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MEROMORPHIC SOLUTIONS OF HIGHER ORDER SYSTEM OF ALGEBRAIC DIFFERENTIAL EQUATIONS

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1. Introduction.

In [1], [2] the first author has employed Nevanlinna's meromorphic function theory to study the existence of meromorphic solutions to the first order system of algebraic differential equations:

$$P_t(W_t', W_2', W_1, W_2, z) = f_t(W_t), \quad t = 1, 2,$$

where P_t is a differential polynomial in W_1 and W_2, with rational functions in z as coefficients, and has obtained Wittich and Malmquist type theorems, (see [3], [4]).

Over the years, Yang, [5], Laine, [6], Bank, and Kaufman, [7], Yosida, [8], Gackstatter, and Laine, [9], He Yuzan and Xiao Xiuzhi, [10], [11], [12], have worked on the question of the existence of algebroid and meromorphic solutions to scalar algebraic differential equations having meromorphic functions in z as coefficients. However, they have not dealt with the case of systems. This paper addresses such a situation and extends results obtained in [1], [2] to higher order cases. If only meromorphic solutions are considered, the result obtained by He Yuzan, [10] is a special case of ours.

Here, we will mainly discuss the following two types of higher order system of algebraic differential equations

$$\Omega_t(W_1, W_2) = f_t[Q_{tt}(W_t, z)], \quad t = 1, 2 \tag{1.1}$$

$$\Omega_t(W_1, W_2) = Q_{t1}(W_1, z) Q_{t2}(W_2, z), \quad t = 1, 2 \tag{1.2}$$

where $\Omega_t(W_1, W_2)$ is a differential polynomial and

$$\Omega_t(W_1, W_2) = \sum_i A_{ti}(z) W_1^{i_{t10}} \ldots [W_1^{(k_{t1k})}]^{i_{t1k_{t1}}} W_2^{i_{t20}} \ldots [W_2^{(k_{t2})}]^{i_{t2k_{t2}2}},$$

$$Q_{ts}(W_s, z) = P_{ts1}(W_s, z)/P_{ts2}(W_s, z) = \sum_j a_{tsj}(z) W_s^j \Big/ \sum_j b_{tsj}(z) W_s^j$$

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$(t, s = 1, 2)$, and the highest degree of W_s in $Q_{ts}(W_s, z)$ is q_{ts}; the coefficients $A_{ti}(z)$, $a_{tsj}(z)$ and $b_{tsj}(z)$ are meromorphic functions in z; and $f_t(\omega_t)$ is a transcendental meromorphic function in ω_t, unless otherwise stated.

To end this section, we give the following definitions;

DEFINITION 1. For the general term

$$A_{ti}(z)W_1^{i_{t10}} \dots [W_1^{(k_{t1})}]^{i_{t1k_{t1}}} W_2^{i_{t20}} \dots [W_2^{(k_{t2})}]^{i_{t2k_{t2}}}$$

in the differential polynomial $\Omega_t(W_1, W_2)$, let

$$\lambda_{ts}(i) = i_{ts0} + i_{ts1} + \dots + i_{tsk_{ts}}, \tag{1.3}$$

$$\mu_{ts}(i) = i_{ts0} + 2i_{ts1} + \dots + (k_{ts} + 1)i_{tsk_{ts}}, \quad t, s = 1, 2, \tag{1.4}$$

$$\lambda_{ts} \max_i \{\lambda_{ts}(i)\} \tag{1.5}$$

$$\mu_{ts} = \max_i \{\mu_{ts}(i)\}. \tag{1.6}$$

DEFINITION 2. Let $\{A_i(z)\}$ be a (finite) family of meromorphic functions and $W(z)$ is a nonconstant meromorphic function. Suppose that $S(r, W)$ is a positive real function in r satisfying

$$S(r, W) = o\{T(r, W)\} \tag{1.7}$$

except possibly for a sequence of intervals Δ_r (on the r-axis) with finite total length. If

$$\sum_i T(r, A_i) = S(r, W), \tag{1.8}$$

then $W(z)$ is said to be *admissible* with respect to the family of meromorphic functions $\{A_i(z)\}$.

DEFINITION 3. Suppose that $\hat{S}(r, W)$ is a positive real function in r and satisfies

$$\hat{S}(r, W) = O\{T(r, W)\} \tag{1.9}$$

except possibly for a sequence of intervals Δ_r (on the r-axis) with finite total length. And $W(z)$ and $\{A_i(z)\}$ are as defined in Def. 2. If

$$\sum_i T(r, A_i) = \hat{S}(r, W), \tag{1.10}$$

then $W(z)$ is said to be *weakly admissible* with respect to the family of meromorphic functions $\{A_i(z)\}$.

DEFINITION 4. Suppose that $W_1(z)$, $W_2(z)$ is a pair of solutions for the system of equations (1.1) ((1.2)) and each of them is a nonconstant meromorphic function such that $Q_{tt}(W_t, z)$ and $f_t(Q_{tt}(W_t, z))$ $(Q_{ts}(W_s, z))$ are (is) nonconstant meromorphic functions (function). Then

(i) if $W_1(z)$ and $W_2(z)$ are (weakly) admissible with respect to the family of the coefficients $\{A_{ti}(z),\ a_{ttj}(z),\ b_{ttj}(z)\}$ $(\{A_{ti}(z),\ a_{tsj}(z),\ b_{tsj}(z)\})$, then they are called a *(weakly) admissible solution of class* I;

(ii) if only $W1(z)$ is admissible with respect to the family of the coefficients $\{A_{ti}(z),\ a_{ttj}(z),\ b_{ttj}(z)\}$ $(\{A_{ti}(z),\ a_{tsj}(z),\ b_{tsj}(z)\})$, then they are called an *admissible solution of class* II_1;

(iii) if only $W_2(z)$ is admissible with respect to the family of the coefficients $\{A_{ti}(z),\ a_{ttj}(z),\ b_{ttj}(z)\}$ $(\{A_{ti}(z),\ a_{tsj}(z),\ b_{tsj}(z)\})$, then they are called an *admissible solution of class* II_2;

(iv) if only $W_1(z)$ is weakly admissible with respect to the family of the coefficients $\{A_{ti}(z),\ a_{ttj}(z),\ b_{ttj}(z)\}$ $(\{A_{ti}(z),\ a_{tsj}(z),\ b_{tsj}(z)\})$, and $W_2(z)$ is not admissible with respect to the same family, then they are called a *weakly admissible solution of class* II_1;

(v) if only $W_2(z)$ is weakly admissible with respect to the family of the coefficients $\{A_{ti}(z),\ a_{ttj}(z),\ b_{ttj}(z)\}$ $(\{A_{ti}(z),\ a_{tsj}(z),\ b_{tsj}(z)\})$, and $W_1(z)$ is not admissible with respect to the same family, then they are called a *weakly admissible solution of class* II_2.

2. Lemmas.

To prove the main theorems given in Section 3, we need the following lemmas:

LEMMA 1. *Given the differential polynomial*

$$\Omega(W_1, W_2) = \sum_i A_i(z) W_1^{i_{10}} \dots [W_1^{(k_1)}]^{i_{1k_1}} W_2^{i_{20}} \dots [W_2^{(k_2)}]^{i_{2k_2}},$$

where the coefficients $A_i(z)$ are meromorphic functions in z. If W_1 $(i = 1,2)$ is a nonconstant meromorphic function and W_1 is admissible with respect to the family of the coefficients $\{A_i(z)\}$, then for any fixed $\eta > 0$, the following inequality holds, except possibly for a sequence of intervals A_r (on the r-axis) with a finite total length,

$$T(r, \Omega(W_1, W_2)) < (\eta + \mu_1)T(r, W_1) + (\eta + \mu_2)T(r, W_2) + S(r, W_1) \tag{2.1}$$

where $\mu_s = \max_i (\mu_s)),$

$$\mu_s(i) = i_{s0} + 2i_{s1} + \dots + (k_s + 1)i_{sk_s}, \quad s = 1,2;$$

$$S(r, W_1) = \sum_i T(r, A_i) = o(T(r, W_1)).$$

PROOF. Suppose $|z| = 1$ is divided into the following subsets,

$$E_+ = \{z \mid |z| = 1, |W_i(z)| \geqq 1, i = 1, 2\},$$

$$E_{+1} = \{z \mid |z| = 1, |W_1(z)| \geqq 1 \text{ and } |W_2(z)| < 1\},$$

$$E_{+2} = \{z \mid |z| = 1, |W_1(z)| < 1 \text{ and } |W_2(z)| \geqq 1\},$$

$$E_- = \{z \mid |z| = 1, |W_i(z)| < 1, i = 1, 2\}.$$

Hence for $z \in E_+$,

$$\begin{aligned}|\Omega(W_1, W_2)| &\leqq \sum_i \{|A_i(z)|\, |W_1|^{\lambda_1(i)}\, |W_1'/W_1]^{i_{11}} \ldots |W_1^{(k_1)}/W_1]^{i_{1k_1}} \\ &\quad \times |W_2|\lambda^{2(i)}\, |W_2'/W_2|^{i_{21}} \ldots |W_2^{(k_2)}/W_2|^{i_{2k_2}}\} \\ &\leqq |W|^{\lambda_1}\, |W_2|^{\lambda_2} \sum_i \{|A_i(z)|\, |W_1'/W_1|^{i_{11}} \ldots |W_1^{(k_1)}/W_1|^{i_{1k_1}} \\ &\quad \times |W_2'/W_2|^{i_{21}} \ldots |W_2^{(k_2)}/W_2|^{i_{2k_2}}\},\end{aligned}$$

where $\lambda_s(i) = i_{s0} + i_{s1} + \ldots + i_{sk_s}$ $(s = 1, 2)$, $\lambda_s = \max_i (\lambda_s(i))$. Therefore, the following inequality holds,

$$\begin{aligned}(2.2)\quad (1/2\pi) \int_{E_+} \ln^+ |\Omega(W_1, W_2)|\, d\phi &\leqq \sum_{s=1}^{2} \Bigg[(\lambda_s/2\pi) \int_{E} \ln^+ |W_s|\, d\phi \\ &\quad + \sum_{t=1}^{k_s} (i_{st}/2\pi) \int_{E_+} \ln^+ |W_s^{(t)}/W_s|\, d\phi \Bigg] \\ &\quad \times \sum_i (1/2\pi) \int_{E_+} \ln^+ |A_1(z)|\, d\phi + O(1)\end{aligned}$$

$$\begin{aligned}(2.3)\quad (1/2\pi) \int_{E_{+1}} \ln^+ |\Omega(W_1, W_2)|\, d\phi &\leqq (\lambda_1/2\pi) \int_{E_{+1}} \ln^+ |W_1|\, d\phi \\ &\quad + \sum_{s=1}^{2} \sum_{t=1}^{k_s} (i_{st}/2\pi) \int_{E_{+1}} \ln^+ |W_s^{(t)}/W_s|\, d\phi \\ &\quad + \sum_i (1/2\pi) \int_{E_{+1}} \ln^+ |A_i(z)|\, d\phi + O(1)\end{aligned}$$

$$(2.4)\quad (1/2\pi)\int_{E_{+2}} \ln^+ |\Omega(W_1, W_2)|\, d\phi \leqq (\lambda_2/2\pi)\int_{E_{+2}} |W_2|\, d\phi$$

$$+ \sum_{s=1}^{2}\sum_{t=1}^{k_s} (i_{st}/2\pi)\int_{E_{+2}} \ln^+ |W_s^{(t)}/W_s|\, d\varphi$$

$$+ \sum_i (1/2\pi)\int_{E_{+2}} \ln^+ |A_i(z)|\, d\phi + O(1)$$

$$(2.5)\quad (1/2\pi)\int_{E_-} \ln^+ |\Omega(W_1, W_2)|\, d\phi \leqq \sum_{s=1}^{2}\sum_{t=1}^{k_s} (i_{st}/2\pi)\int_{E_-} \ln^+ |W_s^{(t)}/W_s|\, d\phi$$

$$+ \sum_i (1/2\pi)\int_{E_-} \ln^+ |A_i(z)|\, d\phi + O(1).$$

Note that W_1 is admissible with respect to the family of the coefficients $\{A_i(z)\}$, and in view of (2.2)-(2.5) and a basic lemma of logarithmic derivative of meromorphic functions:

$$m(r, W_s^{(1)}/W_s) = O\{\ln [rT(r, W_s)]\}$$

holds except possibly a for sequence of intervals Δ_r (on the r-axis) with a finite total length (see [4]). It is not difficult to deduce that

$$(2.6)\quad m(r, \Omega(W_1, W_2)) = (1/2\pi)\sum_{0}^{2\pi} \ln^+ |\Omega(W_1, W_2)|\, d\phi$$

$$= (1/2\pi)\left\{\int_{E_+} + \int_{E_{+1}} + \int_{E_{+2}} + \int_{E_-}\right\} \ln^+ |\Omega(W_1, W_2)|\, d\phi$$

$$\leqq \lambda_1 m(r, W_1) + \lambda_2 m(r, W_2) + \sum_{s=1}^{2} \eta T(r, W_s) + S(r, W_1)$$

where the total length of the sequence of intervals Δ_r of r depends on η. On the other hand, a pole of $\Omega(W_1, W_2)$ must also be a pole of some of its terms, and its multiplicity is less than or equal to the highest order of these terms. By letting the general term in $\Omega(W_1, W_2)$ be

$$F_i(z) = A_i(z) W_1^{i_{10}} \dots [W_1^{(k_1)}]^{i_{1k_1}} W_2^{i_{20}} \dots [W_2^{(k_2)}]^{i_{2k_2}},$$

then

$$n(r, F_i) \leqq n(r, A_i) + i_{10}n(r, W_1) + \dots + i_{1k_1}n(r, W_1^{(k_1)})$$
$$+ i_{20}n(r, W_2) + \dots + i_{2k_2}n(r, W_2^{(k_2)}).$$

Note that $n(r, W_i^{(t)}) \leqq (t + 1)n(r, W_i)$, hence (2.7) becomes

$$n(r, F_i) \leqq n(r, A_i) + \sum_{s=1}^{2} \mu_s(i)n(r, W_s).$$

Thus the following inequality holds

$$n(r, \Omega(W_1, W_2)) \leqq \sum_i n(r, A_i) + \sum_{s=1} \mu_s n(r, W_s).$$

Therefore, we obtain

$$N(r, \Omega(W_1, W_2)) \leqq \mu_1 N(r, W_1) + \mu_2 N(r, W_2) + S(r, W_1). \tag{2.8}$$

From (2.6) and (2.8), we have

$$\begin{aligned} T(r, \Omega(W_1, W_2)) &= m(r, \Omega(W_1, W_2)) + N(r, \Omega(W_1, W_2)) \\ &\leqq (\eta + \mu_1)T(r, W_q) + (\eta + \mu_2)T(r, W_2) + S(r, W_1). \end{aligned} \tag{2.9}$$

This completes the proof to lemma 1.

COROLLARY 1. *If W_2 is admissible with respect to the family of the coefficients $\{A_i(z)\}$, then*

$$T(r, \Omega(W_1, W_2)) \leqq (\eta + \mu_1)T(r, W_1) + (\eta + \mu_2)T(r, W_2) + S(r, W_2) \tag{2.10}$$

where $S(r, W_2) = \sum_i T(r, A_i) = o(T(r, W_2))$.

COROLLARY 2. *If W_1 is weakly admissible with respect to the family of the coefficients $\{A_i(z)\}$, then*

$$T(r, \Omega(W_1, W_2)) \leqq (\eta + \mu_1)T(r, W_1) + (\eta + \mu_2)T(r, W_2) + \hat{S}(r, W_1) \tag{2.11}$$

where $\hat{S}(r, W_1) = \sum_i T(r, A_i) = O(T(r, W_1))$.

The proofs to the above corollaries are similar to that of lemma 1, and we omit the details.

LEMMA 2. *Let $f(\omega)$ be a transcendental meromorphic function in ω and ω is a nonconstant entire function in z. Then for any fixed $m > 0$, there exists $r_0 > 0$, such that when $r > r_0$, the inequality*

$$T(r, f(\omega)) > (m/2)T(r, \omega) + O(1) \tag{2.12}$$

holds.

This is a classical result due to Clunie, (see [12] and [13]).

LEMMA 3. *Let*

$$R(W, z) \equiv P_1(W, z)/P_2(W, z) \equiv \sum_h a_h(z) W^h / \sum_j b_j(z) W^j$$

and the highest degree of W is γ, $\{a_h(z)\}$ and $\{b_j(z)\}$ are meromorphic functions. If $W(z)$ is a nonconstant meromorphic function and is (weakly) admissible with respect to the family of the coefficients $\{a_h(z), b_j(z)\}$, then

$$T(r, R(W, z)) = \gamma T(r, W) + S(r, W) \tag{2.13}$$

where

$$S(r, W) = \sum_h T(r, a_h) + \sum_j T(r, b_j) = \begin{cases} o(T(r, W)). & \text{if } W(z) \text{ is admissible;} \\ O(T(r, W)), & \text{if } W(z) \text{ is weakly admissible.} \end{cases}$$

For the proofs of the lemma, one may consult [15], [16].

LEMMA 4. *Let Δ_r be a sequence of intervals of r which has a (finite) total length μ. Let*

$$a = \exp(4\mu). \tag{2.14}$$

Then for any $\bar{r} \in [a, +\infty) \cap \Delta_r$, there must exist

$$r', r'' \in (1, +\infty) - (1, +\infty) \cap \Delta_r$$

such that

$$\log \bar{r} = (1/2)[\log r' + \log r''] \tag{2.15}$$

i.e.

$$\bar{r} = r' r''. \tag{2.16}$$

PROOF. Under the mapping $x = \log r$ the set $(1 + \infty) \cap \Delta_r$ is mapped into set Δ_1 on the x-axis. Obviously, Δ_1 is also a sequence of intervals with total length less than μ. For any fixed $\bar{r} \in [a, +\infty) \cap \Delta_r$, we can construct an interval $[\log \bar{r} - (1/2)\log a, \log \bar{r}]$ on the x-axis with length $(1/2)\log a = 2\mu$ from (2.14). Under the maping $y = -x + 2\log \bar{r}$ the set (on x-axis)

$$\Delta_2 = [\log \bar{r} - (1/2)\log a . \log \bar{r}] - \Delta_1 \cap [\log \bar{r} - (1/2)\log a, \log \bar{r}],$$

a sequence of intervals with total length greater than μ, is mapped into a sequence of intervals Δ_3 on the x-axis. It is included in the interval $[\log \bar{r}, \log \bar{r} + (1/2)\log a]$ and its total length is the same as Δ_2's, i.e., greater than μ. Thus, the total length of the sequence of intervals

$$\Delta_4 = \Delta_3 - \Delta_1 \cap [\log \bar{r}, \log \bar{r} + (1/2)\log a]$$

is greater than 0. Hence, if we take $y_0 \in \Delta_4$, then $x_0 \in \Delta_2$ but $\notin \Delta_1$ and

$$y_0 = -x_0 + 2\log \bar{r} \tag{2.17}$$

since $y_o \in \Delta_3$ but $\notin \Delta_1$. By letting $r' = \exp(x_0)$ and $r'' = \exp(y_0)$, can be reduced to (2.15) and r', r'' satisfy the following conditions:

$$r' \geqq \exp[\log \bar{r} - (1/2)\log a] \geqq \exp[(1/2)\log a] = \exp(2\mu) > 1,$$

$$r'' \geqq \exp(\log \bar{r}) \geqq \exp(\log a) = \exp(4\mu) > 1, \text{ and } r', r'' \notin (1, +\infty) \cap \Delta_r.$$

Therefore, this completes the proof to lemma 4.

Lemma 5. *Let Δ_r and a be as in lemma* 4, *and $T_1(r)$, $T_r(r)$ be increasing functions in r and convex functions in* $\log r$. *If for $r \notin \Delta_r$, the inequality*

$$T_1(r) \leqq T_2(r) \tag{2.18}$$

holds, then for all $r \in [a, +\infty)$ where a is defined by (2.14), *the following inequality holds*

$$T_1(r) \leqq T_2(r^2). \tag{2.19}$$

Proof. If $r \in (1, +\infty) - (1, +\infty) \cap \Delta_r$, (2.19) clearly holds. So, we only need to consider the case when $r \in [a, +\infty) \cap \Delta_r$.

If $r \in [a, +\infty) \cap \Delta_r$, then by lemma 4, there exist $r', r'' \in (1, +\infty) - (1, +\infty) \cap \Delta_r$, such that $\log r = (1/2)(\log r' + \log r'')$ i.e. $r^2 = r'r''$ ($> r'$ and r''). By the convexity of $T_1(r)$, we have

$$\begin{aligned} T_1(r) = T_1[\exp(\log r)] &= T_1\{\exp[(1/2)(\log r' + \log r'')]\} \\ &\leqq (1/2)\{T_1[\exp(\log r')] + T_1[\exp(\log r'')]\} \\ &\leqq (1/2)[T_1(r') + T_1(r'')]. \end{aligned}$$

From (2.18), together with the fact that $T_2(r)$ is increasing, it follows that

$$\begin{aligned} T_1(r) &\leqq (1/2)[T_2(r') + T_2(r'')] \\ &\leqq (1/2)[T_2(r^2) + T_2(r^2)] = T_2(r^2). \end{aligned}$$

This completes the proof to lemma 5.

3. Main theorems.

We will now proceed to the theorems of the existence or nonexistence of meromorphic solutions to the higher order system of algebraic differential equations (1.1) and (1.2).

THEOREM 1. *There does not exist any admissible solution and weakly admissible solution to the system of equations* (1.1).

PROOF. We only give the proof to the non-existence of weakly admissible solution of class II_1, since other proofs are similar.

Suppose that W_1, W_2 are weakly admissible solutions of class II_1 to the system of equations (1.1). When we substitute them into (1.1), we have the following

Case I. At least one of $Q_{tt}(W_t, z)$ is a non-entire meromorphic function, e.g., $Q_{11}(W_1, z)$, hence $f_1[Q_{11}(W_1, z)]$ will have an essential singularity at finite z but this contradicts the assumption that $f_1[Q_{11}(W_1, z)]$ is meromorphic. Thus, W_1, W_2 cannot be solutions of (1.1).

Case II. $Q_{tt}(W_t, z)$ is a nonconstant entire function. Therefore, by corollary 2 to lemma 1, for any fixed $\eta > 0$, there exists a sequence of intervals Δ_r (on r-axis) with a finite total length, such that for any $r \notin \Delta_r$, the following inequality holds

$$\hat{S}(r, W_1) + (\eta + \mu_{t1})T(r, W_1) + (\eta + \mu_{t2}) > T(r, \Omega_t(W_1, W_2)) \tag{3.1}$$

where $t = 1, 2$. By lemma 2 and 3, for any fixed $m > 0$, the following inequality holds

$$\begin{aligned} T(r, f_t[Q_{tt}(W_t, z)]) &> (m/2)T(r, Q_{tt}(W_t, z)) + O(1) \\ &> (q_{tt}m/2)T(r, W_t) + \hat{S}(r, W_1) \end{aligned} \tag{3.2}$$

where $t = 1, 2$. Combining (1.1), (3.1) and (3.2) we have

$$\hat{S}(r, W_1) + (\eta + \mu_{12})T(r, W_2) > [(q_{11}m/2) - \eta - \mu_{11}]T(r, W_1) \tag{3.3}$$

and

$$\hat{S}(r, W_1) + (\eta + \mu_{2q})T(r, W_1) > [(q_{22}m/2) - \eta - \mu_{22}]T(r, W_2).$$

Now (3.3) and (3.4) imply

$$\begin{aligned} \hat{S}(r, W_1) + (\eta + \mu_{12})(\eta + \mu_{21})T(r, W_1) &> [(q_{11}m/2) - \eta - \mu_{11}] \\ &\times [(q_{22}m/2) - \eta - \mu_{22}]\mathrm{T}(\mathrm{r}, \mathrm{W}_1). \end{aligned}$$

Dividing both sides of the above inequality by $T(r, W_1)$ and then taking limit as $r \to \infty$, $r \notin \Delta_r$ we get

$$O(1) + (\eta + \mu_{12})(\eta + \mu_{21}) \geqq [(q_{11}m/2) - \eta - \mu_{11}][(q_{22}m/2) - \eta - \mu_{22}].$$

But as m can be taken to be sufficiently large, and η can be taken to be sufficiently small, whereas all other terms are constants, so that the above inequality cannot hold thus arriving at a contradiction. This completes the proof to theorem 1.

THEOREM 2. *In the system of equations* (1.1), *if* $f_2[Q_{22}(W_2, z)] \equiv 0$, *and there exists one term in the differential polynomial* $\Omega_2(W_1, W_2)$, e.g., *the* h*th term, such that*

(3.5) $$2h_{220} > \hat{\mu}_{22} + \mu_{22}(h)$$

where h_{220} is the power of W_2 in this term, $\mu_{22}(h)$ is defined by (1.4) *with $i = h$, and $\hat{\mu}_{22} = \max(\mu_{22}(i))$, then the system of equations* (1.1) *has at most admissible solution or weakly admissible solution of class* II_2.

PROOF. It is sufficient to prove that if W_1, W_2 are meromorphic solutions of (1.1) that makes $Q_{11}(W_1, z)$ a entire function, then W_1 is neither admissible nor weakly admissible with respect to the family of coefficients $\{A_{ti}(z), a_{ttj}(z), \{b_{ttj}(z)\}$. Since the steps involved in the proof are similar, we only prove that W_1 is not weakly admissible here.

Now, suppose W_1 is weakly admissible, then from (3.5) it follows that there exists a $\eta > 0$, such that

(3.6) $$2h_{220} > \hat{\mu}_{22} + \mu_{22}(h) + \eta.$$

Thus, we can deduce (3.3) for $t = 1$ as in theorem 1. On the other hand, we rewrite the second equation in (1.1), i.e., $\Omega_2(W_1, W_2) = 0$, as

$$\{\Omega_2(W_1, W_2) - A_{2h}(z)W_1^{h_{210}} \dots [W_1^{(k_{21})}]^{h_{21k_{21}}} W_2^{h_{220}} \dots [W_2^{(k_{22})}]^{h_{22k_{22}}}\}$$
$$/\{-A_{2h}(z)W_1^{h_{210}} \dots [W_1^{(k_{21})}]^{h_{21k_{21}}} [W_2']^{h_{221}} \dots [W_2^{(k_{22})}]^{h_{22k_{22}}}\} = W_2^{h_{220}}.$$

Applying corollary 2 of lemma 1 to the above equation, and after rearrangement (using also (2.13)) we obtain

$$\hat{S}(r, W_1) + (\eta + \mu_{21})T(r, W_1) + [\hat{\mu}_{22} + \mu_{22}(h) + \eta - h_{220}]T(r, W_2) > h_{220}T(r, W_2),$$

i.e.,

(3.7) $$\hat{S}(r, W_1) + (\eta + \mu_{21})T(r, W_1) > [2h_{220} - \hat{\mu}_{22} - \mu_{22}(h) - \eta]T(r, W_2).$$

Comparing (3.7) and (3.3) ($t = 1$), we get

$$\hat{S}(r, W_1) + (\eta + \mu_{12})(\eta + \mu_{21})T(r, W_1) > [(q_{11}m/2) - \eta - \mu_{11}]$$
$$\times [2h_{220} - \hat{\mu}_{22} - \mu_{22}(h) - \eta]T(r, W_1).$$

From this it follows that

$$O(1) + (\eta + \mu_{12}) \geqq [(q_{11}m/2) - \eta - \mu_{11}][2h_{220} - \hat{\mu}_{22} - \mu_{22}(h) - \eta].$$

But for m sufficiently large, the above inequality cannot hold. This completes the proof to theorem 2.

THEOREM 3. *In the system of equations* (1.1), *if $f_t[Q_{tt}(W_t. z)] \equiv 0$ ($t = 1,2$) and there exists one term in the differential polynomial $\Omega_t(W_1, W_2)$ ($t = 1,2$), e.g., the gth term in $\Omega_1(W_1, W_2)$ and the hth term in $\Omega_2(W_1, W_2)$, such that*

$$(3.8) \qquad 2g_{110} > \hat{\mu}_{11} + \mu_{11}(g),$$

$$(3.5) \qquad 2h_{220} > \hat{\mu}_{22} + \mu_{22}(h)$$

$$(3.9) \qquad [2g_{110} - \hat{\mu}_{11} - \mu_{11}(g)][2h_{220} - \hat{\mu}_{22} - \mu_{22}(h)] > \mu_{12}\mu_{21}$$

where g_{110} is the power of W_1 in the gth term of $\Omega_1(W_1, W_2)$, $\mu_{11}(g)$ is defined by (1.4) with $i = g$, $\hat{\mu}_{11} = \max_{i \neq g} [\mu_{11}(i)]$; and all the other notations are the same as in theorem 2, then there exist at most weekly admissible solutions to (1.1).

PROOF. It is sufficient to prove that for the meromorphic solutions W_1, W_2 of (1.1), either W_1 or W_2 cannot be admissible with respect to the family of coefficients $\{A_{ti}(z)\}$. Let us prove the above statement for W_2.

Now, suppose that W_2 is admissible. After we have chosen a small enough $\eta > 0$, so that

$$(3.10) \qquad 2g_{110} > \hat{\mu}_{11} + \mu_{11}(g) + \eta,$$

$$2h_{220} > \hat{\mu}_{22} + \mu_{22}(h) + \eta$$

and

$$[2g_{110} - \hat{\mu}_{11} - \mu_{11}(g) - \eta][2h_{220} - \hat{\mu}_{22} - \mu_{22}(h) - \eta] > (\eta + \mu_{12})(\eta + \mu_{21}),$$

then we can deduce the following, similar to the case of (3.7) in theorem 2,

$$(3.11) \qquad S(r, W_2) + (\eta + \mu_{12})T(r, W_2) > [2g_{110} - \hat{\mu}_{11} - \mu_{11}(g) - \eta]T(r, W_1)$$

$$(3.12) \qquad S(r, W_2) + (\eta + \mu_{21})T(r, W_1) > [2h_{220} - \hat{\mu}_{22} - \mu_{22}(h) - \eta]T(r, W_2).$$

Comparing (3.11) and (3.12) we have

$$S(r, W_2) + (\eta + \mu_{12})(\eta + \mu_{21})T(r, W_2) > [2g_{110} - \hat{\mu}_{11} - \mu_{11}(g) - \eta] \times [2h_{220} - \hat{\mu}_{22} - \mu_{22}(h) - \eta]T(r, W_2).$$

and hence

$$(\eta + \mu_{12})(\eta + \mu_{21}) \geqq [2g_{110} - \hat{\mu}_{11} - \mu_{11}(g) - \eta][2h_{220} - \hat{\mu}_{22} - \mu_{22}(h) - \eta].$$

But this contradicts (3.10). Therefore, this completes the proof.

THEOREM 4. *Suppose that $f_1[R(W_1, W_2)]$ and $f_2(W_1. W_2)$ replace $f_1(W_1)$ and $f_1(W_2)$ respectively in the system of equations (1.1), where $f_1(u)$ is a transcendental meromorphic function in u; $R(W_1, W_2)$ is a rational function in W_1 and W_2, in which the coefficients of W_1 and W_2 are meromorphic functions in z and the degree of W_1 and W_2 are $p > 0$ and $q > 0$ respectively: $f_2(W_1, W_2)$ is a meromorphic function in W_1, W_2 and their derivatives (it may also include z). That is,* (3.13) $$\Omega_1(W_1, W_2) = f_1[R(W_1. W_2)],$$

(3.14) $$\Omega_2(W_1, W_2) = f_2(W_1, W_2)$$

Then (i) *if it has admissible solutions* W_1, W_2 *of class* I, *then each of* W_1 *and* W_2 *is not admissible with respect to each other;*

(ii) *if it has admissible solutions* W_1, W_2 *of class* II_1, *then* W_1 *is not admissible with respect to* W_2*;*

(iii) *if it has admissible solutions* W_1, W_2 *of class* II_2, *then* W_2 *is not admissible with respect to* W_1.

PROOF. Without loss of generality, we only prove case (ii) with $R(W_1, W_2)$ being an entire function in z.

Now, suppose that W_1 is admissible with respect to W_2, i.e.

(3.15) $$T(r, W_2) = S(r, W_1)$$

Then, by lemma 3, we have

(3.16) $$T(r, R(W_1, W_2)) = pT(r, W_1) + S(r, W_1), \quad r \notin \Delta_r.$$

On the other hand, by applying lemma 1,2 to equation (3.13), we can deduce the following inequality, as in the proof of theorem 1,

3.17 $$S(r, W_1) + (\eta + \mu_{11})T(r, W_1) + (\eta + \mu_{12})T(r, W_2) > (m/2)T(r, R(W_1, W_2))$$
$$= (mp/2)T(r, W_1).$$

Combining (3.15), (3.16) and (3.17) we get

$$S(r, W_1) > [(mp/2) - \mu_{11} - \eta]T(r, W_1).$$

Thus, $O \geqq (mp/2) - \mu_{11} - \eta$. But this can not hold when m is sufficiently large. This completes the proof to theorem 4.

COROLLARY. *If* $R(W_1, W_2) \equiv A(z)W_1^p W_2^q$ *where* $A(z)$ *is a meromorphic function in* z, *and* W_1, W_2 *are some class of admissible solutions to equations* (3.13) *and* (3.14), *then* W_1 *and* W_2 *have the same order and type.*

PROOF. Without loss of generality, we just consider the case for admissible solutions of class II_1. Note that

$$S(r, W_1) + T(r, W_2^q) + T(r, W_1^p W_2^q) \geqq T(r, W_1^p),$$

and by lemma 3, we get

$$S(r, W_1) + qT(r, W_2) + T(r, W_1^p W_2^q) \geqq pT(r, W_1).$$

Thus together (3.17) for any arbitrary small $\eta > 0$ and $r \notin \Delta_r$, leads to the following inequality

$$S(r, W_1) + \mu_{11}T(r, W_1) + (\mu_{12} + \eta)T(r, W_2) > (m/2)[pT(r, W_1) - qT(r, W_2)]$$

i.e.

(3.19) $$[(mp/2) + \mu_{11} + \eta]T(r, W_1) > [(mq/2) - \mu_{12} - \eta]T(r, W_2).$$

Let $$M = \min\{[(mp/2) - \mu_{11} - \eta]/[(mq/2) + \mu_{12} + \eta],$$
$$[(mq/2) - \mu_{12} - \eta]/ [(mp/2) + \mu_{11} + \eta]\},$$

it is clear that when m is sufficiently large, then $O < M < 1$. Now, (3.18) and (3.19) can be rewritten as

(3.20) $$T(r, W_2) > MT(r, W_1)$$

(3.21) $$T(r, W_1) > MT(r, W_2).$$

Since $T(r, W_1)$ possesses the increasing property and convexity, (see [13]), as stated in lemma 5, so that, for all sufficiently large r, the following inequalities hold

$$T(r^2, W_2) > MT(r, W_1)$$

and $$T(r^2, W_1) > MT(r, W_2).$$

Therefore, W_1 and W_2 have the same order. Also, when they are of finite order, for all sufficiently large r, (3.20) and (3.21) hold, thus, they are of the same type.

The above four theorems and the corollary are the Rellich-Wittich type theorems, (see [3], [4]). In the system of equations (1.1), if the dervatives of W_1 and W_2 are just of first order, the coefficientes of $\Omega_{ts}(W_1, W_2)$, and $A_{ti}(z)$, are polynomials in z and $Q_{tt}(W_t, z) \equiv W_t$, $R(W_1, W_2) \equiv W_2^q$, then, they reduce to theorems 1–4 and ther corollary in [1]; and if we just consider meromorphic solutions, together with the additional assumption that $\Omega_2(W_2, W_2) \equiv W_2 - W_1'$ and $f_2[Q_{22}(W_2, z)] \equiv 0$, then theorem 2 here is theorem 4 of the [10].

In the following, we will give the Malmquist type theorem to the system of equations (1.2), (see [4], [8]).

THEOREM 5. *In the system of equations* (1.2), *suppose that*

(3.22) $$q_{11} > \mu_{11}, \quad q_{22} > \mu_{22},$$

where μ_{ts} *is defined by* (1.4) *and* (1.6); q_{ts} *is the highest degree of* W_s *in* $Q_{ts}(W_s, z)$. *Then a necessary condition for having admissible solutions* W_1, W_2 *of class* I *is*

(3.23) $$(q_{11} - \mu_{11})(q_{22} - \mu_{22}) \leqq (q_{12} + \mu_{12})(q_{21} + \mu_{21}).$$

PROOF. Choose an arbitrary $\eta > 0$, so that

$$q_{11} > \mu_{11} + \eta \text{ and } q_{22} > \mu_{22} + \eta,$$

and rewrite the system of equations (1.2) into

$$\Omega_1(W_1, W_2)/Q_{12}(W_2, z) = Q_{11}(W_1, z),$$

$$\Omega_2(W_1, W_2)/Q_{21}(W_1, z) = Q_{22}(W_2, z).$$

Then by lemmas 1 and 3, the following inequalities can be obtained,

$$S(r, W_1) + (q_{12} + \mu_{12} + \eta)T(r, W_2) > (q_{11} - \mu_{11} - \eta)T(r, W_1), \tag{3.24}$$

$$S(r, W_1) + (q_{21} + \mu_{21} + \eta)T(r, W_1) > (q_{22} - \mu_{22} - \eta)T(r, W_2). \tag{3.25}$$

Comparing (3.24) and (3.25), we get

$$S(r, W_1 + (q_{12} + \mu_{12} + \eta)(q_{21} + \mu_{21} + \eta)T(r, W_1)$$
$$> (q_{11} - \mu_{11} - \eta)(q_{22} - \mu_{22} - \eta)T(r, W_1).$$

Dividing the above relation by $T(r, W_1)$ and then taking the limit as $r \to \infty, r \notin \Delta_r$, we have

$$(q_{12} + \mu_{12} + \eta)(q_{21} + \mu_{21} + \eta) \geqq (q_1 - \mu_{11} - \eta)(q_{22} - \mu_{22} - \eta).$$

Since the above relation holds for all $\eta > 0$. Thus, by taking the limit as $\eta \to$;0, (3.23) follows. This completes the proof.

COROLLARY 1. *Under the condition* (3.22), *the transcendental and admissible solutions* W_1, W_2 *of class* I *to the system of equations* (1.2) *must be of the same order and type.*

PROOF. In view of (3.24) and (3.25), for large enough $r \notin \Delta_r$, we have

$$2(q_{12} + \mu_{12} + \eta)T(r, W_2) > (q_{11} - \mu_{11} - \eta)T(r, W_1),$$

$$2(q_{21} + \mu_{21} + \eta)T(r, W_1) > (q_{22} - \mu_{22} - \eta)T(r, W_2).$$

Moreover,

$$T(r, W_2) > MT(r, W_1) \tag{3.26}$$

$$T(r, W_1) > MT(r, W_2) \tag{3.27}$$

where
$$M = (1/2)\min\{(q_{11} - \mu_{11} - \eta)/(q_{12} + \mu_{12} + \eta),$$
$$(q_{22} - \mu_{22} - \eta)/(q_{21} + \mu_{21} + \eta)\}$$

and $O < M < 1$ (since (3.23) holds). Since $T(r, W_1)$ is increasing and convex as stated in lemma 5, thus from (3.26) and (3.27), it follows that for all sufficiently large r, the following inequalities hold,

$$T(r^2, W_2) > MT(r, W_1),$$

$$T(r^2, W_1) > MT(r, W_2).$$

Therefore W_1 and W_2 are of the same order.

If they are of finite order, then, for sufficiently large r, (3.26) and (3.27) must hold. Hence, they are also of the same type.

COROLLARY 2. *If* $q_{12} > \mu_{12}$ *and* $q_{21} > \mu_{21}$, *then a necessary condition for the system of equations* (1.2) *to have admissible solutions* W_1 *and* W_2 *of class* I *is*

$$(q_{12} - \mu_{12})(q_{21} - \mu_{21}) \leqq (q_{11} + \mu_{11})(q_{22} + \mu_{22}).$$

Moreover, W_1 *and* W_2 *which are transcendental are of the same order and type.*

COROLLARY 3. *If* $q_{11} > \mu_{11}$ *and* $q_{12} > \mu_{12}$ *or* $q_{21} > \mu_{21}$ *and* $q_{22} > \mu_{22}$ *hold, then the transcendental and admissible solutions* W_1, W_2 *of class* I *to the system of equations* (1.2) *must be of the same order and type.*

Proofs of the above two corrollaries are similar to those of theorem 5 and corollary 1.

Special examples to theorem 5 are theorems 1–3 in [2] but the conditions given here are weaker.

4. Remarks.

1. Without basic changes, the theorems and corollaries stated in the previous sections can be extended to the case when $\Omega_t(W_1, W_2)$ is a quotient of differential polynomials in W_1, W_2.

2. The corollary to theorem 4 still holds when $R(W_1, W_2) \equiv A(z)W_1^p + B(z)W_2^q$ (where $A(z)$ and $B(z)$ are meromorphic functions in z and p, q are positive integers).

3. The results in this paper can also be extended to algebraic system of differential equations having $n \geqq 3$ unknown functions $W_1, W_2, W_3, \ldots,$ and W_n.

$$\Omega_t(W_1, W_2, W_3, \ldots, W_n) = f_t[Q_{tt}(W_t, z)], \tag{4.1}$$

$$\Omega_t(W_1, W_2, W_3, \ldots, W_n) = \prod_{s=1}^{n} Q_{ts}(W_s, z), \quad t = 1, 2, \ldots, n. \tag{4.2}$$

But this concerns the problem of suitably solving the following linear system of inequalities involving $\hat{S}(r, W_1)$ and T (r, W$_1$),

$$\hat{S}(r, W_1) + \sum_{\substack{j=1\\ j\neq 1}}^{n} a_{ij} T(r, W_j) > P_{t1}(m) T(r, W_1), \quad t = 1, 2, \ldots, n.$$

where a_{ij} are positive constants; $P_{t1}(m)$ are at most polynomials in m of the first degree and the coefficient of the term of the first degree is positive.

For example, by using mathematical induction, it is not difficult to solve

$$\hat{S}(r, W_1) + (a_0 m^{n-1} + a_1 m^{n-2} + \dots + a_{n-1}) T(r, W_1)$$
$$> (b_0 m^n + b_1 m^{n-1} + \dots + b_n) T(r, W_1),$$

where $a_0, b_0 > 0$. Thus, we can extend theorem 1 to the system of equations (4.1), that is

THEOREM 6. *There does not exist any kind of admissible and weakly admissible solutions to the system of equations* (4.1).

It is also not difficult to prove that

THEOREM 7. *In the system of equations* (4.1), *if* $f_t[Q_{tt}(W_t, z)] \equiv 0$ $(t = 2, \dots, n)$ *and there exists one term in* $\Omega_t(W_1, W_2, \dots, W_n)$ $(t = 2, \dots, n)$, e.g., the $h^{(t)}$th *term, such that*

$$2h_{tt0}^{(t)} > \hat{\mu}_{tt}^{(t)} + \mu_{tt}(h^{(t)}) \quad t = 2, 3, \dots, n.$$

Then for the meromorphic solutions $W_1, W_2, \dots, W_n$ *to the system of equations* (4.1), W_1 *is neither admissible nor a weakly admissible with respect to the family of the coefficients* $\{A_{ti}(z)\}$, $a_{ttj}(z)\}$, $b_{ttj}(z)\}$ *where* $h^{(t)}$, $h_{tt0}^{(t)}$ *and* $\hat{\mu}_{tt}^{(t)}$ *are the counterparts of* h, h_{220} *and* $\hat{\mu}_{22}$ *in theorem* 2.

4. The existence of other kinds of (weakly) admissible solution to system (1.2) may be considered similarly.

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