MATH. SCAND. 71 (1992), 122–132

# COMPOSITION OF POTENTIALS WITH INNER FUNCTIONS

MANFRED STOLL


## Abstract.

For a regular Borel measure $\mu$ on the unit disc $U$ satisfying $(1 - |w|) \in L^1(\mu)$, let $G_\mu$ denote the Green potential of $\mu$. In the paper we characterize those measures $\mu$ such that $G_\mu \circ f$ is a good plurisuperharmonic function on the unit ball $B$ in $\mathbf{C}^n$ (or $U^n$) for every inner function $f$ on $B$ (or $U^n$). The results are then used to obtain several results concerning the boundary behavior of $G_\mu(f(rt))$.


## 1. Introduction.

For $n \geqq 1$, let $B$ or $B_n$ denote the unit ball in $\mathbf{C}^n$ with boundary $S$. For convenience, when $n = 1$ we will denote the unit disc in $\mathbf{C}$ by $U$, with boundary $T$. A nonconstant bounded holomorphic function $f: B \to U$ is called an *inner function* on $B$ if $|f^*(t)| = 1$ a.e. on $S$, where

$$f^*(t) = \lim_{r \to 1} f(rt).$$

As in [7,9], an inner function $f$ on $B$ is said to be *good* if

$$\lim_{r \to 1} \int_S \log|f(rt)|\, d\sigma(t) = 0, \tag{1.1}$$

where $\sigma$ denotes the normalized rotation-invariant measure on $S$. In the disc, the good inner functions are precisely the Blaschke products.

In the unit disc $U$, a superharmonic function $V$ with $V(z) \geqq 0$ is called a potential on $U$ if the greatest harmonic minorant of $V$ is the zero function. It is well known that this is equivalent to

$$\lim_{r \to 1} \int_0^{2\pi} V(re^{i\theta})\, d\theta = 0, \tag{1.2}$$

Received August 1, 1991.

and if this is the case, then

$$V(z) = G_\mu(z) = \int_U G(z, w)\, d\mu(w), \tag{1.3}$$

where $G$ is the Green function of $U$ given by

$$G(z, w) = \log \left| \frac{1 - \bar{w}z}{w - z} \right|,$$

and $\mu$ is a nonnegative regular Borel measure on $U$ satisfying

$$\int_U (1 - |w|)\, d\mu(w) < \infty. \tag{1.4}$$

The superharmonic function $G_\mu(z)$ as defined by (1.3) is called the *Green potential* of the measure $\mu$. The condition (1.4) is necessary and sufficient that $G_\mu \not\equiv +\infty$.

In [3], M. Heins proved that if $B(z)$ is a Blaschke product in $U$, then

$$\liminf_{r \to 1} (1 - r) \log \frac{1}{|B(rt)|} = 0 \tag{1.5}$$

for all $t \in T$. This result was extended by the author in [10] to arbitrary Green potentials on $U$ as follows:

THEOREM A. *If $G_\mu$ is a Green potential of a measure $\mu$ satisfying* (1.4), *then for all curves $\gamma: [0, 1) \to U$ with* $\lim_{r \to 1} \gamma(r) = 1$,

$$\liminf_{r \to 1} (1 - |\gamma(r)|)\, G_\mu(\gamma(r)t) = 0 \tag{1.6}$$

*for all $t \in T$.*

If $B$ is a Blaschke product with zeros $\{a_n\}$, $a_n \neq 0$, satisfying the Blaschke condition $\sum(1 - |a_n|) < \infty$, then $-\log|B(z)| = G_\mu(z)$ where $\mu = \sum \delta_{a_n}$, and $\delta_{a_n}$ is point mass measure at $a_n$. Thus (1.5) follows from (1.6) with $\gamma(r) = r$. An alternate way to consider the function $-\log|B(z)|$ is as the composition of the potential $G_{\delta_0}$ with the inner function $B$, i.e.,

$$-\log|B(z)| = G_{\delta_0}(B(z)),$$

where $\delta_0$ is point mass at 0. The purpose of the paper is to consider the composition of a potential $G_\mu$ with an inner function $f$ on the ball $B$ (or the polydisc $U^n$), and to consider generalizations of (1.5) for $G_\mu \circ f$.

As we will see in section 2, if $G_\mu$ is the potential of a regular Borel measure $\mu$, then $G_\mu \circ f$ is plurisuperharmonic on $B$ for every nonconstant holomorphic function $f: B \to U$. In analogy with (1.1) and (1.2), we will say that a plurisuperharmonic function $V \geqq 0$ on $B$ is *good* if

$$\lim_{r \to 1} \int_S V(rt)\, d\sigma(t) = 0. \tag{1.7}$$

Since plurisuperharmonic functions are also superharmonic on $B$, any plurisuperharmonic function $V$ satisfying (1.7) can also be expressed as $V = G_\nu$ for some measure $\nu$ on $B$, where $G$ is the euclidean Green function for $B$.

The main result of the paper (Theorem 1) characterizes the regular Borel measures $\mu$ on $U$ such that $G_\mu \circ f$ is good for every inner function $f$ on $B$. Specifically, we will prove that $G_\mu \circ f$ is good for every inner function $f$ if and only $\mu(K) = 0$ for every compact subset $K$ of $U$ of capacity zero. As a consequence of Theorem 1 and Theorem A, for such a measure $\mu$,

$$\liminf_{r \to 1} (1 - r) G_\mu(f(rt)) = 0 \tag{1.8}$$

for every inner function $f: U \to U$ and every $t \in T$. By example, we will show that (1.8) in general is false for $n > 1$. However, in Theorem 6, we will prove that if we restrict ourselves to those $t \in S$ for which $|f^*(t)| = 1$, then (1.8) is valid for every potential $G_\mu$.

The main result of the paper is stated and proved in section 3. In section 4 we give two examples of measures satisfying the hypothesis $\mu(K) = 0$ for every compact subset $K$ of $U$ of capacity zero. Finally, in section 5 we give several applications of the result to boundary limits of $G_\mu(f(rt))$.

## 2. Preliminaries.

Let $\Omega$ be a domain in $\mathbf{C}^n$. A lower semicontinuous function $V: \Omega \to (-\infty, \infty]$, with $V \not\equiv \infty$, is said to be *plurisuperharmonic* on $\Omega$ if for each $a \in \Omega$ and $b \in \mathbf{C}^n$, the function $\lambda \to V(a + \lambda b)$ is superharmonic in a neighborhood of 0 in $\mathbf{C}$. The class of *plurisubharmonic* and the class of *pluriharmonic* functions on $\Omega$ are defined analogously. As in the unit disc, it is easily shown that if $V \geqq 0$ is a good plurisuperharmonic function on $B$, then the greatest pluriharmonic minorant of $V$ is the zero function.

For a compact subset $K$ of $U$, the (*Green*) *capacity* of $K$ is defined to be

$$C_g(K) = \sup\{\mu(K): S_\mu \subset K \text{ and } G_\mu \leqq 1 \text{ on } K\},$$

where $S_\mu$ denotes the *support* of the measure $\mu$. Thus a compact subset $K$ of $U$ has *positive capacity* if there exists a positive measure $\mu$ supported on $K$ such that the Green potential $G_\mu$ is bounded on $U$. An arbitrary set $E$ has positive capacity if some compact subset of $E$ has positive capacity.

For a fixed compact set $K$, let $\mathscr{M}_K^+$ denote the set of Borel measures $\mu$ with

$S_\mu \subset K$ and $\mu(K) = 1$. A measure $\mu \in \mathcal{M}_K^+$ is said to have *finite energy* if $I_g(\mu) < \infty$, where $I_g(\mu)$ denotes the *energy integral* of $\mu$ given by

$$I_g(\mu) = \int_K G_\mu(z)\, d\mu(z).$$

It is well known that if $C_g(K) > 0$, then there exists a unique measure $\mu_K \in \mathcal{M}_K^+$, called the *equilibrium measure* of $K$, which minimizes the energy integral and satisfies

$$I_g(\mu_K) = \frac{1}{C_g(K)}.$$

Finally, for $w \in U$, let $\varphi_w$ be the biholomorphic automorphism of $U$ given by

$$\varphi_w(z) = \frac{w - z}{1 - \bar{w}z}.$$

Then $G(z, w) = -\log|\varphi_w(z)|$.

LEMMA 1. *If $f$ is an inner function on $B$, then*

(1) *$\varphi \circ f$ is an inner function on $B$ for every inner function $\varphi$ on $U$, and*

(2) *$\varphi_w \circ f$ is a good inner function on $B$ for all $w \in U$ except possibly a subset of $U$ of capacity zero.*

PROOF. That *$\varphi \circ f$ is an inner function on $B$* is due to W. Rudin [9, Theorem 17.5]. The proof of (2) is identical to the corresponding result in the unit disc, which is initially due to O. Frostman [2].

LEMMA 2. *Let $G_\mu$ be the Green potential of a regular Borel measure $\mu$ satisfying* (1.4) *and let $f: B \to U$ be a nonconstant holomorphic function. Then*

(1) *$G_\mu(f(z))$ is plurisuperharmonic on $B$, and*

(2) *$G_\mu(f(z))$ is pluriharmonic for all $z \in B$ such that $f(z) \notin S_\mu$.*

PROOF. The proof is an immediate consequence of the fact that for fixed $w \in U$, the function $z \to G(f(z), w)$ is plurisuperharmonic on $B$, and pluriharmonic for all $z$ such that $f(z) \neq w$.

LEMMA 3. *If $V$ is plurisuperharmonic on $B$, then for $0 < \rho < r < 1$,*

$$\int_S V(rt)\, d\sigma(t) \leqq \int_S V(\rho t)\, d\sigma(t). \tag{2.2}$$

PROOF. Since (2.2) holds for superharmonic functions (e.g. [4, p. 103]), it also holds for plurisuperharmonic functions.

REMARK. If $B$ is replaced by the unit polydisc $U^n$ in $\mathbf{C}^n$, then the results of Lemmas 1 through 3 are still valid for inner functions on $U^n$. In this setting, a bounded holomorphic function $f$ on $U^n$ is inner if $|f^*(t)| = 1$ a.e. on $T^n$, where $f^*(t) = \lim_{r\to 1} f(rt)$ [7], and a nonnegative plurisuperharmonic function $V$ on $U^n$ is good, if

$$\lim_{r\to 1}\int_{T^n} V(rt)\,dm_n(t) = 0, \tag{2.3}$$

where $m_n$ is normalized Lebesgue measure on $T^n$.

## 3. Composition of potentials with inner functions.

If $f$ is an inner function on $B$ (or $U^n$), let

$$E_f = \{w \in U : \varphi_w \circ f \text{ is not good}\}.$$

That the set $E_f$ is a Borel set is obtained as follows: For $w \in U$ and $0 < r < 1$, set

$$U_r(w) = \int_S G(f(rt), w)\,d\sigma(t). \tag{3.1}$$

Then $U_r$ is lower semicontinuous on $U$, and thus $U(w) = \lim_{r\to 1} U_r(w)$ is Borel measurable on $U$. Since $E_f = \{w : U(w) > 0\}$, $E_f$ is a Borel set.

THEOREM 1. *Let $G_\mu$ be the potential of a regular Borel measure $\mu$ satisfying* (1.4).

(1) *If $f$ is an inner function on $B$ (or $U^n$), then $G_\mu \circ f$ is good if and only if $\mu(E_f) = 0$.*

(2) *$G_\mu \circ f$ is good for every inner function $f$ on $B$ (or $U^n$) if and only if $\mu(K) = 0$ for every compact subset $K$ of $U$ of capacity zero*

PROOF. (1). Let $f$ be a nonconstant inner function on $B$ (or $U^n$). By Tonelli's Theorem,

$$\int_S G_\mu(f(rt))\,d\sigma(t) = \int_U\int_S G(f(rt), w)\,d\sigma(t)\,d\mu(w) = \int_U U_r(w)\,d\mu(w), \tag{3.2}$$

where $U_r$ is as defined by (3.1). If $f$ is an inner function on $U^n$, the integration is taken over $T^n$.

Suppose $\mu(E_f) = 0$. Since $\varphi_w \circ f$ is good for every $w \in U \sim E_f$,

$$\lim_{r\to 1} U_r(w) = 0 \quad \mu\text{-a.e..}$$

Since $z \to G(f(z), w)$ is plurisuperharmonic on $B$ for each $w \in U$, by Lemma 3 we have $0 \leqq U_r(w) \leqq U_{\frac{1}{2}}(w)$ for all $w \in U$ and all $r$, $\frac{1}{2} \leqq r < 1$. Furthermore,

$$\int_U U_{\frac{1}{2}}(w)\,d\mu(w) = \int_S G_\mu(f(rt))\,d\sigma(t) < \infty,$$

since $G_\mu \circ f$ is superharmonic (e.g. [4, p. 67]). Thus by (3.2) and Lebesgue's Dominated Convergence Theorem,

$$\lim_{r\to 1}\int_S G_\mu(f(rt))\,d\sigma(t) = \lim_{r\to 1}\int_U U_r(w)\,d\mu(w) = \int_U \lim_{r\to 1} U_r(w)\,d\mu(w) = 0.$$

Conversely, suppose $\mu(E_f) > 0$. Since $\mu$ is regular, there exists a compact set $K$ with $K \subset E_f$ such that $\mu(K) > 0$. Thus

$$G_\mu(f(z)) \geqq \int_K G(f(z), w)\,d\mu(w),$$

and with $U_r$ as defined by (3.1),

$$\int_S G_\mu(f(rt))\,d\sigma(t) \geqq \int_K U_r(w)\,d\mu(w).$$

But $U(w) = \lim_{r\to 1} U_r(w) > 0$ for all $w \in K$. Hence by Fatou's Lemma,

$$0 < \int_K U(w)\,d\mu(w) \leqq \lim_{r\to 1}\int_K U_r(w)\,d\mu(w) \leqq \lim_{r\to 1}\int_S G_\mu(f(rt))\,d\sigma(t),$$

and thus $G_\mu \circ f$ is not good.

Proof of (2). Let $f$ be any inner function on $B$ (or $U^n$). Suppose $\mu$ satisfies $\mu(K) = 0$ for every compact subset $K$ of $U$ of capacity zero. By Lemma 1, $E_f$ has capacity zero. Since $\mu$ is regular, $\mu(E_f) = \sup\{\mu(K)\}$, where the sup is taken over all compacts subsets $K$ of $E_f$. Thus $\mu(E_f) = 0$, and thus $G_\mu \circ f$ is good by (1).

Conversely, Suppose $K$ is a compact subset of $U$ of capacity zero with $\mu(K) > 0$. By [1, p. 118] there exists an inner function $\varphi$ on $U$ whose range is precisely $U \sim K$. Let $g$ be any inner function on $B$. Then $f(z) = \varphi(g(z))$ is an inner function on $B$ whose range is contained in $U \sim K$. Since

$$G_\mu(f(z)) \geqq \int_K G(f(z), w)\,d\mu(w),$$

$$\int_S G_\mu(f(rt))\,d\sigma(t) \geqq \int_K\int_S \log\left|\frac{1-\bar{\omega} f(rt)}{w - f(rt)}\right| d\sigma(t)\,d\mu(w).$$

But $F(z) = G(f(z), w)$ is pluriharmonic on $B$ for all $w \in K$. Therefore

$$\int_S G(f(rt), w)\,d\sigma(t) = G(f(0), w),$$

and hence, for all $r, 0 < r < 1$,

$$\int_S G_\mu(f(rt))\,d\sigma(t) \geqq \int_K G(f(0), w)\,d\mu(w),$$

which proves the result.

## 4. Examples.

In this section we provide two examples of measures satisfying (1.4) and $\mu(K) = 0$ for every compact set $K$ with $C_g(K) = 0$. If $\mu$ is a regular Borel measure on $U$, we adopt the notation $\mu \prec\!\!\prec C_g$ to mean that $\mu(K) = 0$ for every compact subset $K$ of $U$ with $C_g(K) = 0$. This notion is sometimes referred to as C-absolute continuity.

EXAMPLE 1. Let $h$ be a nonnegative Borel measurable function on $U$ satisfying

$$\int_U (1 - |w|)h(w)\,dA(w) < \infty, \tag{4.1}$$

where $A$ denotes area measure on $U$. Then $\mu$ defined by

$$\mu(E) = \int_E h(w)\,dA(w) \tag{4.2}$$

is clearly a regular Borel measure on $U$ satisfying (1.4). Since every compact set $K$ of capacity zero has planar measure zero, $\mu \prec\!\!\prec C_g$.

EXAMPLE 2. Let $K$ be any compact subset of $U$ with $C_g(K) > 0$, and let $\mu_K$ be the equilibrium measure of $K$ as defined in section 2. Suppose $C$ is any compact subset of $U$ with $C_g(C) = 0$. Then

$$\mu_K(C) = \mu_K(C \sim K) + \mu_K(C \cap K).$$

Since $C \sim K \subset \tilde{S}_{\mu_K}$, $\mu_K(C \sim K) = 0$. If $\mu_K(C \cap K) > 0$, then $C_g(C \cap K) > 0$, and thus $C_g(C) > 0$, which is a contradiction. Thus $\mu_K \prec\!\!\prec C_g$. In fact, the same argument shows that if $\mu$ is any measure with $S_\mu \subset K$ for which the energy integral $I_g(\mu) < \infty$, then $\mu \prec\!\!\prec C_g$

As a consequence of this example we obtain the following variation of the Theorem of Frostman:

THEOREM 2. *If $E$ is a subset of $U$ with $C_g(E) > 0$, then there exists a measure $\mu$ on $U$ with $S_\mu \subset E$ such that $G_\mu \circ f$ is good for all inner functions $f$ on $B$ (or $U^n$).*

## 5. Applications to Boundary limits.

In this section we will consider several applications of Theorem 1 to boundary limits of $G_\mu(f(rt))$, where $f$ is an inner function on $B$ or $U^n$. The results are new even in the case $n = 1$.

If $G_\mu$ is the potential of a measure $\mu$ on $U$, then by a classical result of Littlewood [5],

$$\lim_{r\to 1} G_\mu(rt) = 0 \tag{5.1}$$

for almost every $t \in T$. It is also known (see [11]) that in general $G_\mu$ need not have a nontangential limit at any point $t \in T$, even if $\mu$ is an absolutely continuous measure. However, for $G_\mu(f(z))$ we have the following:

THEOREM 3. *Let $G_\mu$ be the potential of a regular Borel measure $\mu$ satisfying* (1.4).
(1) *If $f$ is an inner function on $B$ (or $U^n$) for which $\mu(E_f) = 0$, then*

$$\lim_{r\to 1} G_\mu(f(rt)) = 0 \quad \text{a.e. } t \in S \ (\text{or } T^n). \tag{5.2}$$

(2) *If $\mu \lll C_g$, then* (5.2) *holds for every inner function $f$ on $B$ (or $U^n$).*

PROOF. Let $V(z) = G_\mu(f(z))$. Thus by Theorem 1, if either (1) or (2) hold, $V$ is a good plurisuperharmonic function on $B$ (or $U^n$). By [8, Prop. 1.4.7], for $n \geqq 2$,

$$\int_S V(rt)\,d\sigma(t) = \int_S \frac{1}{2\pi}\int_0^{2\pi} V(re^{i\theta}t)\,d\theta\,d\sigma(t).$$

As a consequence, the function $V_t$ defined on $U$ by $V_t(\lambda t)$ is a potential on $U$ for almost every $t \in S$ (or $T^n$). When $n = 1$, $V$ itself is a potential on $U$. Thus (5.2) is now an immediate consequence of (5.1).

COROLLARY 1. *If $B$ is a finite Blaschke product in $U$, then*

$$\lim_{r\to 1} G_\mu(B(rt)) = 0 \quad \text{a.e. } t \in T \tag{5.3}$$

*for every potential $G_\mu$ on $U$.*

PROOF. If $B$ is a finite Blaschke product, then $\varphi_w \circ B$ is good for every $w \in U$. Thus $E_B = \phi$.

Another application of Theorem 1 is the following generalization of the result (1.5) of M. Heins:

THEOREM 4. *Let $G_\mu$ be the Green potential of a regular Borel measure $\mu$ satisfying* (1.4).
(1) *If $f$ is an inner function on $U$ for which $\mu(E_f) = 0$, then*

$$\liminf_{r\to 1}(1 - r)G_\mu(f(rt)) = 0 \quad \text{for all } t \in T. \tag{5.4}$$

(2) *If $\mu \lll C_g$, then* (5.4) *holds for every inner function $f$ on $U$.*

PROOF. If either (1) or (2) hold, then by Theorem 1, $V_f(z) = G_\mu(f(z))$ is a potential on $U$. Thus the result follows from Theorem A with $\gamma(r) = r$.

Theorem 4 can be extended to inner functions on $U^n$ as follows:

THEOREM 5. *If* $\mu \prec\prec C_g$, *then*

$$\liminf_{(r)\to(1)} \left(\prod_{j=1}^{n} (1 - r_j)\right) G_\mu(f(r_1t_1,\ldots,r_nt_n)) = 0 \tag{5.5}$$

*for all* $t \in T^n$, *where* $(r) = (r_1,\ldots,r_n)$.

PROOF. If $f$ is an inner function on $U^n$, then since $V_f(z) = G_\mu(f(z))$ is plurisuperharmonic on $U^n$, $V_f$ is also n-superharmonic on $U^n$, i.e., superharmonic in each variable $z_j$. The result now follows by [6, Lemma 3.1].

The following example shows that for $n \geqq 2$, the analogue of (5.4) is not valid for the unit ball $B$ in $\mathbf{C}^n$.

EXAMPLE 3. There exists a measure $\mu \prec\prec C_g$, an inner function $f$ on $B$, and $t \in S$, such that

$$G_\mu(f(rt)) \equiv \infty \tag{5.6}$$

for all $r$, $0 < r < 1$.

PROOF. Let $\mu$ be an absolutely continuous regular Borel measure given by a nonnegative Borel measurable function $h$ satisfying

$$\int_U h(w)\log\frac{1}{|w|}\,dA(w) = \infty. \tag{5.7}$$

An example of such an $h$ is $h(w) = |w|^{-2}(-\log|w|)^{-2}$ for $|w| \leqq \frac{1}{2}$, and $h(w) = 0$ elsewhere. If $\mu$ is the measure given by (4.2), then $\mu \prec\prec C_g$ and $G_\mu(0) = \infty$. For $z = (z_1,\ldots,z_n)$, let $z' = (z_1,\ldots,z_{n-1})$. By [9, Theorem 8.4], there exists a nonconstant inner function $f$ on $B_n$ such that $f(z',0) = 0$ for all $z' \in B_{n-1}$. Thus if $t \in S$ is such that $t_n = 0$, we have $f(rt) = f(rt',0) = 0$. Thus for this $f$ and $t$, $G_\mu(f(rt)) \equiv \infty$ for all $r$, $0 < r < 1$.

Although the above example shows that the analogue of (5.4) does not in general hold for inner functions on $B_n$, $n \geqq 2$, the result is valid for those boundary points $t \in S$ for which $|f^*(t)| = 1$.

THEOREM 6. *Let* $G_\mu$ *be the potential of a measure* $\mu$ *satisfying* (1.4) *and let* $f$ *be an inner function on* $B$. *Then*

$$\liminf_{r\to 1}(1 - r)\,G_\mu(f(rt)) = 0 \tag{5.8}$$

*for all* $t \in S$ *for which* $|f^*(t)| = 1$.

PROOF. Let $t \in S$ be such that $|f^*(t)| = 1$, and let $\gamma$ be the curve in $U$ given by $\gamma(r) = f(rt)\bar{\beta}$, where $\beta = f^*(t)$. Then $\gamma(r) \to 1$ as $r \to 1$, and $G_\mu(f(rt)) = G_\mu(\gamma(r)\beta)$. Thus by Theorem A,

$$\liminf_{r \to 1} (1 - |f(rt)|)\, G_\mu(f(rt)) = 0.$$

Let $b = f(0)$. By Schwartz's Lemma [8, Theorem 8.1.4],

$$\frac{|1 - \bar{b}f(z)|^2}{1 - |f(z)|^2} \leqq \frac{1 - |b|^2}{1 - |z|^2}$$

for all $z \in B$. As a consequence,

$$\frac{1 - r^2}{1 - |f(rt)|^2} \leqq \frac{1 - |b|^2}{|1 - \bar{b}f(rt)|^2} \leqq \frac{1 + |b|}{1 - |b|}.$$

Thus there exists a constant $C$ such that

$$(1 - r)G_\mu(f(rt)) \leqq C(1 - |f(rt)|)\, G_\mu(f(rt))$$

for all $r$, $0 < r < 1$, from which the result now follows.

REMARKS.

(1) In Theorem 4, we must have $\mu(E_f) = 0$. For example, if we take

$$f(z) = \exp\left(\frac{z + 1}{z - 1}\right),$$

then $0 \in E_f$. Thus if $\mu = \delta_0$, for this $f$,

$$(1 - r)G_{\delta_0}(f(r)) = 1 + r.$$

(2) In Theorem 6 we did not require that $\mu \prec\!\!\prec C_g$. If indeed $\mu \prec\!\!\prec C_g$ and $f$ is an inner function on $B$, then as a consequence of Theorem 4, (5.8) will hold for all $t \in S$ for which $f_t$ is inner.

## REFERENCES


1. S. T. Fisher, *Function Theory on Planar Domains*, John Wiley & Sons, New York, 1983.
2. O. Frostman, *Potential d'équilibre et capacité des ensembles*, Lunds Univ. Mat. Sem. 3 (1935).
3. M. Heins, *The minimum modulus of a bounded analytic function*, Duke Math. J. 14 (1947), 179–215.
4. L. L. Helms, *Introduction to Potential Theory*, Wiley-Interscience, New York, 1969.

5. J. E. Littlewood, *On functions subharmonic in a circle*, II, Proc. London Math. Soc. 28 (1929), 383–394.
6. W. Nestlerode and M. Stoll, *Radial limits of n-subharmonic functions in the polydisc*, Trans. Amer. Math. Soc. 279 (1983), 691–703.
7. W. Rudin, *Function Theory in Polydiscs*, Benjamin, New York, 1969.
8. W. Rudin, *Function Theory in the Unit Ball of* $\mathbb{C}^n$, Springer-Verlag, New York, 1980.
9. W. Rudin, *New Constructions of Functions Holomorphic in the Unit Ball of* $\mathbb{C}^n$, Amer. Math. Soc., Providence, R.I., 1986.
10. M. Stoll, *Boundary limits of Green potentials in the unit disc*, Arch. Math. 44 (1985), 451–455.
11. E. Tolsted, *Limiting values of subharmonic functions*, Proc. Amer. Math. Soc. 1 (1950), 636–647.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF SOUTH CAROLINA
COLUMBIA, SC 29208
U.S.A.