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HA-PLITZ OPERATORS BETWEEN MOEBIUS INVARIANT SUBSPACES

GENKAI ZHANG

§0. Introduction.

Let D be the unit disk in the complex plane equipped with the Lebesgue measure dm. For $\alpha > -1$, denote by $d\mu_\alpha$ the weighted measure $(1-|z|^2)^\alpha\, dm(z)$. The Moebius group SU(1, 1) acts unitarily on the Hilbert space $L^2(D, d\mu_\alpha)$ via

$$U_\phi^{(\alpha+2)}: f(z) \to f(\phi z)\phi'(z)^{\frac{\alpha+2}{2}}, \quad \phi \in \mathrm{SU}(1,1). \tag{0.1}$$

This gives a (projective) representation of the group SU(1, 1) (for α not integer, a genuine representation of the universal covering group of SU(1, 1)). For α not an odd integer, the corresponding irreducible decomposition of the representation and Plancherel formula were given in [10]. It is proved that there are $\left[\frac{\alpha+1}{2}\right] + 1$ discrete irreducible components of the group action. We denoted these spaces by $A_l^{\alpha,2}(D)$, where $l = 0, 1, \ldots, \left[\frac{\alpha+1}{2}\right]$; $A_0^{\alpha,2}(D)$ is just the weighted Bergman space $A^{\alpha,2}(D)$. The group action (0.1) on $A_l^{\alpha,2}(D)$ is equivalent to $U^{(\alpha+2-2l)}$ on $A^{\alpha-2l,2}(D)$. These spaces are eigenspaces of invariant Laplace operators with eigenvalues in the discrete spectrum. They can also be defined via certain iterated Cauchy-Riemann operators (see below). In this paper, we are going to study a certain kind of "higher weight" Ha-plitz operators from $A^{\alpha,2}(D)$ to $A_l^{\alpha,2}(D)$. These operators constitute the l many irreducible components of $\overline{A^{\alpha,2}(D)} \otimes A_l^{\alpha,2}(D)$ viewed as the space of Hilbert-Schmidt operators between these two spaces. We develop the Schatten-von Neumann properties of these operators.

In the companion paper [14] concrete realizations of discrete parts of tensor products of two holomorphic discrete series representations are obtained via the analysis of the Casimir operators. Its connection with the present paper is indicated in §3.

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Let us describe our result in some detail. The proof is given in §§1–2. Denote $\bar{D}$ the invariant Cauchy-Riemann operator introduced in [10],

$$\bar{D} = (1 - |z|^2)^2 \frac{\partial}{\partial \bar{z}}.$$

Thus $f \in A^{\alpha,2}(D)$ if and only if $f \in L^2(D, d\mu_\alpha)$, $\bar{D}^{l+1} f = 0$, and $f \perp \operatorname{Ker} D^l$. Let $\alpha > 1$ and not an odd integer, so that there are $\left[\frac{\alpha+1}{2}\right] + 1 \geqq 2$ discrete parts in the decomposition. For $1 \leqq s \leqq l \leqq \left[\frac{\alpha+1}{2}\right]$, we define the operator $H_b^{l,s}$ from $A^{\alpha,2}(D)$ to $A_l^{\alpha,2}(D)$ by the bilinear formula

$$\langle H_b^{l,s} f, g \rangle = \int_D f(z)\overline{b(z)\bar{D}^s g(z)}\, d\mu_\alpha(z). \tag{0.2}$$

Let S_p be the Schatten-von Neumann class of operators on a Hilbert space, $0 < p < \infty$, and S_∞ be the space of bounded operators. We are going to study the S_p-properties of the operator $H_b^{l,s}$. We note here that when $l = s = 0$, our operator becomes the usual Toeplitz operator; so for b an non-zero analytic function, it can never be a compact operator. However, if $l > 0$, $s = 0$, they are part of the usual big Hankel operator (see [1]) so there are plenty of compact ones. So we can speak of Ha-plitz operators (Nikol'skii).

We also recall the definition of the analytic Besov spaces B_p^t, where $0 < p < \infty$ and $-\infty < t < \infty$. Let $m > t$ be an nonnegative integer. Then B_p^t consists of all analytic functions on D such that

$$f^{(m)}(z)(1 - |z|^2)^{m-t} \in L^p((1 - |z|^2)^{-1} dm(z)).$$

The main result of the paper is the following:

THEOREM. *Let b be analytic. If $\frac{1}{s} < p \leqq \infty$, then the operator $H_b^{l,s}$ is in the Schatten-von Neumann class S_p if and only if $b \in B_p^{\frac{1}{p} - s}$. For $p \leqq \frac{1}{s}$, $H_b^{l,s}$ is in S_p only if $b = 0$.*

To prove our theorem, we use the decomposition theorems of Coifman-Rochberg [3] and the methods developed by Peller [11]. We also note that if we map the unit disk to the upper half plane and perform the Fourier transform, the operators here become some kind of paracommutators. The integral kernels are not of the type studied in [12].

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§1. The cut-off and the boundedness.

Let $U^{(\alpha+2)}$ be the action of SU(1, 1) on $C^\infty(D)$ defined by (0.1). First we study invariance properties of the operator $H_b^{l,s}$. For simplicity, we write it as H_b in this section.

From the known intertwining properties of $\bar{D}$ (see [10]) we see that

$$\bar{D}^s U_\phi^{(\alpha+2)} = U_\phi^{(\alpha+2-2s)} \bar{D}^s.$$

Therefore by the definition of H_b,

$$\begin{aligned}\langle H_b U_{\phi^{-1}}^{(\alpha+2)} f, U_{\phi^{-1}}^{(\alpha+2)} g\rangle &= \langle U_{\phi^{-1}}^{(\alpha+2)} f, b\bar{D}^s U_{\phi^{-1}}^{(\alpha+2)} g\rangle \\ &= \langle U_{\phi^{-1}}^{(\alpha+2)} f, bU_{\phi^{-1}}^{(\alpha+2-2s)} \bar{D}^s g\rangle \\ &= \langle f, (U_\phi^{(2s)} b) \bar{D}^s g\rangle,\end{aligned}$$

where the last equality is simply obtained by change of variables. Thus we have

$$(U_{\phi^{-1}}^{(\alpha+2)})^* H_b U_{\phi^{-1}}^{(\alpha+2)} = H_{U_\phi^{(2s)} b}. \tag{1.1}$$

By the Arazy-Fisher theory of Moebius invariant spaces of analytic functions (see, e.g., [1]) we see that

$$\|H_b\|_2^2 = c\|b\|^2_{B_2^{\frac{1}{2}-s}}.$$

where c is either a constant or ∞. So for the S_2-result of T_b, we need only calculate its S_2 norm for a special symbol. To this end we now calculate the singular numbers of the operator H_z on $A^{\alpha,2}(D)$.

The space $A_l^{\alpha,2}(D)$ has the following orthonormal basis $\{e_n^{(l)}\}_{n\geq -l}$ (see [10]),

$$e_n^{(l)}(z) = c_l \sqrt{\frac{(l+1)_n}{(\alpha+2-l)_n}}\, p_{ln}\left(\frac{|z|^2}{1-|z|^2}\right) z^n,$$

where

$$p_{ln}(t) = \frac{(l+n)!}{n!} F(-l, l-\alpha-1; n+1, -t),$$

and we have used the Pochhammer symbol

$$(a)_n = a(a+1)\dots(a+n-1),$$

and $F(a, b;\ c; t) = {}_2F_1(a, b; c; t)$ is the hypergeometric function. In particular,

$e_n(z) = e_n^{(0)}(z) = c_0 \sqrt{\frac{n!}{(\alpha+2)_n}} z^n$ is the standard orthonormal basis of $A^{\alpha,2}(D)$.

Let $b = \sum_{j=0}^{\infty} \hat{b}(j) z^j$ be an analytic function. We calculate the matrix elements of the operator H_b. Since

$$\bar{D}\left(\frac{|z|^2}{1-|z|^2}\right)^r = r\left(\frac{|z|^2}{1-|z|^2}\right)^{r-1} z,$$

we have

$$\bar{D}^s p_{lm}\left(\frac{|z|^2}{1-|z|^2}\right) z^m = p_{lm}^{(s)}\left(\frac{|z|^2}{1-|z|^2}\right) z^{m+s}.$$

Hence, $\langle H_{z^j} e_n, e_{lm} \rangle = 0$, if $n \neq m + s + j$. Therefore

$$\langle H_b e_n, e_{lm} \rangle = C\overline{\hat{b}(n-m-s)} \sqrt{\frac{(l+1)_n}{(\alpha+2-l)_n}} \sqrt{\frac{n!}{(\alpha+2)_n}} \times$$
$$\times \int_D |z^n|^2 p_{lm}^{(s)}\left(\frac{|z|^2}{1-|z|^2}\right) d\mu_\alpha(z).$$

The integral in the above formula is

$$\pi \int_D p_{lm}^{(s)}\left(\frac{r}{1-r}\right) r^n (1-r)^\alpha \, dr.$$

Changing variables, $t = \frac{r}{1-r}$, we can write it as

$$\int_0^\infty p_{lm}^{(s)}(t) \frac{t^n}{(1+t)^{\alpha+2+n}} \, dt.$$

This integral again can be expressed in terms of gamma functions. We find that it is

$$\frac{(-l)_s (l-\alpha-1)_s \Gamma(n+1)\Gamma(\alpha+1)}{(-1)^s (m+1)_s \Gamma(\alpha+2+n)} {}_3F_2(-l+s, l-\alpha-1+s, n+1;\, m+1+s, -\alpha; 1),$$

where ${}_3F_2$ is the generalized hypergeometric function, see Erdelyi [4], vol. 1. Therefore the matrix elements of H_b are

$$(1.2) \qquad \langle H_b e_n, e_m^l \rangle = \overline{\hat{b}(n-m-s)} r_n \rho_m \times$$
$$\times\ {}_3F_2(-l+s, l-\alpha-1+s, n+1;\, m+1+s, -\alpha; 1),$$

where

$$r_n = C\left(\frac{\Gamma(n+1)}{\Gamma(n+\alpha+2)}\right)^{\frac{1}{2}} \approx n^{-\frac{\alpha+2}{2}},$$

$$\rho_m = \rho_m(s) = \frac{(m+l)!}{m!(m+1)_s}\left(\frac{\Gamma(m+\alpha+2-l)}{\Gamma(m+1+l)}\right)^{\frac{1}{2}} \approx m^{\frac{\alpha+2}{2}-s},$$

and C is a constant.

In particular, we see the singular numbers of the operator H_{z^j} are obtained by rearranging the sequence $\{\langle H_{z^j} e_{m+j+s}, e_{lm}\rangle\}$. Moreover, from the definition, we know that

$$\begin{aligned} &{}_3F_2(-l+s, l-\alpha-1+s, m+j+s+1; m+1+s, -\alpha; 1). \\ &= \sum_{k=0}^{l-s} \frac{(-l+s)_k(l-\alpha-1+s)_k(m+j+s+1)_k}{(m+1+s)_k(-\alpha)_k\, k!} \end{aligned}$$

It follows that if $m \to \infty$, the limit of the above sum is

$$\begin{aligned} &{}_3F_2(-l+s, l-\alpha-1+s, 1; 1, -\alpha; 1) \\ &= {}_2F_1(-l+s, l-\alpha-1+s; -\alpha; 1) \\ &= \sum_{k=0}^{l-s} \frac{(-l+s)_k(l-\alpha-1+s)_k}{(-\alpha)_k}\frac{1}{k!} \end{aligned}$$

Since $1 \leqq s \leqq l \leqq \left[\frac{\alpha+1}{2}\right]$, the terms $(-l+s)_k$, $(l-\alpha-1+s)_k$, $(-\alpha)_k$ in the above expansion are < 0, and consequently the sum $\neq 0$.

Thus,

$$|\langle H^s_{z^j} e_{m+j+s}, e_{lm}\rangle| \approx m^{-s}.$$

Therefore the operator H_{z^j} is in the Schatten-von Neumann classe S_p if and only if $p > \frac{1}{s}$.

The invariance argument (see, e.g., Janson [5]) then implies that H^s_b is in S_p for $p < \frac{1}{s}$ only if $b = 0$.

Next, we will use the known Schur's test to study the boundedness property of the operators ([5]). First we recall the following fact ([5]).

LEMMA 1. *For $\beta > -1$ and $c > 0$, we have*

$$\int_D \frac{1}{|1-\bar{z}w|^{\beta+2+c}}\, d\mu_\beta(w) \approx (1-|z|^2)^{-c}.$$

The space $A^{\alpha,2}_l(D)$ has the reproducing kernel ([10])

$$K_l(z,w) = \frac{\alpha+2-2l}{\alpha+1} K(z,w)\, F\left(-l, l-\alpha-1; 1; -\frac{|z-w|^2}{(1-|z|^2)(1-|w|^2)}\right)$$

where $K(z,w) = \dfrac{\alpha+1}{\pi}(1 - z\bar{w})^{-(\alpha+2)}$ is the reproducing kernel of $A^{\alpha,2}(D)$. For convenience, we put

$$\mathscr{P}(z,w) = \frac{|z-w|^2}{(1-|z|^2)(1-|w|^2)}.$$

By the properties of the hypergeometric functions ([4], vol. 1, p. 58), we have

$$D_z^s F(-l, l-\alpha-1; 1; -\mathscr{P}(z,w))$$
$$= \frac{(-l)_s(l-\alpha-1)_s}{s!} \frac{(z-w)^s(1-z\bar{w})^s}{(1-|w|^2)^s} \times$$
$$\times\ F(-l+s, l-\alpha-1+s; 1+s; -\mathscr{P}(z,w)).$$

So for any $g \in A_l^{\alpha,2}(D)$,

$$\bar{D}^s g(z) = \int_D g(w) K(z,w)\, \bar{D}_z^s F(-l, l-\alpha-1; 1; -\mathscr{P}(z,w)) d\mu_\alpha(w)$$
$$= \frac{(-l)_s(1-\alpha-1)_s}{s!} \int_D g(w) K(z,w) \frac{(z-w)^s(1-z\bar{w})^s}{(1-|w|^2)^s} \times$$
$$\times\ F(-l+s, l-\alpha-1+s; 1+s;\ -\mathscr{P}(z,\omega)) d\mu_\alpha(w).$$

Substituting this equality to (0.2), we see that H_b is an integral operator on $L^2(D,\mu_\alpha)$ with the kernel

$$H_b(w,z_1) = \frac{(-l)_s(l-\alpha-1)_s}{s!} \int_D K(w,z)K(z,z_1)\hat{b}(z) \frac{(\bar{z}-\bar{w})^s(1-\bar{z}w)^s}{(1-|w|^2)^s}$$
$$\times\ F(-l+s, l-\alpha-1+s; 1+s;\ -\mathscr{P}(z,w))\, d\mu_\alpha(z).$$

That is

$$H_b f(w) = \int_D H_b(w,z_1) f(z_1) d\mu_\alpha(z_1), \quad f \in A^{\alpha,2}(D).$$

Now let $b \in B_\infty^{-s}$ and $\|b\|_{B_\infty^{-s}} = 1$, that is

$$|b(z)| \leqq (1-|z|^2)^{-s}.$$

Let us prove that this the boundedness of H_b. We define

$$L_p^s = \{f : f(z)(1-|z|^2)^{-s} \in L^p(D,(1-|z|^2)^{-1}\, dm(z))\}.$$

We have

$$\int_D |H_b(w,z_1)|(1-|z_1|^2)^{-\frac{\alpha+1}{2}}\,d\mu_\alpha(z_1)$$
$$\leqq C\sum_{i=0}^{l-s}\int_D\int_D (1-|z|^2)^{-s}|K(w,z)K(z,z_1)|\frac{|z-w|^s|1-\bar{z}w|^s}{(1-|w|^2)^s}\times$$
$$\times\,\mathscr{P}(z,w)^i(1-|z_1|^2)^{-\frac{\alpha+1}{2}}\,d\mu_\alpha(z_1)d\mu_\alpha(z)$$
$$\leqq C\sum_{i=0}^{l-s}\int_D (1-|z|^2)^{-s}|K(w,z)K(z,w)\frac{|z-w|^s|1-\bar{z}w|^s}{(1-|w|^2)^s}(1-|z|^2)^{-\frac{\alpha-1}{2}}\,d\mu_\alpha(z),$$

where the last inequality is obtained from Lemma 1. Using the fact that $|z-w|\leqq|1-z\bar{w}|$ and the same lemma, we then see that

$$\int_D |H_b(w,z_1)|(1-|z_1|^2)^{-\frac{\alpha+1}{2}}\,d\mu_\alpha(z_1)\leqq C(1-|w|^2)^{-\frac{\alpha+1}{2}}. \tag{1.3}$$

Similarly we have

$$\int_D |H_b(w,z)|(1-|w|^2)^{-\frac{\alpha+1}{2}}\,d\mu_\alpha(w)\leqq C(1-|z|^2)^{-\frac{\alpha+1}{2}}. \tag{1.4}$$

Now if $f\in L_\infty^{-\frac{\alpha+1}{2}}$, and $\|f\|_{L_\infty^{-\frac{\alpha+1}{2}}}=1$, that is $|f(z)|\leqq(1-|z|^2)^{-\frac{\alpha+1}{2}}$. Then

$$|H_bf(w)|(1-|w|^2)^{\frac{\alpha+1}{2}}\leqq\int_D |H_b(w,z)|f(z)d\mu_\alpha(z)\cdot(1-|w|^2)^{-\frac{\alpha+1}{2}}$$
$$\leqq\int_D |H_b(w,z)(1-|z|^2)^{\frac{\alpha+1}{2}}\,d\mu_\alpha(z)\cdot(1-|w|^2)^{\frac{\alpha+1}{2}}$$
$$\leqq C,$$

where the last inequality is obtained from (1.3). Therefore $H_b:L_\infty^{-\frac{\alpha+1}{2}}\to L_\infty^{-\frac{\alpha+1}{2}}$ is bounded. Correspondingly (1.4) gives us $H_b:L_1^{-\frac{\alpha+1}{2}}\to L_1^{-\frac{\alpha+1}{2}}$ is bounded. By interpolation, we find that $H_b:L_p^{-\frac{\alpha+1}{2}}\to L_p^{-\frac{\alpha+1}{2}}$ is bounded. In particular, $H_b:A^{\alpha,2}(D)\to A_l^{\alpha,2}$ is bounded.

§2. The S_p properties, $p<1$.

Now we prove the S_p properties of the operators. We will only consider the case $p\leqq 1$. The results for $p\geqq 1$ follow more or less from the standard arguments (interpolation, duality), see [5], [2]. For an operator T acting between Hilbert spaces, we will denote by $\|T\|_p$ its norm in the Schatten-von Neumann class S_p.

Assume that $1 \geqq p > \frac{1}{s}$, and $b \in B_b^{\frac{1}{p}-s}$. We prove that $H_b^{l,s}$ is in S_p. By the atomic decomposition of the Besov space (Coifman-Rochberg [3]), we know that there exist $\{\xi_i\} \subset D$, $\{\lambda_i\} \subset \mathbf{C}$ and an integer $N > 2s + 1$ such that

$$b(z) = \sum_i \lambda_i (1 - |\xi_i|^2)^{N-s} (1 - \bar{\xi}_i z)^{-N}, \tag{2.1}$$

with

$$\sum_i |\lambda_i|^p < \infty.$$

However, by (1.1), we see that if ϕ is a Moebius transformation, then $H^{l,s}_{(1-\bar{\xi}_i z)^{-N}}$ is unitarily equivalent to the operator $H^{l,s}_{(1-\bar{\xi}_i\phi(z))^{-N}\phi'(z)^s}$. Take $\phi(z) = \frac{\xi_i - z}{1 - \bar{\xi}_i z}$ (ϕ is not in SU(1, 1), but this does not matter of course). Then

$$(1 - \xi_i\phi(z))^{-N}\phi'(z)^s = (1 - \bar{\xi}_i z)^{N-2}s(1 - |\xi_i|^2)^{-N+s}.$$

Since $(1 - \bar{\xi}_i z)^{N-2s}$ is a polynomial in z of degree $N - 2s$ and with uniformly bounded coefficients in i, we get

$$(1 - |\xi_i|^2)^{N-s} \| H^{l,s}_{(1-\bar{\xi}_i z)^{N-2s}} \|_{S_p} \leqq C < \infty.$$

Using (2.1) we then obtain

$$\|H_b^{l,s}\|_{S_p}^p \leqq C \sum_i |\lambda_i|^p < \infty.$$

That is $H_b^{l,b}$ is in S_p.

Now we prove the converse. Let $\mathscr{H}$ be a Hilbert space with an orthonormal basis $\{f_n\}_{n=0}^{\infty}$. It is sufficient to consider the following operator on $\mathscr{H}$, which we still denote by $H_b^{l,s}$, whose matrix elements in the basis $\{f_n\}_{n=0}^{\infty}$ are

$$\langle \mathrm{H}_b^{l,s} f_n, f_m \rangle = \hat{b}(n - m - s) r_n \rho_m \times \tag{2.2}$$
$$\times\, {}_3F_2(-1 + s, l - \alpha - 1 + s, n + 1; m + 1 + s, -\alpha; 1),$$

with $r_n \approx n^{-\frac{\alpha+2}{2}}$, $\rho_m \approx m^{\frac{\alpha+2}{2}-s}$. Our observation is that the matrix of $H_b^{l,s}$ is a lower triangular one. We can divide the matrix into sum of blocks, each of which is a Schur multiplier of the Toeplitz matrix. Therefore we can use the known technique developed by Peller [11] to find the desired S_p-estimate.

We recall here some basic facts about the Besov spaces and Toeplitz matrices.

For a function c on the unit circle ∂D we denote $\|c\|_p$ its norm in $L^p(\partial D)$. For

$t < 0$, the Besov space B_p^t has the following equivalent definition. Choose a function $\phi \in C_0^\infty(\mathsf{R})$ such that supp $\phi \subset \left[2 - \frac{1}{2^4}, 2^2 - \frac{1}{2^4}\right]$; $\phi \geqq 0$; $\frac{1}{2} \leqq \phi(x) \leqq 1$ if $x \in \left[2 - \frac{1}{2^5}, 2^2 - \frac{1}{2^3}\right]$ and $\phi(x) = 1$ if $x \in \left[2, 2^2 - \frac{1}{2^3}\right]$. Define functions ϕ_n on ∂D by $\phi_n(z) = \sum_{i=0}^\infty \phi\left(\frac{i}{2^n}\right) z^i$. An analytic function b on ∂D is then in the Besov space B_p^t if and only if

$$\sum_{n=0}^{\infty} 2^{tpn} \|b * \phi_n\|_p^p < \infty.$$

See [7], [8].

The following lemma can be found in Peller [11]

Lemma 2. *Let $c(z) = \sum_{i=0}^{n-1} \hat{c}(i) z^i$ be a polynomial and T be the following $n \times n$ matrix:*

$$\begin{pmatrix} \hat{c}(0) & \hat{c}(n-1) & \dots & \hat{c}(1) \\ \hat{c}(1) & \hat{c}(0) & \dots & \hat{c}(2) \\ & \dots & \dots & \\ \hat{c}(n-1) & \hat{c}(n-2) & \dots & \hat{c}(0). \end{pmatrix}$$

Then we have $\|T\|_p \approx n^{\frac{1}{p}} \|c\|_p$.

For two matrices T and S, we denote $T * S$ their Schur product (pointwise multiplication of the matrix elements). The following Lemma can be proved by the same method as Lemma 4 in Peller [11]. We omit the proof.

Lemma 3. *Let $c(z) = \sum_{i=0}^{n-1} \hat{c}(i) z^i$ and denote T be the following $n \times n$ Toeplitz matrix with symbol c,*

$$\begin{pmatrix} \hat{c}(0) & 0 & \dots & 0 \\ \hat{c}(1) & \hat{c}(0) & \dots & 0 \\ & \dots & \dots & \\ \hat{c}(n-1) & \hat{c}(n-2) & \dots & \hat{c}(0). \end{pmatrix}$$

Then for any $n \times n$ matrix A, we have

$$\|T * A\|_p \leqq C n^{\frac{1}{p} - 1} \|c\|_p \|A\|_p.$$

To simplify the notation, for a polynomial c, we will use the same notation for the Toeplitz operator T_c on $\mathscr{H}$ with symbol c and its matrix. For a subset S of $\mathbf{Z}^+$, we denote P_s the projection from $\mathscr{H}$ onto span $\{f_j, j \in S\}$.

LEMMA 4. *If $k > k_0$, then we have*

$$\|P_{[2^{k-4},2^{k-1}]} H_b^{l,s} P_{[2^{k+1}+s,2^{k+4}+s]}\|_p^p \geqq C2^{k(1-ps)} \|b * \phi_k\|_p^p,$$

where k_0 does not depend on b.

PROOF. First we prove the Lemma for $s = l$. We denote $I_k = [2^{k+1} + l, 2^{k+2} + l]$, $I'_k = [2^{k-4}, 2^{k-4} + 2^{k+2} - 2^{k+1}]$, and $P_k = P_{I_k}$, $P'_k = P_{I'_k}$. Let $c(z) = \phi_k * b(z)$. Define two operators R and S on H as follows,

$$Rf_j = \begin{cases} r_j^{-1} f_j, & \text{if } j \in I_k \\ 0, & \text{if } j \notin I_k, \end{cases}$$

$$Sf_j = \begin{cases} \rho_j^{-1} f_j, & \text{if } j \in I'_k \\ 0, & \text{if } j \notin I'_k. \end{cases}$$

Then we see that

$$\|S\|^p \|R\|^p \leqq C2^{ksp}.$$

We easily check that

$$P'_k T_c P_k = SP'_k H^{l,l}_{\phi_k * b} P_k R = (SP'_k H_b^{l,l} P_k) * (P_k T_{\phi_k} R). \tag{2.3}$$

Since $P_k T_{\phi_k} P_k$ is a Toeplitz matrix, by Lemma 3 we obtain

$$\|P'_k T_c P_k\|_p^p \leqq C2^{k(1-p)} \|\phi_k\|_p^p \|S\|^p \|R\|^p \|P'_k H_b^{l,l} P_k\|_p^p.$$

Noticing that $\|\phi_k\|_p^p \leqq C2^{k(p-1)}$, we get

$$\|P'_k T_c P_k\|_p^p \leqq C2^{klp} \|P'_k H_b^{l,l} P_k\|_p^p. \tag{2.4}$$

Let $Q^{(1)}$ be the following matrix

$$\begin{pmatrix} \hat{c}(2^{k+2} - 2^{k-4}) & \hat{c}(2^{k+2} - 2^{k-4} - 1) & \dots & \hat{c}(2^{k+1} - 2^{k-4} + 1) \\ 0 & \hat{c}(2^{k+2} - 2^{k-4}) & \dots & \hat{c}(2^{k+1} - 2^{k-4} + 2) \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \hat{c}(2^{k+2} - 2^{k-4}) \end{pmatrix}.$$

Then similarly we can prove that

$$\begin{aligned} &\|Q^{(1)}\|_p^p \\ &\leqq C2^{klp} \|P_{[2^{k-4}+2^{k+2}-2^{k+1}-1,\, 2^{k-4}+2^{k+3}-2^{k+2}-1]} \\ &H_b^{l,l} P_{2^{k+3}-2^{k+1}+l,\, 3\cdot 2^{k+2}-2^{k+2}+l]}\|_p^p. \end{aligned} \tag{2.5}$$

Let us put

$$Q = \begin{pmatrix} \hat{c}(2^{k+2} - 2^{k-4}) & \hat{c}(2^{k+2} - 2^{k-4}) & \dots & \hat{c}(2^{k+1} - 2^{k-4} + 1) \\ \hat{c}(2^{k+1} - 2^{k-4} + 1) & \hat{c}(2^{k+1} - 2^{k-4}) & \dots & \hat{c}(2^{k+1} - 2^{k-4} + 2) \\ \dots & \dots & \dots & \dots \\ \hat{c}(2^{k+2} - 2^{k-4}) & \hat{c}(2^{k+2} - 2^{k-4} - 1) & \dots & \hat{c}(2^{k+2} - 2^{k-4}) \end{pmatrix}.$$

It follows that

$$Q = P_k' T_c P_k + Q^{(1)}.$$

We can apply Lemma 2 to Q and use (2.4), (2.5) to obtain

(2.6)

$$\begin{aligned} &2^{k(1-lp)}\|b * \phi_{2^k}\|_p^p \\ &= 2^{k(l-lp)}\|c\|_p^p \\ &\leqq C2^{-lp}\|Q\|_p^p \leqq C2^{-lp}(\|Q^{(1)}\|_p^p + \|P_k' T_c P_k\|_p \\ &\leqq C\|P_{[2^{k-4}+2^{k+2}-2^{k+1}, 2^{k-4}+2^{k+3}-2^{k+2}-1]} H_b^{l,l} P_{[2^{k+3}-2^{k+1}+l, 3 2^{k+2}-2^{k+2}+l]}\|_p^p. \end{aligned}$$

This proves the Lemma for the case $s = l$.

Next we prove this for case $s = l - 1$. The general is much the same. By (2.2), the matrix elements of $H_b^{l,l-1}$ are

$$\begin{aligned} \langle H_b^{l,l-1} f_n, f_m \rangle &= \hat{b}(n - m - s) r_n \rho_m \, {}_3F_2(-1, 2^l - 1, n + 1; m + l, -\alpha; 1) \\ &= \hat{b}(n - m - s) r_n \rho_m \left(1 + \frac{(n+1)(2l-1)}{(m+l)(-\alpha)}\right) \\ &= c_1^{-1} \hat{b}(n - m - s) r_n (n+1) \rho_m (m+l)^{-1} \left(1 + c_1 \frac{m+l}{n+1}\right), \end{aligned}$$

where $c_1 = \dfrac{-\alpha}{2l-1}$. Therefore if $n - m - l + 1 > k_0$, where k_0 is large enough,

$$\left(1 + c_1 \frac{m+l}{n+1}\right)^{-1} = \sum_{g=0}^{\infty} c_1^g \frac{(m+l)^g}{(n+1)^g},$$

with

$$\sum_{g=0}^{\infty} |c_1|^{gp} \frac{(m+l)^{pg}}{(n+1)^{pg}} < \infty. \tag{2.7}$$

Define operators R_g and S_g on $\mathscr{H}$, as follows

$$R_g f_j = \begin{cases} c_1^g (j+1)^{-g} f_j & \text{if } j \in [2^{k+1} + l - 1, 2^{k+2} + l - 1], \\ 0 & \text{if } j \notin [2^{k+1}, 2^{k+2}]. \end{cases}$$

$$S_g f_j = \begin{cases} (j+l)^g f_j, & \text{if } j \in [2^{k-4}, 2^{k-4} + 2^{k+2} - 2^{k+1}] \\ O; & \text{if } j \notin [2^{k-4}, 2^{k-4} + 2^{k+2} - 2^{k+1}] \end{cases}$$

Then, similarly to (2.3), we see that if $k > k_o$,

$$P_{[2^{k-4}, 2^{k-4}+2^{k+2}-2^{k+1}]} T_c P_{[2^{k+1}+l-1, 2^{k+2}+l-1]}$$

$$= \sum_{g=0}^{\infty} S_g S P^{(l)}_{[2^{k-4}, 2^{k-4}+2^{k+2}-2^{k+1}]} H^{l,l}_{\phi_{2^k} * b} P_{[2^{k+1}+l, 2^{k+2}+l]} T R_g.$$

Moreover by (2.7)

$$\sum_{g=0}^{\infty} \|S_g\|^p \|T_g\|^p < \infty.$$

Therefore we can get (2.4) (with l replaced by $l-1$, r_n by $r_n(n+1)$, ρ_m by $\rho_m(m+l)^{-1}$). Similarly we can prove (2.5). Finally (2.6) follows (with l replaced by $l-1$).

Now suppose $H_b^{l,s} \in S_p$ and $\frac{1}{s} < p \leqq 1$. Let us prove $b \in B_p^{\frac{1}{p}-s}$. To complete we have to estimate the norm $\|b\|^p_{B_p^{\frac{1}{p}-s}}$ in term of $\|H_b^{l,s}\|_p^p$. It suffices to consider the case when b is a polynomial $b(z) = \sum_{j=2^{2M}}^{\infty} \hat{b}(j) z^j$, where M is a large integer to be chosen later; see e.g. Peller [11].

Define

$$T^{(i)} = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} P_{[2^{jM+i-4}, 2^{jM+i-1}]} H_b^{l,s} P_{[2^{kM+i+1}+s, 2^{kM+i+4}+s]}, \; i = 0, 1, \ldots, M-1$$

We put

$$P(j,i) = P_{[2^{jM+i-4}, 2^{jM+i-1}]}, \; Q(k,i) = P_{[2^{kM+i+1}+s, 2^{kM+i+4}+s]}.$$

With these notations,

$$T^{(i)} = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} P(j,i) H_b^{l,b} Q(k,i).$$

Then for M large enough, say $M \geqq 2^{10}$, and for fixed i, we see that the projections in the definition of $T^{(i)}$ are jointly orthogonal. Hence

$$\|T^{(i)}\|_p^p \leqq \|H_b^{l,p}\|_p^p. \tag{2.7}$$

Moreover, since $H_b^{l,s}$ is a lower triangular matrix, we have

$$T^{(i)} = \sum_{k=0}^{\infty} \sum_{j=0}^{k} P(j,i) H_b^{l,s} Q(k,i), \; i = 0, 1, \ldots, M-1.$$

We split this matrix into diagonal and off diagonal parts,

$$T^{(i)} = T_1^{(i)} + T_2^{(i)}$$

with

$$T_1^{(i)} = \sum_{k=0}^{\infty} P(k,i)H_b^{l,s}Q(k,i),$$

and

$$T_2^{(i)} = \sum_{k=0}^{\infty}\sum_{j=0}^{k-1} P(j,i)H_b^{l,s}Q(k,i).$$

We notice that if $M > 10, n - s \in [2^{kM+i+1}, 2^{kM+i+4}], m \in [2^{i-4}, 2^{(k-1)M+i-1}]$, then $\phi_{kM+i} * \hat{b}(n-m-s)$. Therefore

$$\begin{aligned} &\|\sum_{j=0}^{k-1} P(j,i)H_b^{l,s}Q(k,i)\|_p^p \\ &= \|\sum_{j=0}^{k-1} P(j,i)H^{l,s}_{\phi_{2kM+i}*b}Q(k,i)\|_p^p \\ &\leqq \|P_{[2^{i-4},2^{(k-1)M+i-1}]}H^{l,s}_{\phi_{2kM+i}*b}Q(k,i)\|_p^p. \end{aligned}$$

Using (2.2), we see that the matrix $P_{[2^{i-4},2^{(k-1)M+i-1}]}H^{l,s}_{\phi_{2kM+i}*b}Q(k,i)$ is a Schur product of the Toeplitz matrix symbol $\phi_{2kM+i} * b$ and the operator A from $Q(k,i)\mathscr{H}$ to $P_{[2^{i-4},2^{(k-1)M+i-1}]}\mathscr{H}$ with matrix element

$$\begin{aligned} &r_n\rho_m {}_3F_2(-1+s, l-\alpha-1+s, n+1; m+1+s,, -\alpha; 1) \\ &= r_n\rho_m \sum_{d=0}^{l-s} \frac{(-1+s)_d(l-\alpha-1+s)_d(n+1)_d}{(m+1+s)_d(-\alpha)_d}\frac{1}{d!}. \end{aligned}$$

So A is a finite sum of rank one operators. Using the asympotics of the matrix elements, we find

$$\begin{aligned} \|A\|_p^p &\leqq C\sum_{d=0}^{l-s} 2^{(kM+i)p}2^{((k-1)M+i)(\frac{\alpha+2}{2}-s-d)p}2^{(kM+i)(-\frac{\alpha+2}{2}+d)p} \\ &\leqq C2^{-M(\frac{\alpha+2}{2}-l)p}2^{(kM+i)(1-s)p}. \end{aligned}$$

By Lemma 3, we then obtain

$$\begin{aligned} &\|P_{[2^{i-4},2^{(k-1)M+i-1}]}H^{l,s}_{\phi_{2kM+i}*b}Q(k,i)\|_p^p \\ &\leqq C2^{(kM+i)(l-p)}2^{-M(\frac{\alpha+2}{2}-l)p}2^{(kM+i)(1-s)p}\|\phi_{2kM+i}*b\|_p^p \\ &= C2^{-M(\frac{\alpha+2}{2}-l)p}2^{(kM+i)(1-ps)}\|\phi_{2kM+i}*b\|_p^p. \end{aligned}$$

Consequently,

$$\| T_2^{(i)} \|_p^p \leqq C_2 2^{-M(\frac{\alpha+2}{2}-l)p} \|b\|^p_{B_p^{p,\frac{1}{p}-s}}.$$

Now we give a lower bound for $\| T_1^{(i)} \|_p$. Clearly, by Lemma 4 we have

$$\begin{aligned} \| T_1^{(i)} \|_p^p &= \sum_{k=0}^{\infty} \| P(k,i) H_b^{l,s} Q(k,i) \|_p^p \\ &\geqq C_1 \sum_{k=0}^{\infty} 2^{(kM+i)(1-ps)} \| \phi_{kM+i} * b \|_p^p. \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{i=0}^{M} \| T^{(i)} \|_p^p &\geqq \sum_{i=0}^{M} \| T_1^i \|_p^p - \sum_{i=0}^{M} \| T_2^{(i)} \|_p^p \\ &\geqq C_1 \sum_{i=0}^{M} \sum_{k=0}^{\infty} 2^{(kM+i)(1-ps)} \| \phi_{kM+i} * b \|_p^p \\ &\qquad - C_2 M 2^{-M(\frac{\alpha+2}{2}-s)p} \|b\|^p_{B_p^{p,\frac{1}{p}-s}}. \\ &\geqq C_1 \|b\|^p_{B_p^{p,\frac{1}{p}-s}} - C_2 M 2^{-M(\frac{\alpha+2}{2}-s)p} \|b\|^p_{B_p^{p,\frac{1}{p}-s}}. \end{aligned}$$

Let M be large enough such that $M2^{-M(\frac{\alpha+2}{2}-s)p} C_2 < \frac{1}{2} C_1$. Finally using (2.7) we obtain that

$$\| H_b^{l,s} \|_p^p \geqq C \|b\|^p_{B_p^{p,\frac{1}{p}-s}}.$$

This finishes the proof.

§3. Final remarks.

In this section we clarify the relation between the operators $H_b^{l,s}$ and the invariant Toeplitz type operators obtained in [14]. First we note that $A_l^{\alpha,2}(D)$ and $A^{\alpha-2l,2}(D)$, as SU(1, 1)-module, are unitarily equivalent. More explicitly, the following operator

$$V : h \to \sum_{j=0}^{l} (-1)^{l-j} \left(\frac{\bar{z}}{1-|z|^2} \right)^{l-j} \frac{h^{(j)}(z)}{(\alpha+2-2l)_j},$$

from $A^{\alpha-2l,2}(D)$ to $A_l^{\alpha,2}(D)$ is the intertwining operator:

$$V U_\phi^{(\alpha+2-2l)} = U_\phi^{(\alpha+2)} V. \tag{3.1}$$

See [10].

We consider the operator $(H_b^{l,s})^* V$ from $A^{\alpha-2l,2}(D)$ to $A^{\alpha,2}(D)$. We claim that

$$(H_b^{l,s})^* V = \frac{(-l)_s(2s)_{l-s}}{(-\alpha)_{l-s}} T_b^{(\alpha+1-l+s)}, \tag{3.2}$$

where

$$T_b^{(\alpha+1-l+s)}: h \to \sum_{j=0}^{l-s} (-1)^j \binom{l-s}{j} \frac{b^{(j)}(z)}{(2s)} \frac{h^{(l-s-j)}(z)}{(\alpha+2-2l)_{l-s-j}} \tag{3.3}$$

is the invariant Toeplitz operator in [14].

$$(H_b^{l,s})^* V1 = P(b\bar{D}^s V1),$$

where P is the projection from $L^2(D, d\mu_\alpha)$ onto $A^{\alpha,2}(D)$. Since

$$V1(z) = (-1)^l \left(\frac{\bar{z}}{1-|z|^2}\right)^l,$$

and

$$\bar{D}^s \left(\frac{\bar{z}}{1-|z|^2}\right)^l = (-1)^s(-^l)_s \left(\frac{\bar{z}}{1-|z|^2}\right)^{l-s},$$

we easily see from the reproducing property that

$$(H_b^{l,s})^* V1 = P(b\bar{D}^s V1)(z) = \frac{(-1)^{l+s}(-l)_l}{(-\alpha)_s} b^{(l-s)}(z).$$

On the other hand, we see that

$$T_b^{(\alpha+1-l+s)} 1(z) = (-1)^{l-s} \frac{b^{l-s}(z)}{(2s)_{l-s}}.$$

Therefore

$$(H_b^{l,s})^* V1 = \frac{(-l)_s(2s)_{l-s}}{(-\alpha)_{l-s}} T_b^{(\alpha+1-l+s)} 1.$$

This proves our claim.

REFERENCES

1. J. Arazy, S. Fisher and J. Peetre, *Hankel operators on weighted Bergman spaces*, Amer. J. Math. 110 (1988), 989–1054.
2. J. Boman, S. Janson and J. Peetre , *Big Hankel operators of higher weights*, Rend. Circ. Mat. Palermo 38 (1989), 65–78.
3. R. Coifman and R. Rochberg, *Representation theorems for holomorphic and harmonic functions in* L^p, Asterisque 77 (1980), 11–86.

4. A. Erdelyi et al, *Higher Transendental Functions*, vol. 1–3, McGraw-Hill, 1953.
5. S. Janson, *Hankel operators between weighted Berman spaces*, Ark. Mat. 26 (1988), 205–219.
6. S. Janson, J. Peetre, *Paracommutators – boundedness and Schatten-von Neumann properties*, Trans. Amer. Math. Soc. 305 (1987), 463–507.
7. S. Janson, J. Peetre and R. Rochberg, *Hankel forms and Fock spaces*, Rev. Mat. Iberoamecicana. 3 (1987), 61–138.
8. J. Peetre, *New thoughts on Besov spaces*, Duke Univ. Math. Ser., Durham, 1976.
9. J. Peetre, *Fourier analysis of a space of Hilbert-Schmidt operators – New Ha-plitz operators*, Publ. Mat. 34 (1990), 181–197.
10. J. Peetre, L. Peng and G. Zhang, *A weighted Plancherel formula* I. *The case of the unit ball. Application to Hankel operators*, technical report, Stockholm.
11. V. V. Peller, *A description of Hankel operators of class S_p for $p > 0$, an investigation of the rate of rational approximation, and other applications*, Math. USSR-Sb. 51 (1985), 465–494.
12. L. Peng, *Paracommutator of Schatten-von Neumann class S_p*, $0 < p < 1$, Math. Scand. 61 (1987), 68–92.
13. S. Semmes, *Trace ideal criteria for Hankel operators, and applications to Besov spaces*, Integral Euqations Operator Theory 7 (1984), 241–281.
14. G. Zhang, *Tensor products of weighted Bergman space and invariant Ha-plitz operators*, rechnical report, No. 10, Mittag-Leffler Institute, 1990/91.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF STOCKHOLM
BOX 6701
S-113 85 STOCKHOLM
SWEDEN

CURRENT ADDRESS:
INSTITUT FOR MATEMATIK OG DATALOGI
ODENSE UNIVERSITET
DK-5230 ODENSE M
DENMARK