MATH. SCAND. 71 (1992), 96–104

ON A BARRELLED SPACE OF CLASS $\aleph_0$ AND MEASURE THEORY

J. C. FERRANDO and L. M. SÁNCHEZ RUIZ

1. Introduction.

Throughout this paper the word "space" will stand for "Hausdorff locally convex topological vector space defined over the field of real or complex numbers. Let us recall a space E is Baire-like, [8], if given an increasing sequence of closed absolutely convex subsets of E covering E, there is one which is a neighbourhood of the origin. A countable family $\{C_n: n \in \mathbf{N}\}$ of classes of spaces with strong barrelledness conditions is studied in [4] considering as C_0 the class of Baire-like spaces and, for each $n \in \mathbf{N}$, a space E is said to be barrelled of class n, or more briefly $E \in C_n$, if given an increasing sequence of subspaces of E, there is one which belongs to C_{n-1}. So, C_1 is the class of suprabarelled spaces, [11], and C_2 is the class of ordered suprabarrelled spaces, [3]. A space E is barrelled of class $\aleph_0$, [4], if $E \in C_n$ for every $n \in \mathbf{N}$.

Given a σ-algebra $\mathscr{A}$ on a set X, $l_0^\infty(X, \mathscr{A})$ will stand for the normed space over the field of real or complex numbers generated by the characteristic functions $e(A)$, $A \in \mathscr{A}$, whose norm is defined by $\|z\| = \sup\{z(j)|: j \in X\}$.

In [10] $l_0^\infty(X, \mathscr{A})$ was shown to be suprabarrelled, in [6] it was shown to be barrelled of class 2 and in [5], using some duality methods, it was shown to be barrelled of class $\aleph_0$.

In this paper we show the sequential methods used in [6] can be applied to show $l_0^\infty(X, \mathscr{A})$ is barrelled of class $\aleph_0$, clearly separating the algebraic part of the proof (Lemmas 1–4) from the topological part (Theorem 1). Besides, we give new applications of this result to vector measures theory. We think the methods used in this paper are easier to understand than those given in [5], where algebraic and topological parts of the proof are mixed up.

Given $A \in \mathscr{A}$, $l_0^\infty(A, \mathscr{A})$ will denote the linear subspace of $l_0^\infty(X, \mathscr{A})$ generated by $\{e(B): B \in \mathscr{A}$ and $B \subset A\}$. Given a continuous linear form u on $l_0^\infty(X, \mathscr{A})$, $u(A)$ will stand for the restriction of u to $l_0^\infty(A, \mathscr{A})$ and $\|u(A)\|$ for the norm of $u(A)$. $\mathscr{F}$ will stand for the family of all the finite dimensional subspaces of $l_0^\infty(X, \mathscr{A})$. When

Received December 3, 1990; in revised form July 18, 1991.

$X := \mathsf{N}$ and $\mathscr{A}$ is the σ-algebra 2^{N} of all subsets of N, we write l_0^∞ instead of $l_0^\infty(\mathsf{N}, 2^{\mathsf{N}})$.

2. Strong barrelledness of $l_0^\infty(X, \mathscr{A})$.

We shall use the following two results, [6],

LEMMA 1. *Let E be a linear subspace of $l_0^\infty(X, \mathscr{A})$ and $A \in \mathscr{A}$ such that $l_0^\infty(A, \mathscr{A}) \not\subset E + F$ for every $F \in \mathscr{F}$. If $\{P, Q\}$ is a partition of A, $P, Q \in \mathscr{A}$ then either $l_0^\infty(P, \mathscr{A}) \not\subset E + F$ for every $F \in$ or $l_0^\infty(Q, \mathscr{A}) \not\subset E + F$ for every $F \in \mathscr{F}$.*

LEMMA 2. *Let E be a linear subspace of $l_0^\infty(X, \mathscr{A})$ and $A \in \mathscr{A}$ such that $l_0^\infty(A, \mathscr{A}) \not\subset E + F$ for every $F \in \mathscr{F}$. Given a number of elements $x_1, x_2, \ldots, x_r$ of $l_0^\infty(X, \mathscr{A})$ and $q \in \mathsf{N}$, there are q elements $Q_1, Q_2, \ldots, Q_q$ of $\mathscr{A}$ which are a partition of A, such that $e(Q_i) \notin \langle E \cup \{x_1, x_2, \ldots, x_r\}\rangle$ for $i = 1, 2, \ldots, q$.*

For the next result we shall assume given:

1. A positive integer $p > 2$.

2. A number of elements $x_1, x_2, \ldots, x_r$ of $l_0^\infty(X, \mathscr{A})$.

3. q positive integers $m(1) < m(2) < \ldots < m(q)$; for each $i_1 \in \{1, 2, \ldots, q\}$, $q(i_1)$ positive integers $m(i_1, 1) < m(i_1, 2) < \ldots < m(i_1, q(i_1))$. And in general, making $k = 2, \ldots, p-1$, given $i_1 \in \{1, 2, \ldots, q\}$ and $i_j \in \{1, 2, \ldots, q(i_1, i_2, \ldots, i_{j-1})\}$ for $j = 2, \ldots, k$, we assume there are $q(i_1, i_2, \ldots, i_k)$ positive integers $m(i_1, i_2, \ldots, i_{k-1}, 1) < m(i_1, i_2, \ldots, i_{k-1}, 2) < \ldots < m(i_1, i_2, \ldots, i_{k-1}, q(i_1, i_2, \ldots, i_k))$.

4. A family of subspaces E_s of $l_0^\infty(X, \mathscr{A})$ where s takes values in:

i) The set $\mathscr{I} := \{(m(i_1), \ldots, m(i_1, i_2, \ldots, i_p))$: $i_1 \in \{1, 2, \ldots, q\}$ and $i_j \in \{1, 2, \ldots, q(i_1, i_2, \ldots, i_{j-1})\}$ for $j = 2, \ldots, p\}$.

ii) Some set of type (1), where we call sets of type (1) those of the form $\{(m(i_1), m_2, \ldots, m_p)$: $i_1 \in \{1, 2, \ldots, q\}$ and for each one of these $m(i_1)$, m_2 may take infinitely many values; for each one of these m_2, m_3 may take infinitely many values; ... ; and for each of one these m_{p-1}, m_p may take infinitely many values$\}$.

(iii) Some set of type (k) for each $k = 2, \ldots, p-1$, where we call sets of type (k) those of the form $\{(m(i_1), \ldots, m(i_1, i_2, \ldots, i_k), m_{k+1}, \ldots, m_p)$: $i_1 \in \{1, 2, \ldots, q\}$, $i_j \in \{1, 2, \ldots, q(i_1, i_2, \ldots, i_{j-1})\}$ for $j = 2, \ldots, k$ and for each one of these $m(i_1), \ldots, m(i_1, i_2, \ldots, i_k)$, m_{k+1} may take infinitely many values; ... ; and for each one of these m_{p-1}, m_p may take infinitely many values$\}$.

(iv) Some set of type (0), where we call sets of type (0) those of the form $\{(m_1, m_2, \ldots, m_p)$: where m_1 may take infinitely many values; for each one of these m_1, m_2 may take infinitely many values; for each one of these m_2, m_3 may take infinitely many values; ... ; and for each of one these m_{p-1}, m_p may take infinitely many values$\}$.

We shall set $q_1 = q$, $q_2 = \sum_{i_1=1}^{q} q(i_1)$, and for each $j \in \{3, \ldots, p\}$,

$q_j = \sum_{i_1=1}^{q} \sum_{i_2=1}^{q(i_1)} \cdots\cdots \sum_{i_j-1}^{q(i_1,i_2,\ldots,i_{j-2})} q(i_1,i_2,\ldots,i_{j-1})$. Besides, let $s_i = \sum_{j=1}^{i} q_j$ for each $i \in \{1,\ldots,p\}$.

LEMMA 3. *Let $A \in \mathscr{A}$ be such that $l_0^\infty(A,\mathscr{A}) \not\subset E_s + F$ for every $F \in \mathscr{F}$. Then there are q_p pairwise disjoint elements of $\mathscr{A}$ $\{M_s, s \in \mathscr{I}\}$, contained in A such that $e(M_s) \notin \langle E_s \cup \{x_1,x_2,\ldots,x_r\}\rangle$ for each $s \in \mathscr{I}$. Moreover $l_0^\infty(A \setminus \cup\{M_s, s \in \mathscr{I}\}, \mathscr{A})$ $\not\subset E_s + F$ for every $F \in \mathscr{F}$, for each $s \in \mathscr{I}$ and some sets of type* (0)-($p-1$).

PROOF. Pick an $s \in \mathscr{I}$. As $l_0^\infty(A,\mathscr{A}) \not\subset E_s + F$ for every $F \in \mathscr{F}$, by Lemma 2, there is a partition of A, $\{Q_n: n = 1,2,\ldots,s_p + 2\}$, formed by elements of $\mathscr{A}$ such that $e(Q_n) \notin \langle E_s \cup \{x_1,x_2,\ldots,x_r\}\rangle$ for $n = 1,2,\ldots,s_p + 2$.

As $l_0^\infty(A,\mathscr{A}) \not\subset E_s + F$ for every $F \in \mathscr{F}$ when s takes values in some set of type (1) with $i_1 \in \{1,\ldots,q\}$, by Lemma 1, there is some $n_j \in \{1,2,\ldots,s_p + 2\}$, $1 \leqq j \leqq q$, such that $l_0^\infty(Q_{n_j},\mathscr{A}) \not\subset E_s + F$ for every $F \in \mathscr{F}$ when s takes values in some set of type (1), which in general will be a subset of the original set of type (1).

Now, as $l_0^\infty(A,\mathscr{A}) \not\subset E_s + F$ for every $F \in \mathscr{F}$ when s takes values in some set of type (2) with $i_1 \in \{1,\ldots,q\}$, $i_2 \in \{1,2,\ldots,q(i_1)\}$, by Lemma 1, there is some $n_j \in \{1,2,\ldots,s_p + 2\}$, $q + 1 \leqq j \leqq s_2$, such that $l_0^\infty(Q_{n_j},\mathscr{A}) \not\subset E_s + F$ for every $F \in \mathscr{F}$ when s takes values in some set of type (2).

Going on in this way, as $l_0^\infty(A,\mathscr{A}) \not\subset E_s + F$ for every $F \in \mathscr{F}$ when s takes values in some set of type ($p-1$) with $i_1 \in \{1,\ldots,q\}$, $i_j \in \{1,2,\ldots,q(i_1,i_2,\ldots,i_{j-1})\}$ for $j = 2,\ldots,p-1$, by Lemma 1, there is some $n_j \in \{1,2,\ldots,s_p + 2\}$, $s_{p-2} + 1 \leqq j \leqq s_{p-1}$, such that $l_0^\infty(Q_{n_j},\mathscr{A}) \not\subset E_s + F$ for every $F \in \mathscr{F}$ when s takes values in some set of type ($p-1$).

In the same way, as $l_0^\infty(A,\mathscr{A}) \not\subset E_s + F$ for every $F \in \mathscr{F}$ when s takes values in some set of type (0), there is some $n_{s_{p-1}+1} \in \{1,2,\ldots,s_p + 2\}$ such that $l_0^\infty(Q_{n_{s_{p-1}+1}},\mathscr{A}) \not\subset E_s + F$ for every $F \in \mathscr{F}$ when s takes values in some set of type (0).

If we set $Q_0 := \cup\{Q_{n_j}: j = 1,\ldots,s_{p-1} + 1\}$, then $l_0^\infty(Q_0,\mathscr{A}) \not\subset E_s + F$ for every $F \in \mathscr{F}$ when s takes values in some sets of type (0)-($p-1$).

Let $J := \{0,1,2,\ldots,s_p + 2\} \setminus \{n_1,n_2,\ldots,n_{s_{p-1}+1}\}$. Now $\{Q_i: i \in J\}$ is a partition of A and for each $i_1 \in \{1,\ldots,q\}$, $i_j \in \{1,2,\ldots,q(i_1,i_2,\ldots,i_{j-1})\}$ for $j = 2,\ldots,p$, by Lemma 1, there is some $n_j \in J$, $s_{p-1} + 2 \leqq j \leqq s_p + 1$, such that $l_0^\infty(Q_{n_j},\mathscr{A}) \not\subset E_s + F$ for every $F \in \mathscr{F}$ when $s \in \mathscr{I}$.

Therefore, at least two elements of J cannot be contained in $\{n_j: 1 \leqq j \leqq s_p + 1\}$ and some of them, say j_0, must be non zero.

As $l_0^\infty\left(\bigcup_{j=s_{p-1}+2}^{s_p+1} Q_{n_j},\mathscr{A}\right) \not\subset E_s + F$ for every $F \in \mathscr{F}$ when $s \in \mathscr{I}$, $l_0^\infty(A \setminus Q_{j_0},\mathscr{A}) \not\subset E_s + F$ for every $F \in \mathscr{F}$ when $s \in \mathscr{I}$ and this is also true when s lakes values in some sets of type (0)-($p-1$) since $Q_0 \subset A \setminus Q_{j_0}$.

Setting $s(1) := (m(1), \ldots, m(1, \ldots, 1))$, the set $M_{s(1)} := Q_{j_0}$ satisfies $e(M_{s(1)} \notin \langle E_{s(1)} \cup \{x_1, x_2, \ldots, x_r\}\rangle$ and $l_0^\infty(A \backslash M_{s(1)}, \mathscr{A}) \not\subset E_s + F$ for every $F \in \mathscr{F}$, for each $s \in \mathscr{I}$ and some sets of type (0)-($p - 1$).

Picking now another $s \in \mathscr{I}$, $s \neq s(1)$, $A \backslash M_{s(1)}$ instead of A, and repeating the same argument we find another $M_{s(2)} \in \mathscr{A}$ such that $e(M_{s(2)}) \notin \langle E_{s(2)} \cup \{x_1, x_2, \ldots, x_r\}\rangle$ and $l_0^\infty(A \backslash \{M_{s(1)} \cup M_{s(2)}), \mathscr{A}) \not\subset E_s + F$ for every $F \in \mathscr{F}$, for each $s \in \mathscr{I}$ and some sets of type (0)-($p - 1$).

We go on this way until that we find the last $M_{s(q_p)} \in \mathscr{A}$, which establishes the Lemma.

Given $t \in \mathsf{N}^{p+1}$ we shall denote by ${}^{\#}t$ the sum of the coordinates of t. For each $i \in \mathsf{N}$, $\mathsf{N}^{p+1}(p + i) := \{t \in \mathsf{N}^{p+1}: {}^{\#}t = p + i\}$ and let ${}^{\#}N^{p+1}(p + i)$ stand for the cardinal of $\mathsf{N}^{p+1}(p + i)$. We shall consider $\{\alpha(n): n \in \mathsf{N}\}$ the sequence of all the elements of N^{p+1} ordered following the diagonal ordering, i.e., $\alpha(n_1) < \alpha(n_2)$ if either ${}^{\#}\alpha(n_1) < {}^{\#}\alpha(n_2)$ or, when ${}^{\#}\alpha(n_1) = {}^{\#}\alpha(n_1)$, the first coordinate which is smaller than the same coordinate of the other element belongs to $\alpha(n_1)$.

LEMMA 4. *Let $\{E_s : s \in \mathsf{N}^p\}$, $p > 2$, be a countable family of infinite codimensional linear subspaces of $l_0^\infty(X, \mathscr{A})$. Then there is a countable family of pairwise disjoint elements of $\mathscr{A}\{M_t : t \in \mathsf{N}^{p+1}\}$, an increasing sequence of positive integers $\{m(i): i \in \mathsf{N}\}$; for each $i_1 \in \mathsf{N}$, an increasing sequence of positive integers $\{m(i_1, i): i \in \mathsf{N}\}$; and, in general, for $j = 1, \ldots, p - 1$, given $(i_1, i_2, \ldots, i_j) \in \mathsf{N}^j$ there is an increasing sequence of positive integers $\{m(i_1, i_2, \ldots, i_j, i): i \in \mathsf{N}\}$ such that setting for each $t = (i_1, i_2, \ldots, i_{p+1}) \in \mathsf{N}^{p+1}$, $p(t) := (m(i_1), m(i_1, i_2), \ldots, m(i_1, i_2, \ldots, i_p)) \in \mathsf{N}^p$, we have $e(M_t) \notin \langle E_{p(t)} \cup \{e(M_r): r \in \mathsf{N}^{p+1}$ and ${}^{\#}r < {}^{\#}t\}\rangle$ for each $t \in \mathsf{N}^{p+1}$.*

PROOF. Set $q = 1$, $q(1, \overset{(j)}{\ldots\ldots}, 1) = 1$ for $j = 1, \ldots, p - 1$; $m(1) = \ldots = m(1, \overset{(p)}{\ldots\ldots}, 1) = 1$ and $\mathscr{I}_1 = \{(1, \overset{(p)}{\ldots\ldots}, 1)\}$. By Lemma 3, there is some $M_{\alpha(1)} \in \mathscr{A}$ such that $e(M_{\alpha(1)}) \notin E_{p(\alpha(1))}$ and, for each $s \in \mathscr{I}_1$ and some sets of type (0)-($p - 1$), $l_0^\infty(X \backslash M_{\alpha(1)}, \mathscr{A}) \not\subset E_s + F$ for every $F \in \mathscr{F}$.

Let us now define all the $m(i_1, \ldots, i_j)$ such that $i_1 + \ldots + i_j = j + 1$, for $j = 1, 2, \ldots, p$.

Let $m(2)$ be the smallest $m_1 > m(1)$ of the set of type (0) where s may take values; $m(1, 2)$ the smallest $m_2 > m(1, 1)$, depending on $m(1)$, of the set of type (1) where s may take values; $m(2, 1)$ the smallest m_2, depending on $m(2)$, of the set of type (0) where s may take values; $m(1, 1, 2)$ the smallest $m_3 > m(1, 1, 1)$, depending on $m(1)$ and $m(1, 1)$, of the set of type (2) where s may take values; $m(1, 2, 1)$ the smallest m_3, depending on $m(1)$ and $m(1, 2)$, of the set of type (1) where s may take values;...

Once we have defined all the $m(i_1, \ldots, i_j)$ such that $i_1 + \ldots + i_j = j + 1$, $1 \leqq j \leqq p$, let $\mathscr{I}_2 := \{m(i_1, \ldots, i_p): i_1 + \ldots + i_p = p + 1\}$. Bearing in mind that now $q(i_1, \ldots, i_j) = j + 1 - (i_1 + \ldots + i_j)$, $1 \leqq j \leqq p$, then, by Lemma 3, there are $q(1, \overset{(p-1)}{\ldots\ldots}, 1) + q(1 \overset{(p-2)}{\ldots\ldots}, 1, 2) + \ldots + q(2, 1, \overset{(p-2)}{\ldots\ldots}, 1) = 2 + 1 + \overset{(p-1)}{\ldots\ldots},$

$+ 1 = p + 1$ pairwise disjoint elements of $\mathscr{A}$, $\{M_t: t \in \mathsf{N}^{p+1}(p+2)\}$, contained in $X \backslash M_{\alpha(1)}$, such that $e(M_t) \notin \langle E_{p(t)} \cup e(M_{\alpha(1)}) \rangle$ for each $t \in \mathsf{N}^{p+1}(p+2)$ and, for each $s \in \mathscr{I}_2$ and some sets of type (0)-($p-1$), $l_0^\infty(X \backslash \cup \{M_t: t \in \mathsf{N}^{p+1}(p+1) \cup \mathsf{N}^{p+1}(p+2)\}, \mathscr{A}) \not\subset E_s + F$ for every $F \in \mathscr{F}$.

By recurrence, let us assume we have obtained q positive integers $m(1) < \ldots < m(q)$ and, for each $j \in \{1, \ldots, p\}$, we have the $\{m(i_1, \ldots, i_j): i_1 + \ldots + i_j \leqq j - 1 + q\}$ (calling $\mathscr{I}_q$ the corresponding set obtained when $j = p$), such that for each $i_1 \in \{1, \ldots, q\}$, $\{m(i_1, i): i = 1, \ldots, 1 + q - i_1\}$ is strictly increasing; for each $(i_1, i_2, \ldots, i_{p-1}) \in \mathsf{N}^{p-1}$ such that $i_1 + \ldots + i_{p-1} \leqq (p-2) + q$, $\{m(i_1, i_2, \ldots, i_{p-1}, i): i = 1, \ldots, (p-1) + q - (i_1 + i_2 + \ldots + i_{p-1})\}$ is strictly increasing. We shall also assume we have obtained $\sum_{i=1}^{q} {}^{\#}\mathsf{N}^{p+1}(p+i)$ pairwise disjoint elements of $\mathscr{A}$, $\left\{M_t: t \in \bigcup_{i=1}^{q} \mathsf{N}^{p+1}(p+i)\right\}$, such that $e(M_t) \notin \langle E_{p(t)} \cup \{e(M_r): r \in \mathsf{N}^{p+1} \text{ and } {}^{\#}r < {}^{\#}t\} \rangle$ for each $t \in \bigcup_{i=1}^{q} \mathsf{N}^{p+1}(p+i)$ and, for each $s \in \mathscr{I}_q$ and some sets of type (0)-($p-1$), $l_0^\infty\left(X \backslash \cup \left\{M_r: r \in \bigcup_{i=1}^{q} \mathsf{N}^{p+1}(p+i)\right\}, \mathscr{A}\right) \not\subset E_s + F$ for every $F \in \mathscr{F}$.

Let us now define all the $m(i_1, \ldots, i_j)$ such that $i_1 + \ldots + i_j = j + q$, for $j = 1, 2, \ldots, p$.

Let $m(q+1)$ be the smallest $m_1 > m(q)$ of the set of type (0) where s may take values. And for each $j \in \{2, \ldots, p\}$,

a) If $i_j > 1$, we define $m(i_1, \ldots, i_j)$ as the smallest $m_j > m(i_1, \ldots, i_j - 1)$, depending on $m(i_1), \ldots, m(i_1, \ldots, i_{j-1})$ of the set of type ($j-1$) where s may take values,

b) If $i_j = 1$, we define $m(i_1, \ldots, i_j)$ as the smallest $m_j > 1$, depending on $m(i_1)$, $\ldots, m(i_1, \ldots, i_{j-1})$ of the set of type ($k-1$) where s may take values, with k the greatest positive integer smaller than j such that $i_k > 1$.

Let $\mathscr{I}_{q+1} := \{m(i_1, \ldots, i_p): i_1 + \ldots + i_p = p + q\}$

Then by Lemma 3, with $A := X \backslash \cup \left\{M_r: r \in \bigcup_{i=1}^{q} \mathsf{N}^{p+1}(p+i)\right\}$, $q+1$ instead of q, and $x_1, x_2, \ldots, x_r$ equal to each possible $e(M_t)$, $t \in \bigcup_{i=1}^{q} \mathsf{N}^{p+1}(p+i)$, there are $\sum_{i_1=1}^{q+1} \sum_{i_2=1}^{q(i_1)} \cdots\cdots \sum_{i_{p-1}=1}^{q(i_1, i_2, \ldots, i_{p-2})} q(i_1, i_2, \ldots, i_{p-1}) = {}^{\#}\mathsf{N}^{p+1}(p+q+1)$ pairwise disjoint elements of $\mathscr{A}$, $\{M_t: t \in \mathsf{N}^{p+1}(p+q+1)\}$, contained in $X \backslash \cup \left\{M_r: r \in \bigcup_{i=1}^{q} \mathsf{N}^{p+1}(p+i)\right\}$, such that $e(M_t) \notin \left\langle E_{p(t)} \cup \left\{e(M_r): r \in \bigcup_{i=1}^{q} \mathsf{N}^{p+1}(p+i)\right\}\right\rangle$ for each

$t \in \mathsf{N}^{p+1}(p+q+1)$. Moreover, $l_0^\infty\left(X \backslash \cup\left\{M_r: r \in \bigcup_{i=1}^{q+1} \mathsf{N}^{p+1}(p+i)\right\}, \mathscr{A}\right) \nsubseteq E_s + F$ for every $F \in \mathscr{F}$, for each $s \in \mathscr{I}_{q+1}$ and some sets of type (0)-($p-1$).

THEOREM 1. *$l_0^\infty(X, \mathscr{A})$ is barrelled of class $\aleph_0$.*

PROOF. Assume $l_0^\infty(X, \mathscr{A})$ is not barrelled of class $p, p > 2$. Then there is an increasing sequence of subspaces of $l_0^\infty(X, \mathscr{A})$ covering it, $\{F_n: n \in \mathsf{N}\}$, such that no F_n is barrelled of class $p-1$. In the same way, making $j = 1, \ldots, p-1$, for each $(n_1, n_2, \ldots, n_j) \in \mathsf{N}^j$ there is an increasing sequence of subspaces of $F_{n_1, \ldots, n_j}$ covering it, $\{F_{n_1, \ldots, n_j, n}: n \in \mathsf{N}\}$, such that no $F_{n_1, \ldots, n_j, n}$ is barrelled of class $p - (j+1)$. So, for each $s \in \mathsf{N}^p$ there is an increasing sequence of closed absolutely convex subsets of F_s, $\{B_{s,n}: n \in \mathsf{N}\}$, covering F_s such that no $B_{s,n}$ is a neighbourhood of the origin in F_s. We may assume without any loss of generality that $2B_{s,n} \subset B_{s,n+1}$ for each $n \in \mathsf{N}$. Let $R_{s,n}$ be the closure of $B_{s,n}$ in $l_0^\infty(X, \mathscr{A})$ and $E_s := \bigcup_{n=1}^{\infty} R_{s,n}$ for each $s \in \mathsf{N}^p$. Now, $l_0^\infty(X, \mathscr{A})$ being barrelled, for each $s \in \mathsf{N}^p$, $l_0^\infty(X, \mathscr{A}) \nsubseteq E_s + F$ for every $F \in \mathscr{F}$. Therefore, by Lemma 4, there is a countable family of pairwise disjoint elements of $\mathscr{A}$, $\{M_t: t \in \mathsf{N}^{p+1}\}$, an increasing sequence of positive $\{m(i): i \in \mathsf{N}\}$; for each $i_1 \in \mathsf{N}$, an increasing sequence of positive integers $\{m(i_1, i): i \in \mathsf{N}\}$; and, in general, for $j = 1, \ldots, p-1$, given $(i_1, i_2, \ldots, i_j) \in \mathsf{N}^j$ there is an increasing sequence of positive integers $\{m(i_1, i_2, \ldots, i_j, i): i \in \mathsf{N}\}$ sucht that setting for each $t = (i_1, i_2, \ldots, i_{p+1}) \in \mathsf{N}^{p+1}$, $p(t) := (m(i_1), m(i_1, i_2), \ldots, m(i_1, i_2, \ldots, i_p)) \in \mathsf{N}^p$, it holds $e(M_t) \notin \langle E_{p(t)} \cup \{e(M_r): r \in \mathsf{N}^{p+1}$ and $^\# r < {}^\# t\}\rangle$ for each $t \in \mathsf{N}^{p+1}$.

If for each $t = (i_1, i_2, \ldots, i_{p+1}) \in \mathsf{N}^{p+1}$, we set $T_t := R_{m(i_1), m(i_1, i_2), \ldots, m(i_1, i_2, \ldots, i_p), i_{p+1}}$, *then* $e(M_t) \notin 3(T_t + \delta_t \Gamma\{e(M_r): r \in \mathsf{N}^{p+1}$ and $^\# r < {}^\# t\})$, δ_t being card $\{e(M_r): r \in \mathsf{N}^{p+1}$ and $^\# r < {}^\# t\}$, and, by the Hahn-Banach theorem, there is some continuous linear form u_t on $l_0^\infty(X, \mathscr{A})$ such that $|\langle e(M_t), u_t\rangle| > 3$, $\Sigma\{|\langle e(M_r), u_t\rangle|: r \in \mathsf{N}^{p+1}$ and $^\# r < {}^\# t\} \leqq 1$ and $|\langle z, u_t\rangle| \leqq 1$ for every $z \in T_t$, (*).

Next we shall find a decreasing sequence $\{N_{\alpha(n)}: n \in \mathsf{N}\}$ of subsets of N^{p+1} such that for each $s \in \mathsf{N}^p$, there are infinitely many elements in each $N_{\alpha(n)}$ whose first p coordinates are just those of s and satisfy $\|u_{\alpha(n)}(\cup\{M_r: r \in N_{\alpha(n)}\})\| < 1$ for each $n \in \mathsf{N}$, (**).

Set $G := \cup\{M_t: t \in \mathsf{N}^{p+1}\}$ and let m be a positive integer such that $\|u_{\alpha(1)}(G)\| < m$. Let $\{P_r: 1 \leqq r \leqq m\}$ be a partition of N^{p+1} such that in each P_r, given any $s \in \mathsf{N}^p$, there are infinitely many elements whose first p coordinates are just those of s. Clearly now, [10], $\sum_{r=1}^{m} \|u_{\alpha(1)}(\cup\{M_t: t \in P_r\})\| \leqq \|u_{\alpha(1)}(G)\| < m$. Hence, there exists some P_j, $1 \leqq j \leqq m$, such that $\|u_{\alpha(1)}(\cup\{M_t: t \in P_j\})\| < 1$. Then we shall take $N_{\alpha(1)} := P_j$.

Let us find $N_{\alpha(n+1)}$ assuming we have determined $N_{\alpha(n)}$, $n \in \mathbf{N}$.

Let q be a positive integer such that $\|u_{\alpha(n+1)}(G)\| < q$. Let $\{Q_r: 1 \leqq r \leqq q\}$ be a partition of $N_{\alpha(n)}$ such that in each Q_r, given any $s \in \mathbf{N}^p$, there are infinitely many elements whose first p coordinates are just those of s. As $\sum_{r=1}^{q} \|u_{\alpha(n+1)}(\cup\{M_t: t \in Q_r\})\| \leqq \|u_{\alpha(n+1)}(G)\| < q$, there exists some Q_h, $1 \leqq h \leqq q$, such that $\|u_{\alpha(n+1)}(\cup\{M_t: t \in Q_h\})\| < 1$. Then we shall take $N_{\alpha(n+1)} := Q_h$.

Next we shall find a sequence $S = \{t(n): n \in \mathbf{N}\}$ in $\mathbf{N}^{p+1}$ such that for each $n \in \mathbf{N}$,

a) $t(n+1) \in N_{t(n)}$.

b) ${}^{\#}t(n) < {}^{\#}t(n+1)$.

c) $\{T_{t(n)}: n \in \mathbf{N}\}$ covers $l_0^\infty(X, \mathscr{A})$.

In order to do it we take $t(1) := \alpha(1)$ and assuming we have obtained the first $n-1$ elements of S we shall take as $t(n)$ the first element of $N_{\alpha(n-1)}$ whose first p coordinates are the same as those of $t(n-1)$ and the coordinate $p+1$ is such that ${}^{\#}t(n-1) < {}^{\#}t(n)$.

Taking $Q := \cup\{M_{t(n)}: n \in \mathbf{N}\}$, by property c) of S, there exists some $t(n_0) \in S$ such that $e(Q) \in T_{t(n_0)}$. By (*), this implies that $|\langle e(Q), u_{t(n_0)}\rangle| \leq 1$. But S also satisfies property b), so

$$\begin{aligned}\langle e(Q), u_{t(n_0)}\rangle = {} & \langle e(M_{(t(n_0)}), u_{t(n_0)}\rangle + \langle e(\cup\{M_{t(n)}: n < n_0\}), u_{t(n_0)}\rangle \\ & + \langle e(\cup\{M_{t(n)}: n > n_0\}), u_{t(n_0)}\rangle.\end{aligned}$$

Lastly, using property a) of S,

$$\begin{aligned}|\langle e(Q), u_{t(n_0)}\rangle| \geqq {} & |\langle e(M_{t(n_0)}), u_{t(n_0)}\rangle| - \Sigma\{|\langle e(M_r), u_{t(n_0)}\rangle|: {}^{\#}r < {}^{\#}t(n_0)\} \\ & - \|e(\cup\{M_r: r \in N_{t(n_0)})\|.\end{aligned}$$

From this, (*) and (**), it follows that $|\langle e(Q), u_{t(n_0)}\rangle| > 1$. Contradiction.

3. Application to vector measures theory.

DEFINITION. We shall say a family of linear subspaces $\{E_{n_1}, \ldots, n_p: (n_1, n_2, \ldots, n_p) \in \mathbf{N}^p\}$ of E is *p-increasing* if given any $(n_1, n_2, \ldots, n_{p-1}) \in \mathbf{N}^{p-1}$, $\{E_{n_1}, \ldots, n_p: n_p \in \mathbf{N}\}$ is increasing and for each $k \in \{2, \ldots p-1\}$,

$$\left\{ \bigcup_{n_k=1}^{\infty} \bigcup_{n_{k+1}=1}^{\infty} \cdots\cdots \bigcup_{n_p=1}^{\infty} E_{n_1, \ldots, n_{k-1}, n_k \ldots n_p} : n_{k-1} \in \mathbf{N} \right\} \text{ is increasing.}$$

Let us recall a locally convex space E is a Γ_r-space if given any quasi-complete subspace G of $E^*(\sigma(E^*, E))$ such that $G \cap E'$ is dense in $E'(\sigma(E', E))$, then G contains E', and that Γ_r-spaces are the maximal class of locally convex spaces verifying the closed graph theorem when barrelled spaces are considered in the domain, see [9] and [12, Ch. 1 §6.2].

PROPOSITION 1. *Let $\mathscr{W}$ be a p-increasing family of linear subspaces of a space*

E covering E and let f be a linear mapping of $l_0^\infty(X, \mathscr{A})$ *into E with closed graph. If each* $L \in \mathscr{W}$ *has a locally convex topology* τ_L *stronger than the induced topology such that* $L(\tau_L)$ *is a* Γ_r*-space, then there is a* $G \in \mathscr{W}$ *such that* $\operatorname{Im} f \subset G$ *and* f: $l_0^\infty(X, \mathscr{A}) \to G(\tau_G)$ *is continuous.*

PROOF. By Theorem 1, there is some $G \in \mathscr{W}$ such that $F := f^{-1}(G)$ is barrelled and dense in $l_0^\infty(X, \mathscr{A})$. So, given $x \in l_0^\infty(X, \mathscr{A}) \backslash F$, the set $L := \langle \{x\} \cup F \rangle$ is barrelled. Let $h: L \to G$ be a linear extension of the restriction of f to F with closed graph. Then, by the closed graph theorem of Valdivia, h is continuous. Therefore if $\{x_n: n \in \mathsf{N}\}$ is a sequence in F which converges to x in $l_0^\infty(X, \mathscr{A})$, $\{h(x_n): n \in \mathsf{N}\}$ converges to $h(x)$ in G and, the graph of f being closed, $f(x) = h(x) \in G$. Hence $\operatorname{Im} f \subset G$ and f: $l_0^\infty(X, \mathscr{A}) \to G(\tau_G)$ is continuous.

[1, Theorem 1.1, (i) ⇒ (iii)] and [2, Corollary 3, 1.3] were generalized in [6, Theorems 2 and 3]. With the help of Theorem 1 it is easy to strengthen the results obtained in [6].

THEOREM 2. *Let* μ *be a finitely additive measure on* $\mathscr{A}$ *with values in a space E and let H be a* $\sigma(E', E)$*-total subset of* E'. *Let* $\mathscr{W}$ *be a p-increasing family of linear subspaces of E covering E such that each* $L \in \mathscr{W}$ *has a locally convex topology* τ_L, *stronger than the induced topology, such that* $L(\tau_L)$ *is a sequentially complete* Γ_r*-space which does not contain a copy of* l^∞. *If* $u \circ \mu$ *is a countably additive scalar measure for each* $u \in H$, *then there exists some* $G \in \mathscr{W}$ *such that* μ: $\mathscr{A} \to G$ *and* μ *is countably additive.*

THEOREM 3. *Let* μ *be a mapping of* $\mathscr{A}$ *into a space E and let H be a* $\sigma(E', E)$*-total subset of* E', *let* $\mathscr{W}$ *be a p-increasing family of linear subspaces of E such that each* $L \in \mathscr{W}$ *has a locally convex topology* τ_L, *stronger than the induced topology, such that* $L(\tau_L)$ *is a* Γ_r*-space. If* $u \circ \mu$ *is a bounded finitely additive scalar measure for each* $u \in H$, *then there is some* $G \in \mathscr{W}$ *such that* μ: $\mathscr{A} \to G$ *and* μ *is a bounded vector measure.*

ACKNOWLEDGMENT. The authors are very grateful to the referee for his helpful comments.

REFERENCES

1. J. Diestel and B. Faires, *On vector measures*, Trans. Amer. Math. Soc. 198 (1974), 253–271.
2. J. Diestel and J. J. Uhl, *Vector Measures*, Math. Surveys 15. Amer. Math. Soc. Providence 1977.
3. J. C. Ferrando amd M. López-Pellicer, *On ordered subrabarrelled spaces*, Arch. Math. 53 (1989), 405–410.
4. J. C. Ferrando and M. López-Pellicer, *Barrelled spaces of class n and of class n and of class* $\aleph_0$, An. Sem. Mat. UNED, to appear.

5. J. C. Ferrando and M. López-Pellicer, *Strong barrelledness properties in* $l_0^\infty(X, \mathcal{A})$ *and bounded finite additive measures*, Math. Ann. 287 (1990), 727–736.
6. J. C. Ferrando and M. López-Pellicer, *An ordered suprabarrelled space*, J. Austral. Math. Soc., (Series A) 51 (1991), 106–117.
7. P. Perez Carreras and J. Bonet, *Barrelled Locally Convex Spaces*, Math. Studies 131, North Holland, 1987.
8. S. A. Saxon, *Nuclear and product spaces, Baire-like spaces and the strongest locally convex topology*, Math. Ann. 197 (1972), 87–106.
9. M. Valdivia, *Sobre el teorema de la gráfica cerrada*, Collect. Math. 22 (1971), 51–72.
10. M. Valdivia, *On certain barrelled normed spaces*, Ann. Inst. Fourier 29 (1979), 39–56.
11. M, Valdivia, *On suprabarrelled spaces*. Lect. Not. in Math. 843. Func. Anal. Holomorphy and Approximation Theory. Rio de Janeiro 1978, 572–580. Springer Verlag, 1981.
12. M. Valdivia, *Topics in Locally Convex Spaces*. Math. Studies 67, North Holland, 1982.

DEPARTEMENTO DE MATEMÁTICA APLICADA
UNIVERSIDAD POLITÉCNICA DE VALENCIA
APARTADO 22012
46071 VALENCIA
SPAIN