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TENSOR PRODUCTS OF WEIGHTED BERGMAN SPACES AND INVARIANT HA-PLITZ OPERATORS

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§ 0. Introduction.

The Moebius group SU(1, 1) acts unitarily on a weighted Bergman space $A^{\alpha,2}(D)$ (or the complex conjugate $\overline{A^{\alpha,2}(D)}$ of it). If we have two such spaces $A^{\alpha,2}(D)$ and $A^{\beta,2}(D)$, then the same group acts on the space of Hilbert-Schmidt operators from one space into the other (or into its complex conjugate). As to the second alternative, we remark that, quite generally, the study of linear operators from a space of analytic functions into the conjugate of a space of analytic functions is equivalent to the study of bilinear forms between the two spaces.

In the case when $\alpha = \beta$ the decomposition of the space of Hilbert-Schmidt forms into irreducible components was considered by Janson and Peetre [6]. Forms belonging to such an irreducible component were called Hankel forms of higher weight. The corresponding problem for operators was treated in Peetre [10]. Now operators reminding of Toeplitz operators arise. So perhaps it is justified to speak, as a unifying concept, of "Ha-plitz" operators (Nikol'skii, 1984).

In this paper we generalize some of the results of [6] and [10] to the case $\alpha \neq \beta$. We find the irreducible decomposition of the space of forms and establish their Schatten-von Neumann properties. We also study the corresponding problem for operators. There are now only finitely many discrete parts in the decomposition. We find explicit realizations of these discrete parts. The operators in the discrete parts are certain kinds of combinations of fractional integrations (or differentiations) and ordinary Toeplitz operators. The Schatten-von Neumann properties of these operators display the usual cut-off phenomenon. This follows from the results in [19], dealing with an equivalent class of operators.

Note that on the abstract level the problem of irreducible decomposition of tensor products of holomorphic discrete series representations of SU(1, 1) has been studied by Pukánszky [14] and Repka [15].

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§ 1. Hankel forms of higher weight.

Let D be the unit disk in the complex plane equipped with dm the Lebesgue measure. The Moebius group $SU(1, 1)$ consists of the following 2×2 matrices

$$\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}, \; a, b \in \mathbf{C}, \; |a|^2 - |b|^2 = 1.$$

It acts on D via the transformations

$$\phi(z) = \frac{az + b}{\bar{b}z + \bar{a}}, \; \phi = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in SU(1, 1),$$

The group $SU(1, 1)$ has an Iwasawa decomposition $SU(1, 1) = KAN$, where

$$K = \left\{ \begin{pmatrix} e^{i\frac{\varepsilon}{2}} & 0 \\ 0 & e^{-\frac{\varepsilon}{2}} \end{pmatrix}, \varepsilon \in \mathbf{R} \right\},$$

$$A = \left\{ \begin{pmatrix} \cosh(\frac{\varepsilon}{2}) & \sinh(\frac{\varepsilon}{2}) \\ \sinh(\frac{\varepsilon}{2}) & \cosh(\frac{\varepsilon}{2}) \end{pmatrix}, \varepsilon \in \mathbf{R} \right\},$$

$$N = \left\{ \begin{pmatrix} \cosh(\frac{\varepsilon}{2}) & i\sinh(\frac{\varepsilon}{2}) \\ -i\sinh(\frac{\varepsilon}{2}) & \cosh(\frac{\varepsilon}{2}) \end{pmatrix}, \varepsilon \in \mathbf{R} \right\}.$$

The Lie algebra $SU(1, 1)$ is generated by

$$e_3 = \begin{pmatrix} \frac{i}{2} & 0 \\ 0 & -\frac{i}{2} \end{pmatrix}, e_1 = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & \frac{i}{2} \\ -\frac{i}{2} & 0 \end{pmatrix}.$$

The Casimir operator is

$$c = 4(e_1^2 + e_2^2 - e_3^2).$$

Let α and β be nonnegative integers. We consider the weighted measure $d\mu_\alpha(z) = \dfrac{\alpha + 1}{\pi}(1 - |z|^2)\, dm(z)$. Let $A^{\alpha, 2}(D)$ be the weighted Bergman space consisting of the analytic functions on D square integrable with respect to $d\mu_\alpha$. The tensor product $A^{\alpha, 2}(D) \otimes A^{\beta, 2}(D)$ of two such spaces can be realized as the space of Hilbert-Schmidt forms on $A^{\alpha, 2}(D) \times A^{\beta, 2}(D)$. A bilinear form F on $A^{\beta, 2}(D) \times A^{\beta, 2}(D)$ can be written

$$F(f,g) = \int_D \int_D \overline{F(z,w)} f(z) g(w)\, d\mu_\beta(w),\ f \in A^{\alpha,2}(D),\ g \in A^{\beta,2}(D),$$

where F is analytic in z, w and

$$\int_D \int_D |F(z,w)|^2\, d\mu_\alpha(z)\, d\mu_\beta(w) < \infty. \tag{1.1}$$

Here we use the same notation for the form and its kernel.

Denote

$$\nu = \alpha + 2, \kappa = \beta + 2.$$

The group SU(1, 1) acts on the space $A^{\alpha,2}(D)$ unitarily via

$$U_\phi^{(\nu)} f = f(\phi z)(\phi'(z))^{\frac{\nu}{2}}.$$

The corresponding action on $A^{\beta,2}(D)$ is $U^{(\kappa)}$. The representation $U^{(\nu)} \otimes U^{(\kappa)}$ on $A^{\alpha,2}(D) \otimes A^{\beta,2}(D)$ is the following

$$F(z,w) \mapsto F(\phi(z), \phi(w))(\phi'(z))^{\frac{\nu}{2}}(\phi'(w))^{\frac{\kappa}{2}}. \tag{1.2}$$

The infinitesimal actions corresponding to e_3, e_1, e_2 are the following

$$E_3 f = iz\frac{\partial f}{\partial z} + iw\frac{\partial f}{\partial w} + \frac{i}{2}(\nu + \kappa) f.$$

$$E_1 f = \frac{i}{2}\left((1 + z^2)\frac{\partial f}{\partial z} + (1 + w^2)\frac{\partial f}{\partial w} + \nu z f + \kappa w f\right),$$

$$E_2 f = \frac{1}{2}\left((1 - z^2)\frac{\partial f}{\partial z} + (1 - w^2)\frac{\partial f}{\partial w} - \nu z f - \kappa w f\right).$$

The Casimir operator becomes

$$C = -(z - w)^2 \frac{\partial^2 f}{\partial z \partial w} - \nu(z - w)\frac{\partial f}{\partial w} + \kappa(z - w)\frac{\partial f}{\partial z} + (\nu + \kappa)(\nu + \kappa - 2) f.$$

LEMMA 1.1. *Let r be a nonnegative integer and let $b \in \partial D$, where ∂D is the unit circle. Then the function*

$$(z - w)^r (1 - \bar{b} z)^{\nu + r}(1 - \bar{b} w)^{\kappa + r} \tag{1.3}$$

is an eigenfunction of the Casimir operator:

$$Cf = \lambda f, \tag{1.4}$$

with

$$\lambda = (\nu + \kappa + r - 1) r + (\nu + \kappa)(\nu + \kappa - 2).$$

PROOF. It is easy to check that the function $f(z, w) = (z - w)^r$ is an eigenfunction of the Casimir operator with the given eigenvalue λ. Take $z_0 \in D$ and let $g \in SU(1,1)$ be the symmetry interchanging the points 0 and z_0, that is $g(z) = \dfrac{z - z_0}{1 - \bar{z}_0 z}$. Since C commutes with the group action, we see that the function $(g(z) - g(w))^r (g'(z))^{\frac{\nu}{2}} (g'(w))^{\frac{\nu}{2}}$ is an eigenfunction of C with the same eigenvalue λ. A straightforward calculation shows now that this is a constant multiple of the function

$$\frac{(z - w)^r}{(1 - \bar{z}_0 z)^{\nu + r}(1 - \bar{z}_0 w)^{\kappa + r}}.$$

Next let z_0 approach a boundary point b. This function then approaches the function (1.3) (in the distribution sense). Therefore (1.3) is a solution of (1.4). This proves the lemma.

Now we can form the following kernel function

$$H_f^{(r)}(z, w) = (z - w)^r \int_{\partial D} \frac{f(b)}{(b - z)^{\nu + r}(b - w)^{\kappa + r}}\, db, \tag{1.5}$$

where f is an arbitrary analytic function on the disk. It is clear that $H_f^{(r)}$ is also an eigenfunction of the Casimir operator.

The corresponding Hankel form on $A^{\alpha, 2}(D) \times A^{\beta, 2}(D)$ with the "symbol" f is then given by the formula

$$H_f^{(r)}(f_1, f_2) = \int_D \int_D \overline{H_f^{(r)}(z, w)} f_1(z) f_2(w)\, d\mu_\alpha(z)\, d\mu_\beta(w).$$

The group action (1.2) on these forms is equivalent to the following action on the symbols:

$$f(z) \to f(\phi z)\{\phi'(z)\}^{-(r - 1 + \frac{\nu + \kappa}{2})}.$$

It is also easy to check that $H_f^{(r)}$ has finite Hilbert-Schmidt norm for $f(z) = z^{\alpha + \beta + 2r + 4}$. Therefore by the Arazy-Fisher theory of Moebius invariant function spaces (see e.g. [1]), we see that for an analytic function f,

$$\|H_f^{(r)}\|_2^2 = c\, \|f\|^2_{B_2^{\frac{\nu + \kappa}{2} + r - \frac{1}{2}}},$$

with suitable constant c. (Here B_2^s is the usual scale of Besov spaces). It follows that for each nonnegative integer r, the Hilbert-Schmidt forms $H_f^{(r)}$ constitute an irreducible component, which we denote

$$V_r = \{H_f^{(r)}, f \in B_2^{\frac{v+\kappa}{2}}\}.$$

By the same calculation as in [6], we see that V_r is an irreducible SU(1, 1)-module of lowest weight $2r + v + \kappa - 2$, and that

$$A^{\alpha,2}(D) \otimes A^{\beta,2}(D) = \sum_{r=0}^{\infty} {}^{\oplus} V_r. \tag{1.6}$$

Moreover, we have the following theorem.

THEOREM 1.2. *For $p > 0$, the generalized Hankel form $H_f^{(r)}$ is in the Schatten-von Neumann class S_p if and only if $f \in B_p^{\frac{v+\kappa+2(r-1)}{2}+\frac{1}{p}}$.*

PROOF. Mapping D to the upper plane and performing a Fourier transform, we can see that the Hankel form $H_b^{(r)}$ is unitarily equivalent (disregarding a constant) to the following paracomutator on $H^2(\mathsf{R}) \times H^2(\mathsf{R})$,

$$(f_1, f_2) \mapsto \int_{\mathsf{R}} \int_{\mathsf{R}} \overline{\hat{f}(\xi_1 + \xi_2) J(\xi_1, \xi_2)} \hat{f}_1(\xi_1) \hat{f}_2(\xi_2) \xi_1^{-(\alpha+1)} \xi_2^{-(\beta+1)} \, d\xi_1 \, d\xi_2,$$

where

$$J(\xi_1, \xi_2) = \sum_{j=0}^{\alpha+\beta+r+2} \binom{\alpha+\beta+r+2}{j} \binom{r}{\alpha+r-j+1} (-1)^j \xi_1^j \xi_2^{\alpha+\beta+r+2-j}.$$

The S_p-result therefore follows from the general theory of Janson and Peetre [6] (see also Peng [13]). We omit the details here.

§ 2. Invariant Toeplitz operators.

In this section we study the same problem for the tensor product $A^{\alpha,2}(D) \otimes \overline{A^{\beta,2}(D)}$, which can be realized as the space of Hilbert-Schmidt operators F from $A^{\beta,2}(D)$ to $A^{\alpha,2}(D)$. Such an operator can be written

$$Fg(z) = \int_D F(z, w) g(w) \, d\mu_\beta(w), \tag{2.1}$$

where the function $F(z, w)$ is analytic in z and anti-analytic in w and satisfies (1.1).

For simplicity we assume that α and β are nonnegative integers and that $\alpha - \beta$ is an nonnegative even integer. Thus we can write $\alpha - \beta = 2l$ were l is an integer > 0. It is proved in Repka [15] that $A^{\alpha,2}(D) \otimes \overline{A^{\beta,2}(D)}$ has $l = (\alpha - \beta)/2$ discrete parts. We will find an explicit realization of the discrete parts and study their Schatten-von Neumann properties.

The tensor product $U^{(\nu)} \otimes \overline{U^{(\kappa)}}$ (where $\overline{U^{(\kappa)}}$ acts on $\overline{A^{\beta,2}(D)}$ by taking formally ecomplex conjugate in the expression for $U^{(\kappa)}$ on $A^{\beta,2}(D)$ is the following:

$$F(z,w) \mapsto F(\phi(z), \phi(w))\{\phi'(z)\}^{\frac{\nu}{2}}\{\overline{\phi'(w)}\}^{\frac{\kappa}{2}}, \ \phi \in \mathrm{SU}(1,1).$$

The generators corresponding to the Lie algebra elements e_3, e_1, e_2, are (see [10] for the case $\nu = \kappa$):

$$E_3 = iz\frac{\partial}{\partial z} - i\overline{w}\frac{\partial}{\partial \overline{w}} + \frac{i}{2}(\nu - \kappa),$$

$$E_1 = \frac{1}{2}\left((1 - z^2)\frac{\partial}{\partial z} + (1 - \overline{w}^2)\frac{\partial}{\partial \overline{w}} - \nu z - \kappa\overline{w}\right),$$

$$E_2 = \frac{i}{2}\left((1 + z^2)\frac{\partial}{\partial z} - (1 - \overline{w}^2)\frac{\partial}{d\overline{w}} + \nu z - \kappa\overline{w}\right).$$

Then we find that the corresponding Casimir element is, omitting a constant term, the following operator (if $\nu = \kappa$ see again [10])

$$C = (1 - z\overline{w})^2\frac{\partial^2}{\partial z\,\partial \overline{w}} - \nu\overline{w}(1 - z\overline{w})\frac{\partial}{\partial \overline{w}} - \kappa z(1 - z\overline{w})\frac{\partial}{\partial z} + \nu\kappa z\overline{w}.$$

We study the eigenvalue problem

$$CF = \lambda F. \tag{2.2}$$

LEMMA 2.1. *Let $b \in \partial D$ and $s \in \mathrm{R}$. Then the following function,*

$$e_{s,b}(z,w) = \frac{(1 - z\overline{w})^{-s}}{(1 - z\overline{b})^{\nu - s}(1 - b\overline{w})^{\kappa - s}}, \tag{2.3}$$

is a solution of (2.2) *with the eigenvalue*

$$\lambda = (\nu - s)(\kappa - s) + s. \tag{2.4}$$

PROOF. We use an idea in Helgason [8] (Chapter IV, § 2, pp. 402–403). First we observe that $e_{s,b}(z,w) = e_{s,1}(bz, bw)$, so we need only to prove that $e_{s,1}$ is an eigenfunction. If $g \in \mathrm{SU}(1,1)$, we have the known identity

$$(1 - g(z)\overline{g(w)}) = (g'(z))^{\frac{1}{2}}(\overline{g'(w)})^{\frac{1}{2}}(1 - z\overline{w}).$$

From this it is not difficult to prove the following transformation formula for $e_{s,1}$:

$$\int_{\partial D} e_{s,1}(g(kz), g(kw))(g'(kz))^{\frac{\nu}{2}}(\overline{g'(kw)})^{\frac{\kappa}{2}}\,|dk|$$

$$= (g'(0))^{\frac{\nu}{2}}(\overline{g'(0)})^{\frac{\kappa}{2}}e_{s,1}(g(0), g(0)) \int_{\partial D} e_{s,1}(kz, kw)\,|dk|.$$

Since the operator C commutes with the group action, it follows that

$$\int_{\partial D} Ce_{s,1}(g(kz), g(kw))(g'(kz))^{\frac{\nu}{2}}\overline{(g'(kw))}^{\frac{\kappa}{2}}\,|dk|$$

$$= (g'(0))^{\frac{\nu}{2}}\overline{(g'(0))}^{\frac{\kappa}{2}} e_{s,1}(g(0), g(0)) \int_{\partial D} Ce_{s,1}(kz, kw)\,|dk|.$$

Therefore, putting $z = w = 0$ gives

$$Ce_{s,1}(g(0), g(0)) = Ce_{s,1}(0,0)e_{s,1}(g(0), g(0)).$$

Since $Ce_{s,1}$ and $e_{s,1}$ are analytic in the first variable and anti-analytic in the second varible, we have

$$Ce_{s,1}(z, w) = Ce_{s,1}(0,0)e_{s,1}(z, w).$$

The eigenvalue $Ce_{s,1}(0,0)$ can be computed directly and it is (2.4).

We can also write down the radial part of the Casimir operator. Suppose $F(z, w) = f(z\overline{w})$. Writing $t = z\overline{w}$, then one finds $\frac{\partial F}{\partial z} = \overline{w}\frac{df}{dt}$, $\frac{\partial F}{\partial \overline{w}} = z\frac{df}{dt}$, and $\frac{\partial^2 F}{\partial z \partial \overline{w}} = \frac{df}{dt} + z\overline{w}\frac{d^2 f}{d^2 t}$. The equation (2.2) then becomes

$$(1 - t)^2(tf'' + f') - (\nu + \kappa)t(1 - t)f' + \nu\kappa t f = \lambda f. \tag{2.5}$$

Integrating the function (2.3) over the unit circle we then find that

$$(1 - t)^{-s}\,{}_2F_1(\nu - s, \kappa - s; 1; t),$$

is a solution of (2.5) with λ as in (2.4), where ${}_2F_1$ is the hypergeometric function. Expanding it as a power series $\sum_{n=0}^{\infty} p_n(\lambda)t^n$, we arrive at the following recursion formula for p_n (*cf.* [10]):

$$(n + 1)^2 p_{n+1}(\lambda) + (n - 1 + \nu)(n - 1 + \kappa)p_{n-1}(\lambda) - n(2n + \nu + \kappa)p_n(\lambda) = \lambda p_n(\lambda).$$

One can also prove that

$$p_n(\lambda) = \sum_{k=0}^{n} \frac{(\nu - s)_k(\kappa - s)_k}{(k!)^2}\frac{(s)_{n-k}}{(n - k)!}$$

$$= \frac{(s)_n}{n!}\,{}_3F_2(-n, \nu - s. \kappa - s; 1, 1 - \mathrm{n} - \mathrm{s}; 1),$$

where ${}_3F_2$ is the generalized hypergeometric function ([4]), $(a)_k = a(a + 1)\ldots(a + k - 1)$, and λ and s are connected as in formula (2.4) (we have not indicated ν and κ in the notation for p_n).

These polynomials are orthogonal with respect to the spectral measure of the selfadjoint operator in (2.5), but we have not determined this measure. They seem to be connected with certain orthogonal polynomials studied in [18].

However, we are more interested in the discrete spectrum of the Casimir operator, which corresponds the discrete parts of the irreducible decomposition. Given an analytic function f on the unit disk, we form the following kernel

$$\mathrm{T}_f^{(s)}(z,w) = (1 - z\bar{w})^{-s} \int_{\partial D} \frac{\bar{b}^{\nu - s} f(b)}{(1 - \bar{b}z)^{\nu - s}(1 - b\bar{w})^{\kappa - s}}\, db.$$

where $s = \alpha - l + 2, \alpha - l + 1, \ldots, \alpha + 1$. The corresponding λ's obtained from (2.4) then give the discrete spectrum of C.

To illustrate the invariance property of $T_f^{(s)}$, we put $s = \alpha + 1 - \sigma$ where thus $\sigma = 0, 1, \ldots, l - 1$. Let us further put $t = l - \sigma$. The group action on $T_f^{(s)}$ is then equivalent to the following action on f: if $\phi \in \mathrm{SU}(1,1)$,

$$f(z) \mapsto f(\phi z)(\phi'(z))^t. \tag{2.6}$$

Note thus that f transforms like a form of degree t (weight $2t$).

REMARK. Using the reproducing property of the Szegö kernel $(1 - b\bar{w})^{-1}$, we can also write

$$T_f^{(s)}(z,w) = (1 - z\bar{w})^{-s} \left(\frac{\partial}{\partial z}\right)^{\sigma} [f(z)(1 - z\bar{w})^{s - \kappa}].$$

The transformation properties of this kernel can be read off directly from "Bol's lemma" ([5]): notice that the expression within brackets has weight $1 - \sigma$ so $\left(\frac{\partial}{\partial z}\right)^{\sigma}$ applied to it gives weight $1 + \sigma$. (If α and β are not integers we get here fractional derivatives.)

The corresponding operator $T_f^{(s)}$ from $A^{\beta,2}(D)$ to $A^{\alpha,2}(D)$ is

$$T_f^{(s)} g(z) = \int_D T_f^{(s)}(z,w) g(w)\, d\mu_\beta(w), \quad g \in A^{\beta,2}(D).$$

It is easy to calculate the integral. We find that

$$T_f^{(s)} g(z) = \sum_{j=0}^{\sigma} \binom{\sigma}{j} \frac{(\sigma)_{\sigma - j}}{(\kappa)_{\sigma - j}} D^j f(z) D^{\sigma - j} g(z).$$

Recall the relationship

$$(a)_\sigma = (-1)^{\sigma - j}(a)_j (1 - a - \sigma)_{\sigma - j}.$$

Using this we can transform the expression, up to a constant, to the form

$$\sum_{j=0}^{\sigma}(-1)^j\binom{\sigma}{j}\frac{D^jf(z)}{(2t)_j}\cdot\frac{D^{\sigma-j}g(z)}{(\beta+2)_{\sigma-j}}.$$

Thus we have an instance of a bilinear differential covariant, the transvectant, studied in [5]. The covariance of $T_f^{(s)}g$ is again manifest.

Let $\{e_n^{(\alpha)}\}, e_n^{(\alpha)}(z) = \frac{z^n}{\|z^n\|_\alpha}$, be the standard orthonormal basis of $A^{\alpha,2}(D)$ and $\{e_n^{(\beta)}\}$ the corresponding one of $A^{\beta,2}(D)$. The matrix elements of $T_{z^k}^{(s)}$ are given by

$$\begin{aligned}\langle T_{z^k}^{(s)}e_m^{(\beta)}, e_n^{(\alpha)}\rangle &= \delta_{m+k-\sigma,n}(-1)^\sigma\frac{\|z^n\|_\alpha}{\|z^m\|_\beta}\\ &\quad\times\sum_{j=0}^{\sigma}\binom{\sigma}{j}\frac{(\beta+2-s)_{\sigma-j}}{(\beta+2)_{\sigma-j}}(-k)_j(-m)_{\sigma-j}\\ &= \delta_{m+k-\sigma,n}(-1)^\sigma(-1)^\sigma\frac{\|z^n\|_\alpha}{\|z^m\|_\beta}\frac{(\beta+2-s)_\sigma(-m)_\sigma}{(\beta+2)_\sigma}\\ &\quad {}_3F_2(-\sigma,-1-\sigma-\beta,-k;1+m-\sigma,\alpha-\beta-2\sigma;1).\end{aligned}$$

So we see that the only possible non-zero elements are

$$\begin{aligned}&\langle T_{z^k}^{(s)}e_{n+\sigma-k}^{(\beta)}, e_n^{(\alpha)}\rangle\\ &= (-1)^\sigma\frac{\|z^n\|_\alpha}{\|z^{n+s-k}\|_\beta}\frac{(\beta+2-s)_\sigma(-n-\sigma+k)_\sigma}{(\beta+2)_\sigma}\\ &\quad\times {}_3F_2(-\sigma,-1-\sigma-\beta,-k;1+n-k,\alpha-\beta-2\sigma;1).\end{aligned}$$

Since σ is a positive integer, the hypergeometric term ${}_3F_2(-\sigma,-1-\sigma-\beta,$ $-k;1+n-\mathrm{k},\alpha-\beta-2\sigma;1)$ has at most σ non zero terms in its expansion. Moreover if $n\to\infty$, the term approaches to the constant 1. So the singular numbers of the operator are

$$\langle T_{z^k}^{(s)}e_{n+\alpha+1-s-k}^{(\beta)}, e_n^{(\alpha)}\rangle \approx \frac{1}{m^{s-\frac{\alpha+\beta}{2}-1}}.$$

Using (2.6) and the invariance argument we can then prove the following

THEOREM 2.2. *Then the operators $T_f^{(s)}$ is in the Schatten-von Neumann class S_2 if and only if $f\in B_2^{\frac{1}{2}-t}$, where $t=s+l-\alpha-1$.*

It follows from the theorem that the spaces

$$\{T_f^{(s)}, f\in B_2^{\frac{1}{2}-t}\},$$

with $s=\alpha+1-\sigma$, $t=l-\sigma$, $\sigma=0,1,\ldots,l-1$, constitute all the discrete parts of the irreducible decomposition of $A^{\alpha,2}(D)\otimes\overline{A^{\beta,2}(D)}$.

The S_p-properties of those operators are studied in [19].

§ 3. Concluding remarks.

We indicate here a simple application of our decomposition in § 1. Let $\mathscr{A}(D)$ be the disk algebra. We note here since $A^{\alpha,2}(D)$ and $A^{\beta,2}(D)$ are Hilbert modules over $\mathscr{A}(D)$, so we can form the tensor product $A^{\alpha,2}(D) \otimes_{\mathscr{A}(D)} A^{\beta,2}(D)$, see [3]. It can be viewed as a space of analytic function on D (identifying D with the diagonal in $D \times D$). We claim that

$$A^{\alpha,2}(D) \otimes_{\mathscr{A}(D)} A^{\beta,2}(D) = A^{\nu+\kappa-2,2}(D).$$

In fact, from (1.5) we see that if $H \in V_r$, $r \geqq 1$, then

$$\begin{aligned} H_f^{(r)}(z,w) = z(z-w)^{r-1} &\int_{\partial D} \frac{f(b)}{(b-z)^{\nu+r}(b-w)^{\kappa+r}}\,db \\ -w(z-w)^{r-1} &\int_{\partial D} \frac{f(b)}{(b-z)^{\nu+r}(b-w)^{\kappa+r}}\,db \end{aligned}$$

and

$$(z-w)^{r-1} \int_{\partial D} \frac{f(b)}{(b-z)^{\nu+r}(b-w)^{\kappa+r}}\,db \in A^{\alpha,2}(D) \otimes A^{\beta,2}(D).$$

Hence $H \equiv 0$ in $A^{\alpha,2}(D) \otimes_{\mathscr{A}(D)} A^{\beta,2}(D)$ by definition. If $H \in V_0$, then $H = H_f^{(0)}$ for some $f \in B_2^{\frac{\nu+\kappa}{2}-\frac{1}{2}}$ and

$$H_f^{(0)}(z,z) = D^{\nu+\kappa-1} f(z).$$

By the Moebius invariance of $A^{\alpha,2}(D) \otimes_{\mathscr{A}(D)} A^{\beta,2}(D)$, we see that

$$\|H_f^{(0)}\|^2_{A^{\alpha,2}(D) \otimes_{\mathscr{A}(D)} A^{\beta,2}(D)} = c\,\|f\|^2_{B_2^{\frac{\nu+\kappa}{2}-\frac{1}{2}}} = c\,\|D^{\nu+\kappa-1} f\|^2_{A^{\nu+\kappa-2,2}(D)}.$$

Therefore the map

$$H_f^{(0)}(z,w) \mapsto H_f^{(0)}(z,z)$$

induces an $\mathscr{A}(D)$*-isomorphismfrom* $A^{\alpha,2}(D) \otimes_{\mathscr{A}(D)} A^{\beta,2}(D)$ to $A^{\nu+\kappa-2,2}(D)$. A more general result for the module tensor products of two Hilbert spaces of functions on planar domains was announced in [3], Chapter 5, p. 85.

It is also interesting to work out the explicit decomposition in § 1 for the unit ball in several variables, or more generally, a bounded symmetric domain. This problem has been studied by Repka [16] from the point of view of representation theory.

For a bounded symmetric domain of higher rank, it is known that there are no non-trival compact big Hankel operators on the Bergman space. The invariant Toeplitz operator obtained in § 2 may also shed some light on the construction of compact Hankel-type operators on a symmetric domain. Eventually, we would like to obtain the Schatten-von Neumann property of a Hankel-type operator (or the so called Hausdorff-Young theorem for integral operators).

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