\bigvee_{q<p} \mathcal{T}_q^B\right) \vee \mathcal{T}_\infty^B \vee \mathcal{Q}_\infty^B$. This implies that $f\colon N \to I$ factors through any of the tubes $\mathcal{T}_p^B(\lambda)$, Therefore, the vector space category $\operatorname{Hom}_B(N, \operatorname{mod} B)$ contains infinitely many pairwise orthogonal objects of the form $\operatorname{Hom}_B(N, X_\lambda)$, with $X_\lambda \in \mathcal{T}_p^B$. Hence $B[N]$ is wild, a contradiction to the tameness of $B[M]$.

3.3. LEMMA. *Let B a truncated branch extension (respectively, coextension) of a tame concealed algebra, M be a B-module whose indecomposable direct summands lie in $\mathcal{T}_0^B$ (respectively, $\mathcal{T}_\infty^B$) and assume that $B[M]$ is cycle-finite. Then M is indecomposable and lies on the mouth of a tube in $\mathcal{T}_0^B$ respectively, $\mathcal{T}_\infty^B$).*

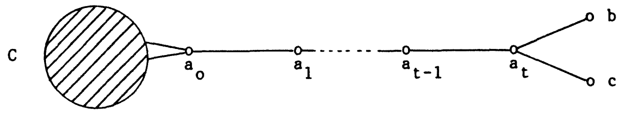
PROOF. Suppose that M is decomposable. It follows from the argument in [20] (3.6) and the tameness of $B[M]$ that $M = M_1 \oplus M_2$, where both M_1 and M_2 lie on the mouths of tubes in $\mathcal{T}_0^B$. Then the vector space category $\operatorname{Hom}_B(M, \operatorname{mod} B)$ contains a subcategory formed by the disjoint union of two linearly ordered sets $\operatorname{Hom}_B(M, M_1) \to \operatorname{Hom}_B(M, N)$ and $\operatorname{Hom}_B(M, L_1) \to \operatorname{Hom}_B(M, L_2) \to \operatorname{Hom}_B(M, L_3) \to \dots$ where N is such that there exists an irreducible morphism $M_1 \to N$, and $L_1, L_2, L_3 \dots$ lie on the sectional path starting in M_2 and pointing to infinity.

By the same argument as in [2] (3.1), we obtain a contradiction to the cycle-finiteness of $B[M]$. This shows that M is indecomposable.

Assume now that M does not lie on the mouth of a tube in Γ_B. Again, it follows from the argument in [20] (3.6) and the tameness of $B[M]$ that the Auslander-Reiten sequence in mod B starting with M has two indecomposable middle terms one of which lies on the mouth of a tube. Then, by [1] (3.1), we obtain a contradiction to the cycle-finiteness of $B[M]$. The proof is complete.

3.4. LEMMA. *Let C be a tame concealed algebra, and D be a subcategory of the form*

Fig. 10.



where $t \geqq 0$, the full subcategory of D formed by the objects a_t, b, c is hereditary, unoriented arrows may be oriented arbitrarily, and the full subcategories D_b and D_c of D consisting of all objects of D except b and c, respectively, are truncated branch extensions (respectively, coextensions) of C. Then D is not cycle-finite.

PROOF. We may assume, by duality, that a_0 is an extension point of c. We shall first show that we can assume the arrow between a_t and c to be oriented as $c \to a_t$. Indeed, if this is not the case, let T_D be the APR-tilting module corresponding to the sink c, and $D' = \operatorname{End} T_D$. Then the full subcategories of $\operatorname{mod} D$ and $\operatorname{mod} D'$ consisting of those modules not having $S(c)$ as a summand are equivalent. Since $S(c)_D$ is simple projective, while $S(c)_{D'}$ is simple injective, they never occur in cycles in the respective module categories. Consequently, D is cycle-finite if and only if D' is. This shows our claim.

Then D is a one-point extension of $B = D_c$ by an indecomposable module M lying in a tube $\mathscr{T}$ of $\mathscr{T}_0^B$, and the full subcategory of $\operatorname{Hom}_B(M, \operatorname{mod} B)$ consisting of all objects $\operatorname{Hom}_B(M, X) \neq 0$, with $X \in \mathscr{T}$, is of the form

$$\begin{array}{ccccccc} & & \operatorname{Hom}_B(M, L_0) & \to & \operatorname{Hom}_B(M, L_1) & \to & \operatorname{Hom}_B(M, L_2) \to \dots \\ & & \uparrow & & \uparrow & & \uparrow \\ \operatorname{Hom}_B(M, M) & = & \operatorname{Hom}_B(M, N_0) & \to & \operatorname{Hom}_B(M, N_1) & \to & \operatorname{Hom}_B(M, N_2) \to \dots \end{array}$$

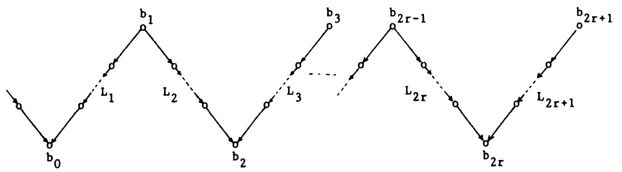
If then follows from [1] (3.1) that $D = B[M]$ is not cycle-finite.

3.5. LEMMA. *Let B a domestic truncated branch extension (respectively, coextension) of a tame concealed algebra C, and M be an indecomposable preinjective (respectively, preprojective) B-module. Assume that $[M]B$ (respectively, $B[M]$) is cycle-finite. Then $[M]B$ (respectively, $B[M]$) is a truncated extension (respectively, coextension) of C.*

PROOF. Assume that B is a domestic truncated branch extension of C, M is preinjective, and $[M]B$ is cycle-finite. We claim that the largest C-submodule $M|_C$ of M is zero, and thus the support of M lies on a branch K of B. Indeed, if $M|_C \neq 0$, it follows from the structure of the preinjective component of B (see [21]) that $M|_C$ is a preinjective C-module. But this implies, by (3.1), that $[M|_C]C$ is wild. Therefore $[M]B$ is also wild, a contradiction to the fact that it is cycle-finite. This completes the proof of our claim.

Let $\mathscr{T}$ be the tube of $\mathscr{T}_0^B$ containing the indecomposable projective modules $P(x)$, $x \in K_0$. Each point of K is connected to C by a unique (non-zero) walk in K. We may thus assume that the support of M lies within the interval in K from b_0 to b_{2r+1},

Fig. 11.



where $r \geqq 0$, and b_0 is closer to C than all $b_j (j \geqq 1)$. For $1 \leqq i \leqq 2r$, let L_i denote the walk in the above interval between b_{i-1} and b_i. Let a (respectively, c) be the point in Supp M whose distance to C is minimal (respectively, maximal). We may assume that a lies on L_1 or L_2, and c on L_{2r} or L_{2r+1}.

First, suppose that either $a \in L_1 \backslash \{b_0, b_1\}$ of else $a = b_0$ and is an extension point of C (that is, rad $P(a)$ has a simple regular C-module as an indecomposable direct summand). If $a = c$ and rad $P(a)$ is indecomposable then $[M]B$ is a one-point coextension of B by the simple module $S(a)$, hence is a truncated branch extension of C and we are done. Otherwise, we have two cases to consider. Suppose $a \neq c$ and rad $P(a)$ is indecomposable. Then let $[M]B$ have e as a coextension point. The full subcategory of $[M]B$ consisting of C, e together with the walk in K from C to the neighbour a' of a such that there exists an arrow $a' \to a$, satisfies the conditions of (3.4), which gives, by (2.5), a contradiction to the cycle-finiteness of $[M]B$.

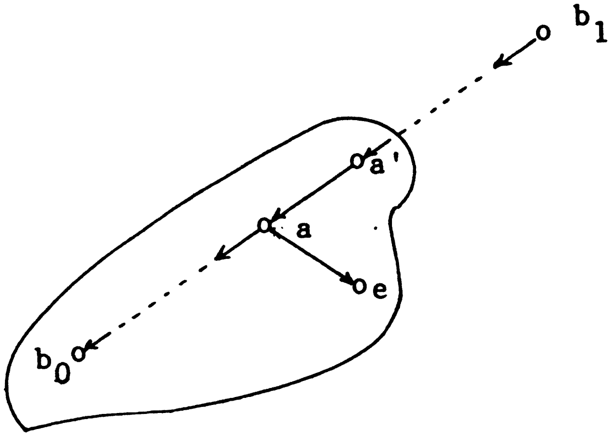


Fig. 12.

If now rad $P(a)$ is decomposable, then a is the source (in B!) of two arrows, one of which lies on L_1 and the other, $a \to a''$, outside. Then the full subcategory of $[M]B$ consisting of C, e together with the walk in K from C to a'' (containing b_0 and a) satisfies again the conditions of (3.4), hence another contradiction (see fig. 13 below).

We may assume that $a \in L_2$ (indeed, the case where $a = b_0$ but is not an extension point of C, is equivalent to the case where $a = b_2$). Suppose that $a = b_1$. We claim that $c = b_2$ and is a sink of B. Indeed, if this is not the case, then there are two possibilities. If $c \notin L_2$, then there is an arrow in $[M]B$ of the form $b_2 \to e$ such that, if $b'_2 \to b_2$ is the arrow on L_3 with target b_2, then the composition $b'_2 \to b_2 \to e$ is non-zero. Thus the full subcategory of $[M]B$ consisting of C, e and the walk in K from C to b'_2 satisfies the conditions of (3.4), hence again a contradiction.

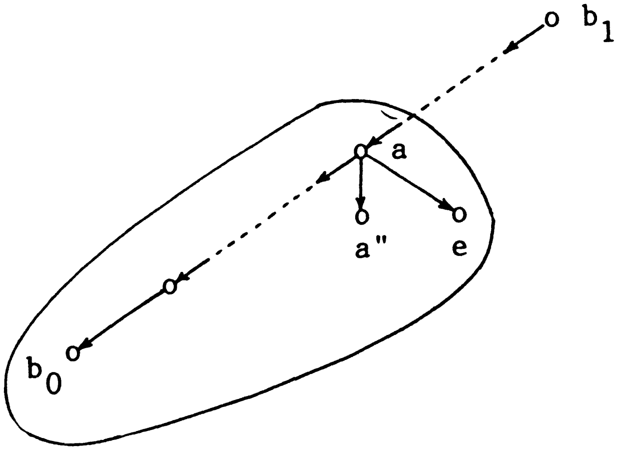


Fig. 13.

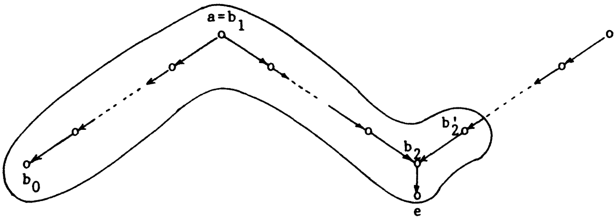


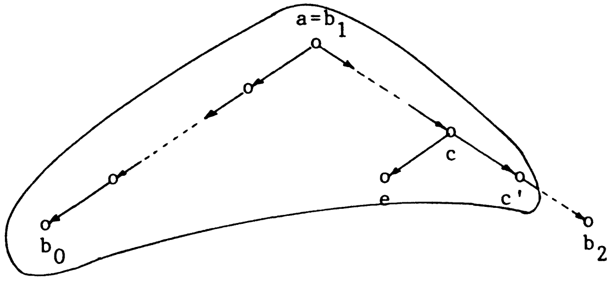
Fig. 14

If $c \in L_2$ and is not a sink, then, since K is truncated, there exists in B an arrow $c \to c'$ such that $B(a, c') \neq 0$. Therefore, $[M]B$ contains an additional arrow $c \to e$ and the full subcategory of $[M]B$ consisting of C, e and the walk in K from C to c' satisfies the conditions of (3.4), hence again a contradiction.
We have shown that $c \in L_2$ and is a sink, thus $c = b_2$. But this clearly implies that $[M]B$ is a truncated branch extension of C. This completes the proof in the case where $a = b_1$.

There remains to consider the case where $a \in L_2 \setminus \{b_1\}$. We shall show that in this case, M is a regular B-module, which would contradict the hypothesis. Let Σ denote the sectional path starting from $P(a)$ and pointing to the mouth. There are two cases to consider.

(i) If $c \in L_{2r+1} \setminus \{b_{2r}\}$, let $U = 0$ if $P(c)$ is uniserial and $U = P(d)$ if K contains an

Fig. 15.



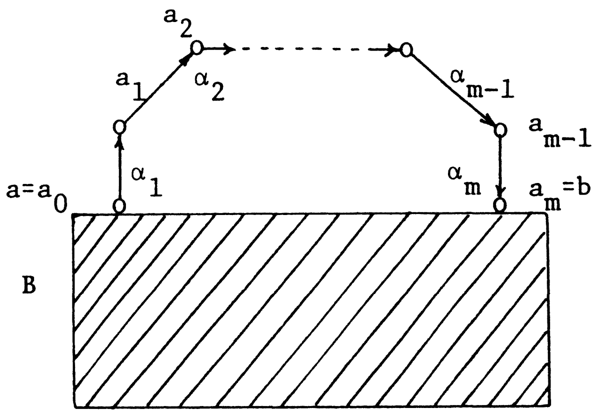
arrow $c \to d$ not on L_{2r+1}. Then M lies at the intersection of Σ and the sectional path starting from $P(c)/U$ and pointing to infinity. This follows from the structure of the tube $\mathscr{T}$ which is obtained from a stable tube of Γ_C by successive ray insertions involving the projectives corresponding to the points of K, see [14].

(ii) If $c \in L_{2r} \backslash \{b_{2r-1}\}$ (indeed, if $c = b_{2r-1}$, then we are in a case equivalent to (i) with $c = b_{2r+1}$), then, similarly, M lies on the intersection of Σ and the sectional path starting from $S(c)$ and pointing to infinity.

The proof of the lemma is now complete.

3.6. LEMMA. *Let B be a tame truncated branch extension of a tame concealed algebra C, and let a, b be two distinct objets of B. Let D be a triangular category of the form*

Fig. 16.



where $a_1, \ldots, a_{m-1} \notin B_0$ for $m \geqq 1$. Then D is not cycle-finite.

PROOF. Let $\beta_1, \ldots, \beta_p$ be the set of all arrows of source $a_m = b$, and $\bar{D}$ be the quotient of D by the ideal generated by $\alpha_m \beta_i (1 \leqq i \leqq p)$ and $\alpha_r \alpha_{r+1} (1 \leqq r < m)$. Clearly, if $\bar{D}$ is not cycle-finite, then D is not cycle-finite either. We shall thus construct a cycle in mod $\bar{D}$ one of whose morphisms lies in the infinite power of the radical. Since B is tame, it is either a representation-infinite tilted algebra of euclidean type having a complete slice in its preinjective component, or a tubular algebra (1.3). Hence there exists in $\operatorname{mod} B$ a path

$$P_B(a) \xrightarrow{g} N_0 \to N_1 \to \dots N_s \to I_B(b)$$

where $g \in \operatorname{rad}^\infty(\operatorname{mod} B)$. Since $\operatorname{mod} B$ is fully embedded in $\operatorname{mod} \bar{D}$, this is also a path in $\operatorname{mod} \bar{D}$, and $g \in \operatorname{rad}^\infty(\operatorname{mod} \bar{D})$. We shall consider three cases:

(i) Assume that $m > 1$. Then we have a path in $\operatorname{mod} \bar{D}$

$$I_B(b) \to I_{\bar{D}}(b) \to S(a_{m-1}) \to P_{\bar{D}}(a_{m-2}) \to S(a_{m-2}) \to \dots \to S(a_1) \to P_{\bar{D}}(a) \to P_B(a)$$

where $I_{\bar{D}}(b)$ (respectively, $I_B(b)$) denotes the injective envelope of $S(b)$ in $\operatorname{mod} \bar{D}$ (respectively, $\operatorname{mod} B$), mod $P_{\bar{D}}(a_i)$ denotes the projective cover of $S(a_i)$ in $\operatorname{mod} \bar{D}$, for each $0 \leqq i < m-1$, and $P_B(a)$ denotes the projective cover of $S(a)$ in mod B. Composing with the first path, we obtain the required cycle.

(ii) Assume that $m = 1$ and $\operatorname{Hom}_B(S(b)) = 0$. By construction of $\bar{D}$, the simple module $k\alpha_1$ is a summand of $\operatorname{rad} P_{\bar{D}}(a)$ and we have an exact šequence

$$\xi\colon\ 0 \to S(b) \xrightarrow{i} P_{\bar{D}}(a) \xrightarrow{p} P_B(a) \to 0$$

with the image of i equal to $k\alpha_1$. On the other hand, $S(b) \simeq \operatorname{soc} I_B(b)$, hence a monomorphism $j\colon S(b) \to I_B(b)$. Consider the exact commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & S(b) \arrow[r, "i"] \arrow[d, "j"'] & P_{\bar{D}}(a) \arrow[r, "p"] \arrow[d, "u"'] & P_B(a) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & I_B(b) \arrow[r, "v"] & M \arrow[r, "w"] & P_B(a) \arrow[r] & 0
\end{tikzcd}$$

where the lower sequence is $\operatorname{Ext}^1_{\bar{D}}(P_B(a), j)(\xi)$. We claim that M is an indecomposable $\bar{D}$-module. Observe that, by the Five Lemma, u is a monomorphism and induces an isomorphism $\operatorname{soc} P_{\bar{D}}(a) \simeq \operatorname{soc} M$. We shall show that, if $f\colon P_{\bar{D}}(a) \to M$ is such that $wf \neq 0$, then f is a monomorphism and consequently $\operatorname{soc}(\operatorname{Im} f) \simeq \operatorname{soc} M$. Since $P_{\bar{D}}(a)$ is projective, there exists an exact commutative diagram in $\operatorname{mod} \bar{D}$

$$\begin{tikzcd}
 & P_{\bar{D}}(a) \arrow[dl, "{\left[\begin{smallmatrix} g \\ h \end{smallmatrix}\right]}"'] \arrow[d, "f"] & \\
I_B(b) \oplus P_{\bar{D}}(a) \arrow[r, "{[v\,u]}"] & M \arrow[r] & 0
\end{tikzcd}$$

Since $wf \neq 0$ and $wv = 0$, we have $h \neq 0$. But then $h \in \operatorname{End} P_{\bar{D}}(a)$ is an isomorphism (because $\bar{D}$ is triangular). Moreover $I_B(b)$ is a B-module, and so $g(\alpha_1) = 0$. This implies that $f(\alpha_1) = vg(\alpha_1) + uh(\alpha_1) = uh(\alpha_1) \neq 0$, since uh is a monomorphism. Now the morphisms $fi, ui\colon S(b) \to M$ are non-zero, and $\operatorname{Hom}_{\bar{D}}(S(b), M) \simeq k$. Thus there exists $0 \neq \lambda \in k$ such that $fi = \lambda \cdot ui$ (and hence $wfi = \lambda \cdot wui = \lambda \cdot wvj = 0$). On the other hand, the triangularity of D *and* $wf \neq 0$ imply that wf induces an isomorphism between the simple tops of $P_{\bar{D}}(a)$ and $P_B(a)$, and therefore wf is surjective. The first exact commutative diagram above gives that soc $P_{\bar{D}}(a) = i(S(b)) \oplus N$, where $N \simeq \operatorname{soc} P_B(a)$. Since $wfi = 0$ implies that the restriction of wf to soc $P_{\bar{D}}(a)$ annihilates $i(S(b))$ and $\dim P_{\bar{D}}(a) = 1 + \dim P_B(a)$, then the restriction of wf to N is a monomorphism. This implies that wf induces an isomorphism between N and soc $P_B(a)$. Let now f' denote the restriction of f to N. Then $f'(N) \subseteq \operatorname{soc} M = u(N) \oplus ui(S(b))$, so $f' = \begin{bmatrix} g' \\ h' \end{bmatrix}$, with $g'\colon N \to u(N)$ and $h'\colon N \to ui(S(b))$. Since $wh'(N) \subseteq wui(S(b)) = 0$ and wf' is an isomorphism, then g' is a monomorphism. This implies that the restriction of f to $N \oplus i(S(b)) = \operatorname{soc} P_{\bar{D}}(a)$ is a monomorphism, and consequently so is f.

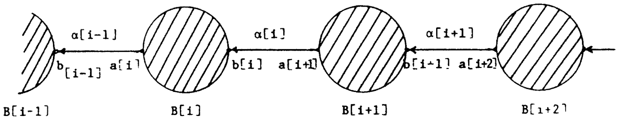
Finally, suppose that $M = M' \oplus M''$ with $M', M'' \neq 0$. Let $q\colon P_B(a) \to S(a)$ be the canonical epimorphism. Then either $qw(M') \neq 0$ or $qw(M'') \neq 0$. Assume the former. Then there exists $w'\colon P_{\bar{D}}(a) \to M'$ such that $qww' \neq 0$. Hence $ww' \neq 0$. The above argument implies that w' is a monomorphism. In particular $\dim_k(\operatorname{soc} M') \geqq \dim_k(\operatorname{soc} P_{\bar{D}}(a)) = \dim_k(\operatorname{soc} M)$ and consequently $\operatorname{soc} M' = \operatorname{soc} M$. This contradicts the facts that $M' \cap M'' = 0$ and $M'' \neq 0$. We have thus shown that M is indecomposable and that we have a cycle in $\operatorname{mod} \bar{D}$

$$P_B(a) \xrightarrow{g} N_0 \to N_1 \to \dots \to N_S \to I_B(b) \xrightarrow{v} M \xrightarrow{w} P_B(a)$$

where $g \in \operatorname{rad}^\infty(\operatorname{mod} \bar{D})$.

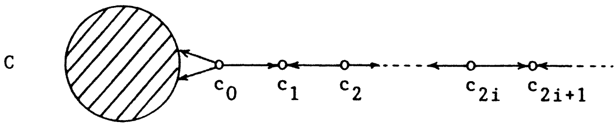
(iii) Assume that $m = 1$ and $\operatorname{Hom}_B(S(b), P_B(a)) \neq 0$. There is thus a non-zero path γ inside B from a to b. Moreover, since the indecomposable projective B-modules lie in $\mathscr{P}_0^B \vee \mathscr{T}_0^B$ and $\operatorname{Hom}_B(\mathscr{G}_\infty^B, \mathscr{P}_0^B \vee \mathscr{T}_0^B) = 0$, then $S(b) \in \mathscr{P}_0^B \vee \mathscr{T}_0^B$. Consider the Galois covering $F\colon R \to \bar{D}$ with infinite cyclic group of the form

Fig. 17.



where $B[i]$ (respectively, $a[i]$, $b[i]$, $\alpha[i]$) denotes the copy of B (respectively, a, b α) indexed by $i \in \mathbb{Z}$. We claim that R is not cycle-finite which will imply (upon applying the push-down functor $F_\lambda\colon \operatorname{mod} R \to \operatorname{mod} \bar{D}$) that $\bar{D}$ is not cycle-finite either. Let Λ denote the full subcategory of R consisting of all objects of $B = B[0]$ and $a[1]$. By definition of $\bar{D}$, the restriction of $P_R(a[1])$ to $B[0]$ is the simple module $S(b) = S(b[0])$ at $b = b[0]$. Hence $\Lambda = B[S(b)]$. If $S(b) \in \mathscr{P}_0^B$, then $\Lambda = B[S(b)]$ is wild and consequently R is not cycle-finite. Assume that $S(b)$ belongs to a tube $\mathscr{T}$ of $\mathscr{T}_0^B$. If $S(b)$ does not lie on the mouth of $\mathscr{T}$, then, by (3.3), Λ is not cycle-finite and consequently neither is R. Thus let $S(b)$ lie on the mouth of $\mathscr{T}$. By (1.4), Λ is a truncated branch extension of C. Now, since, for each $i \in \mathbb{Z}$, there is a non-zero path $\gamma[i]$ inside $B[i]$ from $a[i]$ to $b[i]$, let E denote the full subcategory of R consisting of $B = B[0]$, $b = b[0]$ and all $a[i]$, $b[i]$ with $i \geqq 1$. This subcategory may contain double arrows (in case the full subcategory of B consisting of a and b has as quiver a double arrow). Replacing each such double arrow by a single arrow, E contains a subcategory L of the form

Fig. 18.



where the largest C-submodule of $P_L(C_0)$ is simple regular. Then, by the argument in [2] (3.3), we conclude that L is not cycle-finite. Since L is a specialisation of E in the sense of [20] (1.2), we infer that E is not cycle-finite, and so neither is R. The proof of the lemma is now complete.

3.7. Lemma. *Let B be a tame truncated branch extension (respectively, coextension) of a tame concealed algebra C, M a B-module having an indecomposable direct summand Y lying in $\mathcal{T}_0^B$ (respectively, $\mathcal{T}_\infty^B$) and assume that $B[M]$ (respectively, $[M]B$) is cycle-finite. Then $M = Y$.*

Proof. Assume that B is a tame truncated branch extension of C, $M = Y \oplus M'$ with $Y \in \mathcal{T}_0^B$ and $B[M]$ is cycle-finite. Observe first that M has no indecomposable direct summand lying in $\mathcal{P}_0^B = \mathcal{P}_0^C$. For, if this were the case, and $M|_C$ is the largest C-submodule of M, then, by (3.1), $C[M|_C]$ is a wild full subcategory of $B[M]$, a contradiction to the cycle-finiteness of $B[M]$. Then $M = M_1 \oplus M_2$, where all indecomposable direct summands of M_1 lie in $\mathcal{T}_0^B$, and M_2 has no direct summand lying in $\mathcal{T}_0^B$. Since the vector space category $\operatorname{Hom}_B(M, \operatorname{mod} B)$ fully contains the vector space category $\operatorname{Hom}_B(M_1, \operatorname{mod} B)$, it follows from (3.3) that $M_1 = Y$ and lies on the mouth of a tube in $\mathcal{T}_0^B$. Moreover, by (1.4), $B[Y]$ is a truncated branch extension of C and, since $B[M]$ is tame, so is $B[Y]$. Suppose that $M_2 \neq 0$ and let a denote the extension point of B by Y. Then $B[M]$ is obtained from $B[Y]$ by adding finitely many arrows $\alpha = \gamma_1, \gamma_2, \ldots, \gamma_t$ $(t \geqq 1)$ from a to B. Let D be the specialisation of $B[M]$ obtained by deleting the arrows $\gamma_2, \ldots, \gamma_t$ see [20] (1.2). Then $\operatorname{mod} D$ is a full subcategory of $\operatorname{mod} B[M]$. By (2.5), we deduce that D is cycle-finite. On the other hand, (3.6) implies that D is not cycle-finite, a contradiction. Then $M = Y$.

3.8. Lemma. *Let B be a domestic truncated branch extension (respectively, coextension) of a tame concealed algebra C, M be a B-module having a preinjective (respectively, preprojective) indecomposable direct summand Y and assume that $[M]B$ (respectively, $B[M]$) is cycle-finite. Then $M = Y$.*

Proof. Assume that B is a domestic truncated branch extension of C,

$M = Y \oplus M'$, with Y indecomposable preinjective and $[M]B$ is cycle-finite. This implies that $[Y]B$ is cycle-finite. Consequently, by (3.5), $[Y]B$ is a truncated branch extension of C and obviously is tame. Let a be the coextension point of $[Y]B$. Then $[M]B$ is obtained from $[Y]B$ by adding finitely many arrows $\beta = \delta_1, \delta_2, \ldots, \delta_s (s \geqq 1)$ from B to a. Let D to the specialisation of $[M]B$ obtained by deleting the arrows $\delta_2, \ldots, \delta_s$, see [20] (1.2). Then mod D is a full subcategory of mod $[M]B$. By (2.5), we deduce that D is cycle-finite. On the other hand, (3.6) implies that D is not cycle-finite, a contradiction. Then $M = Y$.

4. Proofs of the main results.

4.1. THEOREM. *Let A be a multicoil algebra. The following conditions are equivalent:*

(i) *There exists a sincere indecomposable A-module X lying in a stable tube of Γ_A*

(ii) *A has sincere tubes*

(iii) *A is either tame concealed or tubular.*

PROOF. The implication (ii) $\Rightarrow$ (i) is trivial, and (iii) $\Rightarrow$ (ii) follows from [21] (4.3) (5.2). We shall show that (i) implies (iii). The proof will be done in several steps. Let A be a multicoil algebra having a sincere indecomposable module X lying in a stable tube of Γ_A (in particular, A is representation-infinite). By (2.3) (c)(d), A is triangular and contains a tame concealed full convex subcategory.

4.2. Let C be a fixed tame concealed full convex subcategory of A. Let $\mathscr{B}_e$ (respectively, ${}_c\mathscr{B}$) dende the set of all truncated branch extensions (respectively, coextensions) of C which are full convex subcategories of A. Since A is finite, so are $\mathscr{B}_e$ and ${}_c\mathscr{B}$. Let thus B_e (respectively, ${}_cB$) be a fixed maximal element of $\mathscr{B}_e$ (respectively, ${}_c\mathscr{B}$). Since A is cycle-finite, so are B_e and ${}_cB$. In the notation of (1.3), we have ind $B_e = \mathscr{P}_0^{B_e} \vee \mathscr{T}_0^{B_e} \vee \mathscr{Q}_e^{B}$ and ind ${}_cB = \mathscr{P}_\infty^{cB} \vee \mathscr{T}_\infty^{cB} \vee \mathscr{Q}_\infty^{cB}$. Finally, let B denote the full subcategory of A consisting of all objects of ${}_cB$ and B_e.

PROPOSITION. *With the above notation, the following statements hold:*

(i) *B is a convex subcategory of A and any path in B either lies in B_e or ${}_cB$ or intersects C*

(ii) ind $B = \mathscr{P}_\infty^{cB} \vee (\mathscr{T}' \vee \mathscr{T} \vee \mathscr{T}'')$ *where $\mathscr{T}'$ consists of all tubes of $\mathscr{T}_\infty^{cB}$ containing injective modules (if such tubes exist), $\mathscr{T}''$ consists of all tubes of $\mathscr{T}_0^{B_e}$ containing projective modules (if such tubes exist) and $\mathscr{T}$ consists of those tubes of $\mathscr{T}_0^C$ which do not contain modules lying in $\mathscr{T}'$ or $\mathscr{T}''$.*

(iii) *For any $x \notin B_0$, $P(x)|_B$ is either zero or a direct sum of indecomposable modules lying in $\mathscr{Q}_0^{B_e}$.*

(iv) *For any* $x \notin B_0$, $I(x)|_B$ *is either zero or a direct sum of indecomposable modules lying in* $\mathscr{P}_{\infty}^{cB}$.

(v) *If* ${}_cB \neq C$ *or there exists a one-point coextension of* B *in* A, *then* $B_e = C$ *and there is no one-point extension of* $B = {}_cB$ *(hence* C*) inside* A.

(vi) *If* $B_e \neq C$ *or there exists a one-point extension of* B *in* A, *then* ${}_cB = C$ *and there is no one-point coextension of* $B = B_e$ *(hence* C*) inside* A.

In particular, $\mathscr{B}_e$ *(respectively,* ${}_c\mathscr{B}$*) contains only one maximal element.*

PROOF. We start by showing that for any $x \notin (B_e)_0$, the restriction $P(x)|_{B_e}$ does not contain an indecomposable direct summand lying in $\mathscr{P}_0^{B_e} \vee \mathscr{T}_0^{B_e}$. Let first $y \notin (B_e)_0$ be a neighbour of B_e such that $R(y) = P(y)|_{B_e}$ has an indecompoable direct summand $Y \in \mathscr{P}_0^{B_e} \vee \mathscr{T}_0^{B_e}$. Then y is an extension point connected to B_e by at least one arrow. The full subcategory $B_e[R(y)]$ of A is clearly cycle-finite. Then, by (3.7) (3.3), $R(y) = Y$ and lies on the mouth of a tube of $\mathscr{T}_0^{B_e}$. But this implies by (1.4) that $B_e[R(y)]$ is also a truncated branch extension of C. We shall show that $B_e[R(y)]$ is convex in A. Suppose that this is not the case, and let E denote its convex hull. Fix a zero path from y to B_e which does not belong to $B_e[R(y)]$, and denote by F the category obtained from E by deleting the remaining zero paths from y to B_e which do not belong to $B_e[R(y)]$. Since A is cycle-finite, so are also E and F. On the other hand, F satisfies the conditions of (3.6) and so is not cycle-finite, a contradiction. We have shown that $B_e[R(y)]$ is a truncated branch extension of C which is a full convex subcategory of A, contradiction to the maximality of B_e. Let now $y_1, \ldots, y_r$ be the set of all neighbours of B_e in A which are extension points of B_e, and let D be the full subcategory of A consisting of B_e together with the objects $y_1, \ldots, y_r$. Let x be an arbitrary object of A not in B_e, and let $R(x) = P(x)|_{B_e}$. Since any path from x to B_e factors through one of the y_i, $1 \leqq i \leqq r$, there exists an epimorphism

$$\bigoplus_{i=1}^{r} P(y_i)^{n_i}|_D \to P(x)|_D$$

which implies that there exists an epimorphism

$$\bigoplus_{i=1}^{r} R(y_i)^{n_i} \to R(x).$$

But ind $B_e = \mathscr{P}_0^{B_e} \vee \mathscr{T}_0^{B_e} \vee \mathscr{Q}_0^{B_e}$ and the above discussion implies that the indecomposable summands of all $R(y_i)$, $1 \leqq i \leqq r$, lie in $\mathscr{Q}_0^{B_e}$. Since we have no non-zero morphism from modules lying in $\mathscr{Q}_0^{B_e}$ to modules lying in $\mathscr{P}_0^{B_e} \vee \mathscr{T}_0^{B_e}$, the existence of the above epimorphism implies that all indecomposable direct summands of $R(x)$ are in $\mathscr{Q}_0^{B_e}$. This completes the proof of our claim.

Dually, for any $x \notin ({}_cB)_0$, the restriction $I(x)|_{{}_cB}$ does not contain an indecomposable direct summand lying in $\mathscr{T}_{\infty}^{cB} \vee \mathscr{Q}_{\infty}^{cB}$.

We now claim that, for any $x \notin (B_e)_0$, the restriction $J(x) = I(x)|_{B_e}$ is zero or has its indecomposable summands lying in $\mathscr{P}_0^{B_e} \vee \mathscr{T}_0^{B_e}$. If B_e is tubular, this follows at once from (3.2) and the tameness of $[J(x)]B_e$. Hence assume that B_e is a domestic truncated branch extension of C. Let first $z \notin (B_e)_0$ be a neighbour of B_e and supose that $J(z) = I(z)|_{B_e}$ has an indecomposable direct summand Y lying in $\mathscr{Q}_0^{B_e}$, which is now the preinjective component of B_e. By (3.8) (3.5), $[J(z)]B_e$ is again a tame truncated branch extension of C. Applying (3.6) as above, we deduce that $[J(z)]B_e$ is also convex in A, a contradiction to the maximality of B_e. Thus $J(z)$ is either zero, or a direct sum of modules lying in $\mathscr{P}_0^{B_e} \vee \mathscr{T}_0^{B_e}$. Let $z_1, \dots, z_t$ be the set of all neighbours of B_e which are coextension points of B_e, and let D be the full subcategory of A consisting of B_e together with $z_1, \dots, z_t$. Let x be an arbitrary object of A which does not belong to B_e. Since any path from B_e to x factors through one of the points $z_1, \dots, z_t$, there exists a monomorphism

$$I(x)|_D \to \bigoplus_{i=1}^{t} I(z_i)^{m_i}|_D$$

which implies that there exists a monomorphism

$$J(x) \to \bigoplus_{i=1}^{t} J(z_i)^{m_i}.$$

But $B_e = \mathscr{P}_0^{B_e} \vee \mathscr{T}_0^{B_e} \vee \mathscr{Q}_0^{B_e}$ and the above discussion implies that the indecomposable summands of all $J(z_i)$, $1 \leqq i \leqq t$, lie in $\mathscr{P}_0^{B_e} \vee \mathscr{T}_0^{B_e}$. Since $J(x)$ is a submodule of $\bigoplus_{i=1}^{t} J(z_i)$, then $J(x)$ is either zero or a direct sum of modules lying in $\mathscr{P}_0^{B_e} \vee \mathscr{T}_0^{B_e}$. This completes the proof of our claim.

Dually, for any $x \notin ({}_cB)_0$, the restriction $P(x)|_{{}_cB}$ is zero or has its indecomposable summands lying in $\mathscr{T}_\infty^{{}_cB} \vee \mathscr{Q}_\infty^{{}_cB}$.

It follows from (2.10) that, if $x \notin ({}_cB)_0$ and $P(x)|_{{}_cB}$ has an indecomposable summand Y in $\mathscr{T}_\infty^{{}_cB}$, then Y lies in a stable tube of $\mathscr{T}_\infty^{{}_cB}$, and therefore $P(x)|_{{}_cB}$ is a C-module. Similarly, if $x \notin (B_e)_0$ and $I(x)|_{B_e}$ has an indecomposable summand Z in $\mathscr{T}_0^{B_e}$, then Z lies in a stable tube of $\mathscr{T}_0^{B_e}$, and hence $I(x)|_{B_e}$ is a C-module. This implies that B is convex, and any path in B either lies in B_e or ${}_cB$ or intersects C, hence statement (i).

Moreover, ind $B = \mathscr{P}_\infty^{{}_cB} \vee (\mathscr{T}' \vee \mathscr{T} \vee \mathscr{T}'') \vee \mathscr{Q}_0^{B_e}$, where $\mathscr{T}'$ consists of all tubes of $\mathscr{T}_\infty^{{}_cB}$ containing injective modules (if such tubes exist), $\mathscr{T}''$ consists of all tubes of $\mathscr{T}_0^{B_e}$ containing projective modules (if such tubes exist) and $\mathscr{T}$ consists of those tubes in $\mathscr{T}_0^C$ which do not contain modules lying in $\mathscr{T}'$ and $\mathscr{T}''$. Further, for any $x \notin B_0$, $P(x)|_B$ is zero or has its indecomposable summands lying in $\mathscr{Q}_0^{B_e}$. Similarly, for any $x \notin B_0$, $I(x)|_B$ is zero or has its indecomposable summands lying in $\mathscr{P}_\infty^{{}_cB}$. We have thus shown (ii), (iii), (iv).

Our next claim is that, if there exists a one-point coextension of B in A, then $B_e = C$ and there is no one-point extension of B (hence C) in A. For, if this is not the case, then there exists an indecomposable injective A-module I with an indecomposable summand J of $I/\mathrm{soc}\, I$ lying in $\mathscr{P}_\infty^{{}_cB}$, an indecomposable summand R of rad P lying in $\mathscr{T}_0^{B_e} \vee \mathscr{Q}_0^{B_e}$ and a path in mod A

$$I \to J \to \ldots \to R \to P.$$

Since, by hypothesis, A has a sincere indecomposable module X lying in a stable tube of Γ_A, there exists a cycle in mod A

$$I \to J \to \ldots \to R \to P \xrightarrow{f} X \xrightarrow{g} I$$

where $f, g \in \mathrm{rad}^\infty(\mathrm{mod}\, A)$, a contradiction to the cycle-finiteness of A.

Similarly, if there exists a one-point extension of B in A, then ${}_cB = C$ and there is no one-point coextension of B (hence C) in A.

Assume now that ${}_cB \neq C$. By the above argument, there is no one-point extension of B in A. Therefore A can be obtained from B by repeated one-point coextensions using modules whose restrictions to B are zero or have their indecomposable summands lying in $\mathscr{P}_\infty^{{}_cB}$, and extensions using modules whose restrictions to B are zero. Then, since A has a sincere indecomposable X, we deduce that $B_e = C$ and consequently $B = {}_cB$.

Similarly, if $B_e \neq C$, then ${}_cB = C$, $B = B_e$ and there is no one-point coextension of B inside A.

This proves (v) and (vi). Moreover, the previous analysis shows also that B_e and ${}_cB$ are the unique maximal elements of $\mathscr{B}_e$ and ${}_c\mathscr{B}$, respectively. This completes the proof.

4.3. Lemma. *With the notation of* (4.2), *assume that* B_e (*respectively*, ${}_cB$) *is a tubular algebra. Then* $A = B_e$ (*respectively*, $A = {}_cB$).

Proof. By duality, we may assume that B_e is tubular. By (4.2) (vi), ${}_cB = C$ and $B = B_e$. From [21] (5.2), B is also cotubular, that is, is a truncated branch coextension of a tame concealed full convex subcategory $C' \neq C$ of A. Moreover $\mathrm{ind}\, B = \mathscr{P}_0^B \vee \mathscr{T}_0^B \vee \left(\bigvee_{q \in Q^+} \mathscr{T}_q^B \right) \vee \mathscr{T}_\infty^B \vee \mathscr{Q}_\infty^B$. Since, by (4.2) (vi), B has no coextensions inside A, then B is maximal truncated branch coextension of C'. By (3.2) and (2.10), for any $x \notin B_0$, $P(x)|_B$ is zero or has its indecomposable summands lying in stable tubes of $\mathscr{T}_\infty^B$ or in $\mathscr{Q}_\infty^B$. By (4.2) (v) applied to C', we infer that B has no one-point extensions in A, and therefore $A = B$ is tubular.

4.4. It follows from (4.3) that we may assume that A does not contain a tubular algebra as a full convex subcategory. Therefore, for any tame concealed full convex subcategory C of A, the algebra B_e (respectively, ${}_cB$) is a representation-infinite tilted algebra of euclidean type with a complete slice in its preinjective) component. Theorem (4.1) is now an immediate consequence of (4.2), (4.3) and the following proposition.

PROPOSITION. *Let B be a full convex subcategory of A which is a tilted algebra of euclidean type having a complete slice in its preinjective (respectively, preprojective) component. Assume that B has no coextensions (respectively, extensions) in A and that, for any one-point extension $B[M]$ (respectively, coextension $[M]B$) of B in A, M is preinjective (respectively, preprojective). Then $A = B$ and is tame concealed.*

PROOF. We may assume, by duality, that B has a complete slice in its preinjective component, no coextensions in A, and for any extension $B[M]$ of B in A, M is preinjective. Then the preprojective and regular components of Γ_B are full components of Γ_A. Let $a \in A_0$, $a \notin B_0$, be such that there exists at least one arrow $a \to b$, with $b \in B_0$ and the restriction $M = P(a)|_B$ is non-zero and preinjective. We may assume that M has an indecomposable summand which is minimal with respect to the natural order of the preinjective component. Since A has only finitely many indecomposable projectives, there exists a complete slice $\mathscr{S}$ in the preinjective component of Γ_B such that, for any projective A-module $P(c)$ with $P(c)|_B$ non-zero and preinjective, the direct summands of $P(c)|_B$ are successors of all modules in $\mathscr{S}$.

Let Γ be the component of Γ_A containing $P(a)$. Then Γ fully contains the translation subquiver of the preinjective component of Γ_B consisting of $\mathscr{S}$ and its predecessors. Since this subquiver is infinite, and, by (2.9), multicoils in Γ_A are standard tubes, Γ is not a multicoil. Thus Γ consists of directing modules. On the other hand, by (2.10), since $P(a) \in \Gamma_0$, then Γ contains no injective A-modules. In particular, for any $U \in \Gamma_0$ and $s \geq 0$, we have $\tau^{-s}U \neq 0$. By [24], Γ has only finitely many τ-orbits. Since Γ has no cycles, some projectives and no injectives, it must contain a full translation subquiver Σ such that Σ_0 is a set of representatives of the τ-orbits of points in Γ, and Σ_1 is a set of representatives of the σ-orbits of arrows in Γ, that is, Σ is a section in the sense of [8] (2.5). Let F be the support algebra of Σ. We shall show that F is a tilted algebra such that the set of all indecomposables $U \in \Sigma_0$, considered as F-modules, is a complete slice in mod F. In order to show it, we shall first prove that Γ is a full component of Γ_F and is convex in mod F.

Let $K \in \Gamma_0$ be a predecessor of Σ. If K is not an F-module, there exists $x \in \operatorname{Supp} K$ such that $x \notin F_0$. However, $\operatorname{Hom}_A(K, I(x)) \neq 0$ and in fact there exists a non-zero morphism $f\colon K \to I(x)$ lying in $\operatorname{rad}^\infty$ (mod A). By [19] there exists, for each $t > 0$, a path in mod A

$$K = K_0 \xrightarrow{f_1} K_1 \xrightarrow{f_2} K_2 \to \dots \to K_{t-1} \xrightarrow{f_t} K_t \xrightarrow{g_t} I(x)$$

with the f_i irreducible and $g_t f_t \dots f_2 f_1 \neq 0$. Since there clearly exists $t > 0$ with $K_t \in \Sigma_0$, we obtain a contradiction to the assumption that $x \notin F_0$. Similarly, let $L \in \Gamma_0$ be a successor of Σ. If L is not an F-module, then there exists $y \in \operatorname{Supp} L$ such that $y \notin F_0$. Since $\operatorname{Hom}_A(P(y), L) \neq 0$, we have two cases. If any non-zero morphism $P(y) \to L$ belongs to $\operatorname{rad}^\infty (\operatorname{mod} A)$, the argument dual to the one above yields a contradiction. Otherwise, if there exists a non-zero morphism $P(y) \to L$ lying in a finite power of the radical of $\operatorname{mod} A$, then in particular $P(y) \in \Gamma_0$ and there exists a path of irreducible morphisms from $P(y)$ to L with non-zero composition which certainly factors through a module in Σ. We have thus shown that Γ consists of F-modules, and so is a full component.

We shall now show that F is convex as a full subcategory of A. If this is not the case, then there exists a path $a_1 \xrightarrow{\alpha_1} a_2 \to \dots \xrightarrow{\alpha_{m-1}} a_m (m \geqq 3)$ in Q_A such that $a_1, a_m \in F_0$ but $a_i \notin F_0$ for all $1 < i < m$. Let $\alpha_1 = \beta_1, \beta_2, \dots \beta_2$ be all the arrows in Q_A from a_1 to a_2 and $\alpha_{m-1} = \gamma_1, \gamma_2, \dots, \gamma_t$ be all the arrows in Q_A from a_{m-1} to a_m. Denote by I' the two-sided ideal of $k\,Q_A$ generated by all paths of the form $\beta_i \delta$, $\delta \gamma_j$ for $\delta \in (Q_A)_1$, $1 \leqq i \leqq s$, $1 \leqq j \leqq t$. Let I denote the defining ideal of A and consider $A' = kQ_A/(I + I')$. Any indecomposable F-module is also an A'-module. Let $P'(a_1)$ denote the projective cover of $S(a_1)$ in $\operatorname{mod} A'$, and $I'(a_m)$ denote the injective envelope of $S(a_m)$ in $\operatorname{mod} A'$. Let $U_l (1 < l < m - 1)$ denote the uniserial A'-module of length two having $S(a_l)$ as a top, and $S(a_{l+1})$ as a socle. Then we have a path in $\operatorname{mod} A'$

$$I'(a_m) \to S(a_{m-1}) \to U_{m-2} \to S(a_{m-2}) \to \dots \to U_2 \to S(a_2) \to P'(a_1)$$

where the morphisms are the obvious ones. Since F is the support algebra of Σ and $a_1 \in F_0$, then there exist an indecomposable $V \in \Sigma_0$ and a non-zero morphism $P'(a_1) \to V$ in $\operatorname{mod} A'$ (and thus in $\operatorname{mod} A$). Similarly, there exist $W \in \Sigma_0$ and a non-zero morphism $W \to I'(a_m)$ in $\operatorname{mod} A'$ (and thus in $\operatorname{mod} A$). Both morphisms actually lie in $\operatorname{rad}^\infty (\operatorname{mod} A)$. Indeed, $a_2 \in \operatorname{Supp} P'(a_1)$ but $a_2 \notin F_0$ so that $P'(a_1)$ is not an F-module. In particular, $P'(a_1) \notin F_0$. Similarly, $I'(a_m) \notin \Gamma_0$. By [19], there exists, for each $t > 0$, a path in $\operatorname{mod} A$

$$W = W_0 \xrightarrow{g_1} W_1 \xrightarrow{g_2} W_2 \to \dots \to W_{t-1} \xrightarrow{g_t} W_t \xrightarrow{f_t} I'(a_m)$$

with the g_i irreducible and $f_t g_t \dots g_2 g_1 \neq 0$. Since Γ has finitely many τ-orbits, no periodic τ-orbits and no injectives, then there exist $s, t > 0$ and a path in Γ

$$V = V_0 \to V_1 \to \dots \to V_s = W_t.$$

We have thus constructed a cycle in $\operatorname{mod} A$

$$I'(a_m) \to S(a_{m-1}) \to U_{m-2} \to \dots \to S(a_2) \to P'(a_1) \to V =$$
$$V_0 \to \dots \to V_s = W_r \xrightarrow{f_t} I'(a_m)$$

where $f_t \in \operatorname{rad}^\infty (\operatorname{mod} A)$, a contradiction to the cycle-finiteness of A.

We shall now deduce that Σ is convex in mod F.

Let $W_0 \xrightarrow{h_1} W_1 \xrightarrow{h_2} \dots \to W_{m-1} \xrightarrow{h_m} W_m$ be a path in mod F such that $W_0, W_m \in \Sigma_0$ but $W_i \notin \Sigma_0$. Since F is convex in A, all the W_i are also A-modules and this path also lies in mod A. Since Σ is convex in Γ, there exists $1 \leqq j \leqq m$ such that $W_0, \dots W_{j-1} \in \Gamma_0$ but $W_j \notin \Gamma_0$. Then $h_j \in \operatorname{rad}^\infty(\operatorname{mod} A)$ and there exists, for each $t > 0$, a path in mod F

$$W_{j-1} = U_0 \xrightarrow{g_1} U_1 \xrightarrow{g_2} U_2 \to \dots U_{t-1} \xrightarrow{g_t} U_t \xrightarrow{f_t} W_j$$

with the g_j irreducible, $f_t \in \operatorname{rad}^\infty(\operatorname{mod} A)$ and $f_t g_t \dots g_2 g_1 \neq 0$. Since Γ has finitely many τ-orbits, no periodic τ-orbits and no injectives, there exist $s, t \geqq 0$ and a path in Γ

$$W_m = V_0 \to V_1 \to \dots \to V_s = U_t.$$

We have thus constructed a cycle in mod A

$$W_m = V_0 \to V_1 \to \dots \to V_s = U_t \xrightarrow{f_t} W_j \xrightarrow{h_{j+1}} \dots \to \mathrm{W}_m$$

with $f_t \in \operatorname{rad}^\infty(\operatorname{mod} A)$, a contradiction.

This completes the proof that Σ is convex in mod F. Since it clearly satisfies the remaining conditions for a complete slice, F is a tilted algebra. Since F is a full convex subcategory of A, it is a tame tilted algebra having a complete slice in a component Γ having finitely many τ-orbits, no periodic τ-orbits and no injectives. It then follows from [17] (4.1) (4.2) that the right end algebra of F is connected and is a tilted algebra of euclidean type with a complete slice isomorphic to Σ in the preprojective component. However, since Σ properly contains a full connected subquiver with the same underlying graph as the slice $\mathcal{S}$ of the preinjective component of Γ_B, then Σ is a wild slice.

This contradiction shows that $A = B$. Since A has a sincere indecomposable lying in a stable tube, and B is a tilted algebra of euclidean type having a complete slice in its preinjective component, then $B = A$ is tame concealed (by [21] (4.9)).

4.5. Corollary. *Let A be a multicoil algebra, and $\mathcal{T}$ be a stable tube of Γ_A. Then the support algebra of $\mathcal{T}$ is a full convex subcategory of A which is tame concealed or tubular, and has $\mathcal{T}$ as a full component.*

Proof. The support algebra B of $\mathcal{T}$ is a full subcategory of A which is, by (2.3) (b), also convex. By (2.3) (a), B is a multicoil algebra and obviously has the tube $\mathcal{T}$ as a full component, and a sincere indecomposable module in $\mathcal{T}$. By (4.1), B is tame concealed or tubular.

4.6. Theorem. *Let A be a multicoil algebra. Then A is of polynomial growth.*

Proof. We may obviously assume that A is representation-infinite. Since A is a multicoil algebra, it is tame. Let $\underline{d} \in K_0(A)$ be such that $\mu_A(\underline{d}) > 0$ and B denote the support algebra of $\underline{d}$. By [12], B has a homogeneous tube containing a sincere indecomposable module. By (4.5), B is a full convex subcategory of A which is tame concealed or tubular. In particular, $\mu_A(\underline{d}) = \mu_B(\underline{d}) \leqq \left(\sum_{i \in B_0} d_i\right)^{m_B}$ for some $m_B > 0$. Thus, let m denote the maximum of m_B when B ranges over all tame concealed or tubular full convex subcategories of A. Since A_0 is finite, so is the set of all such m_B. Then, for any $\underline{d} \in K_0(A)$, we have $\mu_A(\underline{d}) \leqq \left(\sum_{i \in A_0} d_i\right)^m$.

4.7. Corollary. *A multicoil algebra A is domestic if and only if it does not contain a tubular algebra as a full convex subcategory.*

Proof. If A contains a tubular algebra as full convex subcategory, it is not domestic (by [21] (5.2) or [23] (3.6)). Conversely, if A does not contain a tubular algebra as a full convex subcategory, then all full convex subcategories of A with sincere tubes are tame concealed algebras. It then follows from the proof of (4.6) that A is domestic.

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