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APPROXIMATION OF PLURISUBHARMONIC FUNCTIONS AND THE DIRICHLET PROBLEM FOR THE COMPLEX MONGE-AMPÈRE OPERATOR

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0. Introduction.

In section 1 of this paper, we show that every globally defined plurisubharmonic function can be approximated from above by functions that are plurisubharmonic and real analytic.

In section 2, we use the Perron method and approximation with bounds to study the Dirichlet problem for the complex Monge-Ampère operator.

By $\mathrm{PSH}(\Omega)$ we denote the cone of plurisubharmonic functions on Ω and by $A(\Omega)$ the real analytic function on Ω. The Lebesgue measure is denoted by dV.

1. Approximation by real analytic functions.

Let Ω be an open subset of $\mathbf{C}^n$, B the unit ball. To smooth (regularize) a plurisubharmonic function, a standard method is to pick a non-negative testfunction K with $\int_{\mathbf{C}^n} K\,dV = 1$ and consider the regularization u_δ defined by

$$(1)\qquad u_\delta(z) = \int u(z+\delta w)K(w)\,dV = \frac{1}{\delta^{2n}}\int u(w)K\left(\frac{w-z}{\delta}\right)dV, \quad \delta > 0.$$

Then u_δ is both plurisubharmonic and C^∞ on Ω_δ, where

$$\Omega_\delta = \{z \in \Omega,\ d(z, \mathbf{C}^n \backslash \Omega) > \delta\}$$

and $u_\delta(z) \searrow u(z)$, $\delta \searrow 0$ $\forall z \in \Omega$. If Ω is not equal to $\mathbf{C}^n$, then $\Omega_\delta \neq \Omega$ so u_δ is not globally defined, and there are examples of sets Ω where global approximation is impossible (cf Fornaess [8] and Cegrell [4], p. 321, Ex. 2).

However, if Ω is pseudoconvex, then plurisubharmonic functions on Ω can be approximated by C^∞-functions that are plurisubharmonic [12]. See also Fornaess and Wiegerinck [9].

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The global smoothing of plurisubharmonic functions is based on the following theorem.

THEOREM 1.1 [12] *Suppose $M \subset \mathbf{C}^n$ is a closed complex submanifold. Then, if u is any plurisubharmonic function on M, there is a plurisubharmonic function w on $\mathbf{C}^n$, such that $w|_M = u$.*

If M is a Stein manifold of dim n, then it can be imbedded in $\mathbf{C}^{2n+1}$ as a complex submanifold (cf. [10]). We can therefore consider every plurisubharmonic function on M as the trace of a plurisubharmonic function defined on $\mathbf{C}^{2n+1}$. So if we approximate a given plurisubharmonic w with a C^∞-function $w_\delta \in \mathrm{PSH} \cap C^\infty(\mathbf{C}^n)$ and then take the restriction $\omega_\delta|_M$, we get a global approximation of u with

$$u_\delta = w_\delta|_M \in \mathrm{PSH} \cap C^\infty(M).$$

Thus, Theorem 1 reduces to the question of global approximation of plurisubharmonic functions on $\mathbf{C}^n$. Here, we prove that every plurisubharmonic function on $\mathbf{C}^n$ can be approximated from above by plurisubharmonic and real-analytic functions.

THEOREM 1.2. *For every $u \in \mathrm{PSH}(\mathbf{C}^n)$, there exists a sequence of real-analytic, plurisubharmonic functions*

$$u_j \in \mathrm{PSH}(\mathbf{C}^n) \cap A(\mathbf{C}^n), \quad j = 1, 2, \ldots,$$

such that $u_j(z) \searrow u(z)$ for $j \geqq j_0(z)$, where $j_0(z)$ is a locally bounded $\mathbf{N}$-valued function.

PROOF. First we assume that

$$0 \leqq u(z) \leqq Me^{|z|^C}, \quad z \in \mathbf{C}^n, \quad \text{where} \quad M, C \text{ are constants}, C \in \mathbf{N}.$$

As a kernel function $K(z)$ we take

$$K(z) = \alpha e^{-|z|^{2C}}$$

where α is chosen so that $\int K(z)\,dV = 1$. We put

$$u_\delta(z) = \int u(z + w\delta)K(w)\,dV(w) = \frac{1}{\delta^{2n}} \int u(w)K\left(\frac{w-z}{\delta}\right) dV(w).$$

We note that $u_\delta(z)$ is plurisubharmonic and the integral on the right side is uniformly convergent not only on $E \subset\subset \mathbf{C}^n \approx \mathbf{R}^{2n}$, but on each compact $F \subset\subset \mathbf{R}^{2n} + \mathbf{R}^{2n}i$. In fact, let $z = (x_1, x_2, \ldots, x_{2n})$, $w = (\omega_1, \omega_2, \ldots, \omega_{2n})$ be coordinates on $\mathbf{R}^{2n}$.

Let $z^* = (y_1, y_2, \ldots, y_{2n})$. Then

$$K\left(\frac{w-z}{\delta}\right) = \alpha \cdot e^{-\frac{1}{\delta^{2C}}[(\omega_1 - x_1)^2 + \ldots + (\omega_{2n} - x_{2n})^2]^C}$$

is a real-analytic kernel function in $\mathbf{C}^n \approx \mathbf{R}^{2n}$. We have

$$K\left[\frac{w-(z+iz^*)}{\delta}\right] = \alpha e^{-\frac{1}{\delta^{2C}}[(\omega_1 - x_1 - iy_1)^2 + \ldots + (u_{2n} - x_{2n} - iy_{2n})^{2C}},$$

and

$$\begin{aligned}&[(\omega_1 - x_1 - iy_1)^2 + \ldots + (\omega_{2n} - x_{2n} - iy_{2n})^{2]C}\\ &= \{(\omega_1 - x_1)^2 + \ldots + (\omega_{2n} - x_{2n})^2 - y_1^2 - y_2^2 - \ldots - y_{2n}^2 - 2i[y_1(\omega_1 - x_1)\\ &+ \ldots + y_{2n}(\omega_{2n} - x_{2n})]\}^C\\ &= |w|^{2C}[1 + a(w,z,z^*)] + ib(w,z,z^*), |w| \geqq 1\end{aligned}$$

where a, b are real functions, such that for every compact $F \subset \mathbf{R}^{2n}(z) + i\mathbf{R}^{2n}(z^*)$ the norm $\|a(w,z,z^*)\|_F$ tends to zero as $w \to \infty$.

This shows that the integral

$$\int u(w)K\left(\frac{w-z}{\delta}\right) dV(w)$$

is convergent on every $F \subset\subset \mathbf{R}^{2n}(z) + i\mathbf{R}^{2n}(z^*)$ and defines a holomorphic function in $\mathbf{R}^{2n}(z) + i\mathbf{R}^{2n}(z^*)$. The restriction to $\mathbf{R}^{2n}(z)$ is then a real analytic function.

We have (cf. [11])

$$u_\delta(z) = \int u(z+\delta w)K(w)\,dV(w) = \int_0^\infty dt \int_{S_t(0)} u(z+\delta w)K(w)\,d\sigma(w)$$

$$= \alpha \int_0^\infty e^{-t^{2C}}\,dt \int_{S_t(0)} u(z+\delta w)\,d\sigma(w) = \alpha\sigma_{2n} \int_0^\infty e^{-t^{2C}} t^{2n-1} \mathcal{M}_u(z, t\delta)\,dt,$$

where $S_t(0)\colon |z| = t$ is a sphere with radius t,

$$\mathcal{M}_u(z,\delta) = \frac{1}{\sigma_{2n}\delta^{2n-1}} \int_{S_\delta(0)} u(z+\xi)\,d\sigma(\xi)$$

is the spherical mean of u on the sphere $|\xi - z| = \delta$, and σ_{2n} denotes the area of the unit sphere in $\mathbf{C}^n$. Since $\mathcal{M}_u(z,\delta) \searrow u(z)$, if $\delta \searrow 0$ (cf. [11]) u_δ is decreasing in δ and

$$\lim_{\delta \to 0} u_\delta = u(z)\alpha\sigma_{2n} \int_0^\infty t^{2n-1} e^{-t^{2C}} dt = u(z).$$

which completes the proof in our case.

To complete the proof of the theorem we can assume $u(z) \geqq 0$ (we can take $\max\{u(z), -M\} + M$ and then $M \to \infty$).

We put $v_R(z) = M_{eR} \cdot \ln \frac{|z|}{R}$, where $M_R > \sup_{|z| \leqq R} u(z)$ is some constant. Then

$$w_R(z) = \frac{1}{R} + \begin{cases} \sup\{v_R(z), u(z)\}, & \text{if } |z| \leqq eR \\ v_R(z), & \text{if } |z| > eR \end{cases}$$

is plurisubharmonic in $\mathbf{C}^n$ and $w_R(z) = u(z) + \frac{1}{R}$ in $|z| \leqq R$.

We note that $w_R(z)$ satisfy the above growth condition. And we can construct $u_j^{(R)}(z) \searrow w_R(z), j \to \infty$, where $u_j^{(R)}(z)$ are real-analytic and plurisubharmonic functions. Using $u_j^{(R)}(z), j, R = 1, 2, \ldots$, it is not difficult to construct a sequence $u_j^{R_j}(z)$ so that $u_j^{R_j}(z) \to u(z)$ for every $z \in \mathbf{C}^n$, and for j large enough $u_j^{(R_j)}(z) \searrow u(z)$.

COROLLARY. *For every pseudoconvex domain D there exists a sequence of strictly pseudoconvex domains $D_j \subset D_{j+1}$, with real-analytic boundary so that $D = \bigcup_{j=1}^{\infty} D_j$.*

2. The Dirichlet problem for the complex Monge-Ampère operator.

Let Ω be an open and bounded subset of $\mathbf{C}^n$. If $u_j \in C^2(\Omega)$, $1 \leqq j \leqq n$, then the Monge-Ampère operator operates on $(u_1, \ldots, u_n)$;

$$MA(u_1, \ldots, u_n) = dd^c u_1 \wedge \ldots \wedge dd^c u_n \quad \text{where} \quad d = \partial + \bar{\partial} \quad \text{and} \quad d^c = i(\bar{\partial} - \partial).$$

It was shown in [2] that MA is well-defined on $(L^\infty_{\text{loc}} \cap \text{PSH})^n$ and $MA \geqq 0$.

Suppose that Ω is strictly pseudoconvex, that $h \in C(\partial\Omega)$ and that μ is a positive measure on Ω. We are then interested in the Dirichlet problem:

$$\text{(i)} \qquad \begin{aligned} &\varphi \in \text{PSH} \cap L^\infty(\Omega) \\ &(dd^c \varphi)^n = \mu \quad \text{on} \quad \Omega \\ &\lim_{z \to \xi} \varphi(z) = h(\xi), \quad \forall \xi \in \partial\Omega. \end{aligned}$$

Here, we face several problems. It was proved in [2] that if $\mu = (\mathrm{dd}^c\varphi)^n$ for some $\varphi \in L^\infty_{\text{loc}} \cap \text{PSH}$ then μ vanishes on pluripolar sets. But there are more restrictions on μ for (i) to have a solution. This is a consequence of the following lemma.

LEMMA 2.1. *Let B be the unit ball, $R > 1$. Then there is a constant c such that*

$$\int_B -u(dd^c\varphi)^n \leqq c \int_B -u\,dV, \quad \forall 0 \geqq u \in \mathrm{PSH}(RB),$$

$$\forall \varphi \in \mathrm{PSH}(RB), \quad -1 \leqq \varphi \leqq 0.$$

PROOF. See [5, p. 57].

If we take u to be a negative and plurisubharmonic function on RB, unbounded on B, then there is a $f \in L^1(RB)$ with $\int_B uf\,dV = -\infty$ since the dual of L^1 is L^∞.

It follows now from Lemma 2.1 that there is no $\varphi \in \mathrm{PSH} \cap L^\infty(RB)$ with $(dd^c\varphi)^n \geqq f\,dV$.

Since Ω is strictly pseudoconvex, there is always a function $\varphi \in \mathrm{PSH} \cap C(\bar{\Omega})$ with $\varphi = h$ on $\partial\Omega$, but another difficulty in solving (i) comes from the following: Consider the class

$$B(h, \mu) = \{\varphi \in \mathrm{PSH} \cap L^\infty_{\mathrm{loc}}(\Omega);\ \ (dd^c\varphi)^n \geqq \mu\colon \overline{\lim_{z\to\xi}}\, \varphi(z) \leqq h(\xi)\,\forall \xi \in \partial\Omega\}.$$

Then $B(h, \mu)$ may be non-empty, but containing no element with $\lim_{z\to\xi} \varphi(z) = h(\xi)$, $\forall \xi \in \partial\Omega$.

EXAMPLE. Fix $\xi_0 \in \partial\Omega$ and choose $z^j \in \Omega, j \in \mathsf{N}$ so that ξ_0 is the only limit point of $(z^j)_{j=1}^\infty$. Choose $r_j > 0$ so that the balls $B(z^j, r_j)$ are pairwise disjoint and contained in Ω.

Furthermore, we choose $0 < s_j < r_j$ so small that

$$h_j(z) \geqq \frac{1}{2^j} \quad \text{on} \quad \Omega \backslash B(z^j, r^j)$$

where $h_j(z) = \sup\{\varphi \in \mathrm{PSH}(\Omega); 0 \geqq \varphi, \varphi|_{B(z^j, s_j)} = -1\}$. Then $H = \sum_{j=1}^\infty h_j \in \mathrm{PSH}(\Omega)$, $-2 \leqq H \leqq 0$ and $\overline{\lim_{z\to\xi}}\, H(z) = 0$, $\forall \xi \in \partial\Omega$. But if $\varphi \in \mathrm{PSH} \cap L^\infty_{\mathrm{loc}}(\Omega)$, $\varphi \leqq 0$ and $(dd^c\varphi)^n \geqq (dd^cH)^n$ then $\varphi \leqq h_j\ \forall j \in \mathsf{N}$ by the comparison principle (cf. [1], [2] or [5]) which means that $\underline{\lim}_{z\to\xi}\, \varphi(z) \leqq -1$.

Since $\operatorname{supp}(dd^c h_j)^n = \partial B(z^j, s_j)$ and since each h_j is continuous we can smooth h_j near $\partial B(z^j, s_j)$ so that $(dd^c\tilde{h}_j)^n$ is a compactly supported function for every j. Since $h_j = \tilde{h}_j$ near $\partial\Omega$ then $\int (dd^c h_j)^n = \int (dd^c \tilde{h}_j)^n$ which is small if s_j is small; if we chose s_j so small that $\int dd^c \tilde{h}_j^n \leqq \frac{1}{2^j}$, then $\tilde{H} = \sum \tilde{h}_j \in \mathrm{PSH}(\Omega)$, $-2 \leqq \tilde{H} \leqq 0$, $(dd^c\tilde{H})^n \geqq \sum (dd^c\tilde{h}_j)^n = f\,dV$ where $f \in L^1(\Omega)$. As above we see that if

$\varphi \in \mathrm{PSH} \cap L^{\infty}_{\mathrm{loc}}(\Omega)$; $0 \geqq \varphi$ and $(dd^c\varphi)^n \geqq f\,dV$ then $\varphi \leqq \tilde{h}_j$, $\forall j$ and so $\underline{\lim}_{z \to \xi_0} \varphi(z) \leqq -1$ while $\overline{\lim}_{z \to \xi} \tilde{H}(z) = 0$ $\forall \xi \in \partial\Omega$.

In the positive direction, it was shown in [1], that if $\mu = f\,dV, 0 \leqq f \in C(\bar{\Omega})$ then (i) has a unique solution $u \in \mathrm{PSH} \cap C(\bar{\Omega})$. This was extendedin [6]: If $\mu =$ f dV, $0 \leqq f \in L^{\infty}(\Omega)$ then (i) has a unique solution $u \in PSH \cap L^{\infty}(\Omega)$.

Here we prove:

THEOREM 2.1. *Assume that* $0 \leqq f \in L^1_{\mathrm{loc}}(\Omega)$ *and that there exists a function* $w \in \mathrm{PSH} \cap L^{\infty}(\Omega)$ *with* $(dd^cw)^n \geqq f\,dV$.

Then, for every $h \in C(\partial\Omega)$ *there exists a function* $\varphi \in \mathrm{PSH} \cap L^{\infty}(\Omega)$ *with* $(dd^c\varphi)^n = f\,dV$ *and* $\overline{\lim}_{z \to \xi} \varphi(z) = h(\xi)$, $\forall \xi \in \partial\Omega$.

PROOF. Put $\mu_M = \inf(f, M)\,dV$. By the above remark there exists $u_M \in \mathrm{PSH} \cap L^{\infty}(\Omega)$ with $\lim_{z \to \xi} u_M(z) = h(\xi)$, $\forall \xi \in \partial\Omega$ and $(dd^cu_M)^n = \inf(f, M)\,dV$. If $w \in \mathrm{PSH} \cap L^{\infty}_{\mathrm{loc}}(\Omega)$ with $\overline{\lim}_{z \to \xi} w(z) \leqq h(\xi)$ $\forall \xi \in \partial\Omega$ and $(dd^cw)^n \geqq f\,dV$ it follows from the comparison principle that $w \leqq u_M$ and that $\{u_M\}_{M=1}^{\infty}$ is a decreasing sequence of functions. Therefore, $u = \lim u_M \in \mathrm{PSH} \cap L^{\infty}(\Omega)$, $(dd^cu)^n = f\,dV$(cf. [1], [2] or [5]) and $\overline{\lim}_{z \to \xi} u(z) \leqq h(\xi)$, $\forall \xi \in \partial\Omega$.

It remains to prove that

$$\overline{\lim_{z \to \xi}}\, u(z) = h(\xi),\ \forall \xi \in \partial\Omega.$$

To get a contradiction, assume that $\overline{\lim}_{z \to \xi_0} u(z) = h(\xi_0) - \varepsilon$ for a point $\xi_0 \in \partial\Omega$ and a $\varepsilon > 0$. Then there is a ball $B(\xi_0, r)$ so that $u(z) \leqq h(\xi) - \dfrac{\varepsilon}{2}$ on $\partial\Omega \cap B(\xi_0, r)$ so we can find a continuous function τ; $0 \leqq \tau \leqq \dfrac{\varepsilon}{2}$ on $\partial\Omega$ with support in $\partial\Omega \cap B(\xi_0, r)$ with $\tau(\xi_0) = \dfrac{\varepsilon}{2}$. We then solve the Dirichlet problem (i) for a function $0 \leqq \psi \in \mathrm{PSH} \cap C(\bar{\Omega})$ with $\mu \equiv 0$ and $h = \tau$. Then $u + \psi \in \mathrm{PSH} \cap L^{\infty}(\Omega)$, $(dd^c(u + \psi))^n \geqq (dd^cu)^n = f\,dV$ and $\overline{\lim}_{z \to \xi} (u(z) + \psi(z)) \leqq h(\xi), \forall \xi \in \partial\Omega$. We have proved that then $u(z) + \psi(z) \leqq u_M(z)$, for every M which is a contradiction since $\psi > 0$. The proof is complete.

REMARK 1. It was proved in [7], that if $\mu = f\,dV, f \in L^2(\Omega)$ then (i) has a unique solution. We do not know if this is true in general for $f \in L^p(\Omega)$; $p > 1$.

REMARK 2. There exists a strictly pseudoconvex domain Ω with real-analytic boundary, a real-analytic function $h \in \partial\Omega$ such that the solution u to the Dirichlet problem (i) is not real-analytic.

For let first U be the unit disc in the plane $h(\not\equiv 0)$ a real analytic and non-negative function on ∂U with at least three zeros. Denote by $V(z) = \sup\{v(z) \in CVX(U);\ \overline{\lim}_{z \to \xi} v(z) \leqq h(\xi)\ \forall \xi \in \partial U\}$. Then V is convex with boundary values h. Since V vanishes on an open set, V cannot be real analytic. Since V is convex, through every point $(z_0, V(z_0))$ in the graph of V, there is at least one tangent plane H below (or in) the graph. Since V is also maximal, the convex hull of $\{\xi \in \partial U;\ h(\xi) = H(\xi)\}$ must contain z_0. Therefore through every point $z \in U$, there is a line segment on which V is affine.

If we now consider the strictly pseudoconvex domain $\Omega = \{(z, w) \in \mathbb{C}^2;\ (\log|z|^2)^2 + (\log|w|^2)^2 < 1\}$, then $\tilde{h}(z, w) = h(\log|z|^2, \log|w|^2)$ is real analytic on $\partial\Omega$, $\tilde{V}(z, w) = V(\log|z|^2, \log|w|^2)$ is continuous on $\bar{\Omega}$, with boundary values $\tilde{h}$, plurisubharmonic on Ω with vanishing Monge-Ampère. But $\tilde{V}$ is not real analytic.

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