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SINGLY SUPPORTED C^*-ALGEBRAS

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Abstract.

We show that all support algebras of a σ-unital C^*-algebra A are $*$-isomorphic provided A has an approximate identity consisting of projections. Other C^*-algebras without projections are also shown to be singly supported.

0. Introduction

Let A be a σ-unital C^*-algebra. The concept of support algebra S of A was introduced in [Li 1] and also studied in [Li 2]. We redefine it as follows:

DEFINITION 0.1. A subalgebra S of A is called a *support* algebra of A if

(1) S is a dense $*$-subalgebra of A;

(2) $S = \bigcup_{n=1}^{\infty} S_n$, where each S_n is a hereditary C^*-subalgebra of A;

(3) for each n there is $e_n \in S$ such that $0 \leqq e_n \leqq 1$ and

$$xe_n = e_n x = x \quad \text{for all} \quad x \in S_n.$$

When A is commutative, so that $A = C_0(X)$, where X is a locally compact, σ-compact Hausdorff space, then $S = C_{00}(X)$, the set of continuous functions with compact support. In fact, in both [Li 1] and [Li 2], we view support algebras as analogues of $C_{00}(X)$. We notice that S has the following properties:

(i) For any $h \in S_+$, there is a $a \in S$ such that $[h] \leqq a$, where $[h]$ is the range projection of h in A^{**}, the enveloping von Neumann algebra of A. So every element in S has a "compact" support.

(ii) There is a subsequence $\{n_k\}$ such that $\{e_{n_k}\}$ forms an approximate identity for A and $e_{n_{k+1}} e_{n_k} = e_{n_k} e_{n_{k+1}} = e_{n_k}$.

Both (i) and (ii) are easily proved.

DEFINITION 0.2. Let a be a strictly positive element of A. Define

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$$f_n(t) = \begin{cases} 0 & 0 \leqq t \leqq 1/n+1 \\ \text{linear} & 1/n+1 < t < 1/n. \\ 1 & 1/n \leqq t < \infty \end{cases}$$

So f_n is a continuous function defined on $[0, \infty)$. Set $a_n = f_n(a)$. Then $\{a_n\}$ forms an approximate identity for A. Moreover, $a_{n+1}a_n = a_n a_{n+1} = a_n$. So $a_{n+1} \geqq [a_n]$, $n = 1, 2, \ldots$.

Let a and a_n be as in 0.2. Set $S = \bigcup_{n=1}^{\infty} a_n A a_n$. Then it is easy to see that S is a support algebra. We may write this support algebra $S = S(a)$. In fact, every support algebra arises this way. Suppose that S is a support algebra of A. By (ii), we may assume that $\{e_n\}$ forms an approximate identity for A and $e_{n+1}e_n = e_n e_{n+1} = e_n$. Set $a = \sum_{n=0}^{\infty} 2^{-n} e_n$. Then a is a strictly positive element of A. A straightforward computation involving functional calculus shows that

$$S = S(a) = \bigcup_{n=1}^{\infty} a_n A a_n.$$

Moreover, if $\{\varepsilon_n\}$ forms an approximate identity for A and $\varepsilon_{n+1}\varepsilon_n = \varepsilon_n \varepsilon_{n+1} = \varepsilon_n$, then

$$S(\{\varepsilon_n\}) = \bigcup_{n=1}^{\infty} \varepsilon_n A \varepsilon_n$$

is also a support algebra of A.

From [Li 2] we know that for a σ-unital C^*-algebra A, there may be more than one support algebra S in A. Following [Li 2], we say that a σ-unital C^*-algebra A is *singly supported* if every pair of support algebras are *-isomorphic. The purpose of this note is to show that many σ-unital C^*-algebras are singly supported. Notably, every C^*- algebra with an approximate identity consisting of a sequence of projections is singly supported. We also show that some projectionsless C^*-algebras are also singly supported.

Throughout this note, $M(A)$ denotes the multiplier algebra of A (see [Pe 3,3.12.4]).

1. C^*-algebras with approximate identities consisting of projections.

LEMMA 1.1 ([Li 2, 7.6]). *Let A be a σ-unital C^*-algebra and let $\{e_n\}$, $\{p_n\}$ be two approximate identities consisting of projections. Suppose that $S_1 = S(\{e_n\})$ and $S_2 = S(\{p_n\})$. Then there is a unitary $u \in M(A)$ such that*

$$u^* S_1 u = S_2.$$

Moreover, we may choose the unitary u such that

$$\|u - 1\| < 1.$$

PROOF. From the proof of [Li 2, 7.6] we see that we can make

$$\|u - 1\| < 1$$

or even smaller.

THEOREM 1.2. *Let A be a C*-algebra with an approximate identity* $\{e_n\}$ *consisting of projections. Then A is singly supported. Moreover, if S is a support algebra of A, there is a unitary* $u \in M(A)$ *such that* $u^* S u = \bigcup_{n=1}^{\infty} e_n A e_n$ *and* $\|u - 1\| < 1$.

PROOF. Suppose that S is a support algebra of A and $S = S(a)$, where a is a strictly positive element with $0 \leqq a \leqq 1$. Set $a_n = f_n(a)$ as in 0.2. Let p_n denote the spectral projection of a (computed in the enveloping von Neumann algebra) corresponding to the interval $(1/n, 1]$. For each n, there is m such that

$$\|e_m a_n - a_n\| < 1/4.$$

Since $\{a_n\}$ forms an approximate identity for A, there is $k(\geqq n)$ such that

$$\|a_k e_m a_k - e_m\| < 1/4.$$

Thus, by [Ef, A8], there is a projection q_n in $B_k = (a_k A a_k)^-$ such that

$$\|q_n - e_m\| < 1/2.$$

So

$$\begin{aligned}\|q_n a_n - a_n\| &\leqq \|q_n a_n - e_m a_n\| \\ &+ \|e_m a_n - a_n\| < 3/4.\end{aligned}$$

Working now in the unitized algebra $\tilde{B}_k$ (with 1 denoting the identity in $\tilde{B}_k$), we set $x_n = (1 - a_n)^{1/2}(1 - q_n)$. Then

$$\|(1 - q_n) - x_n^* x_n\| = \|(1 - q_n)a_n(1 - q_n)\| < 3/4.$$

So $x_n^* x_n$ is invertible in $(1 - q_n)\tilde{B}_k(1 - q_n)$. Let $v = x_n|x_n|^{-1}$. Then $x_n = v|x_n|$ is the polar decomposition of x_n in $\tilde{B}_k$. Thus $v^*v = 1 - q_n$ and $d_n = vv^*$ is a projection in $\tilde{B}_k$. Moreover, $d_n \in (x_n \tilde{B}_k x_n^*)^-$. Since $(1 - a_n)p_{n-1} = p_{n-1}(1 - a_n) = 0$, we obtain that $d_n \leqq 1 - p_{n-1}$. Since $v = 1 + s$ for some s in B_k, it follows that $g_n = 1 - vv^* = 1 - q_n$ is a projection in B_k. Moreover,

$$a_{k+1} \geqq g_n \geqq p_{n-1} \geqq a_{n-1},$$

for $n = 2, 3, \ldots$.

Hence there is a subsequence $\{n_k\}$ and a sequence of projections $\{\varepsilon_k\}$ in A such that

$$a_{n_{k+1}} \geqq \varepsilon_k \geqq a_{n_k} \quad \text{for all} \quad k.$$

Therefore

$$\bigcup_{k=1}^{\infty} \varepsilon_k A \varepsilon_k = \bigcup_{n=1}^{\infty} a_n A a_n = S(a).$$

It follows from Lemma 1.1, there is a unitary $u \in M(A)$ such that

$$u^* S(a) u = \bigcup_{n=1}^{\infty} e_n A e_n, \text{ and } \|u - 1\| < 1.$$

This completes the proof.

REMARK 1.3. Some of the arguments used in the proof arose in a private communication with Larry Brown.

REMARK 1.4. The original proof of 1.2 in the case $A = C(S^1) \otimes K$ (where K as usual denotes the algebra of compact operators on l^2) was much more complicated. Let $f \in C(S^1) \otimes K$ ($\cong C(S^1, K)$) and $[f] \leqq b$ for some $b \in C(S^1) \otimes K$. It was not clear (at least to the author then) that there should be a projection p in $C(S^1) \otimes K$ such that $f \leqq p$.

COROLLARY 1.5. *Let A be a separable C^*-algebra with an approximate identity consisting of projections. If $a \in A_+$ such that $[a] \leqq b$ for some $b \in A$ then there is a projection $p \in A$ such that $p \geqq a$.*

REMARK 1.6. Many σ-unital C^*-algebras has an approximate identity consisting of projections. For example, C^*-algebras with real rank zero and C^*-algebras with the form $B \otimes K$, where B is unital. There are many more. Among them we would like to present the following examples:

EXAMPLE 1.7. Let B be a σ-unital simple C^*-algebra with a non-trivial projection and let $A = B \otimes K$. Then A has an approximate identity consisting of projections. Consequently, A is singly supported.

In fact, by [Pe 1], B has LP (see [SZ 1, 1.]). It follows from [SZ 1, 1.1] that A has an approximate identity consisting of projections.

EXAMPLE 1.8. Let

$$0 \to I \to A \to C \to 0$$

be a short exact sequence of σ-unital C^*-algebras. Suppose that C has an approximate identity consisting of projections and I is a C^*-algebra of real rank zero and $K_1(I) = 0$. Then A has an approximate identity consisting of projections. This follows easily from the proof of [SZ 2, 2.3]. Therefore A is singly supported.

2. C^*-algebras which may not have projections.

In this section we show that some C^*-algebras without approximate identities consisting of projections or even without any (non-trivial) projections are singly supported.

LEMMA 2.1 (cf. [Ef, A.8]). *Let A be a C^*-algebra and I a closed ideal of A. Suppose that $\pi: A \to A/I$ is the quotient map and $a, e \in A_{sa.}$, e is a projection. If $\pi(e) = \pi(a)$, and*

$$\|e - a\| < 1/2,$$

then there is a projection p in the C^*-subalgebra of A generated by a such that

$$\|e - p\| < 2\|e - a\|$$

and $\pi(p) = \pi(a) = \pi(e)$.

PROOF. Let $\delta = \|e - a\|$. Since $\mathrm{sp}(e) = \{1, 0\}$, we obtain that

$$\mathrm{sp}(a) \subset [-\delta, \delta] \cup [1 - \delta, 1 + \delta].$$

Then the function $\chi = \chi_{(\delta, 1+\delta]}$ is continuous on $\mathrm{sp}(a)$. So $p = \chi(a)$ is a projection in the C^*-subalgebra generated by a. Since

$$\pi(\chi(a)) = \chi(\pi(a)) = \chi(p)),$$
$$\pi(\chi(\mathrm{a})) = \pi(\mathrm{a}) = \pi(p).$$

The rest of the proof is exactly the same as that of [Ef, A8.1].

LEMMA 2.2 (cf. [Ef, A8.2 and A8.3]). *Let A be a C^*-algebra and I a closed ideal of A. Suppose that $\pi: A \to A/I$ is the quotient map and e, p are projections in A. If*

$$\|e - p\| < 1$$

and $\pi(e) = \pi(p)$, then there is a unitary u in $\tilde{A}$ such that

$$u^* e u = p, \ \ \|u - 1\| \leqq 2\|e - p\|$$

and $\pi(u) = 1$.

PROOF. Let $x = pe$. Notice that

$$\pi(x) = \pi(pe) = \pi(p) = \pi(e)$$

and $\pi(x|x|^{-1}) = \pi(p) = \pi(e)$. The proof is the same as that of [Ef, A8.2 and A8.3].

LEMMA 2.3. *Let A be a σ-unital C^*-algebra with an approximate identity $\{e_n\}$ consisting of projections, I a closed ideal of A and S a support algebra of A. Suppose that π is the quotient map from A onto A/I and $\bar{u} \in M(A/I)$ such that*

$$\bar{u}^*\pi(S)\bar{u} = \bigcup_{n=1}^{\infty} \pi(e_n)(A/I)\pi(e_n)$$

and $\|\bar{u} - 1\| < 1$. *Then there is a unitary* $u \in M(A)$ *such that*

$$u^*Su = \bigcup_{n=1}^{\infty} e_n A e_n,$$
$$\pi(u) = \bar{u}, \text{ and}$$
$$\|u - 1\| < 1.$$

PROOF. It follows from 1.2 that $S = \bigcup_{n=1}^{\infty} p_n A p_n$, where $\{p_n\}$ is an approximate identity for A consisting of projections. Since

$$\|u - 1\| < 1,$$

there is $\bar{h} \in M(A/I)_{\mathrm{sa}}$. such that $\bar{u} = \exp(i\bar{h})$ with $\|\bar{h}\| \leqq \pi$. By Pedersen's Tietze-theorem [Pe 2, 10], there is $h \in M(A)_{\mathrm{sa}}$. such that $\pi(h) = \bar{h}$ and $\|h\| = \|\bar{h}\|$. Set $u_0 = \exp(ih)$. Since $\|\bar{h}\| = \|h\|$, by considering the commutative C^*-subalgebra generated by h, we have

$$\|u_0 - 1\| = \|\bar{u} - 1\| < 1.$$

Set $\delta = (1/2)(1 - \|\bar{u} - 1\|)$, $\varepsilon_n = u_0 e_n u_0^*$ and $S_1 = \bigcup_{n=1}^{\infty} \varepsilon_n A \varepsilon_n$. Let a_n and b_n be in S and S_1, respectively, such that $0 \leqq a_n \leqq 1$ and $0 \leqq b_n \leqq 1$, $\pi(a_n) = \pi(\varepsilon_n)$ and $\pi(b_n) = \pi(p_n)$.

Now there is an integer n_1 such that

$$\|(1 - p_{n_1})\varepsilon_1\| < \delta/64$$

and

$$\|p_{n_1} e_1 p_{n_1} - \varepsilon_1\| < \delta/32.$$

Since $a_1 \in S = \bigcup_{n=1}^{\infty} p_n A p_n$, we may assume that

$$p_{n_1} a_1 = a_1 p_{n_1} = a_1.$$

So

$$\pi(p_{n_1}\varepsilon_1 p_{n_1}) = \pi(p_{n_1})\pi(\varepsilon_1)\pi(p_{n_1})$$
$$= \pi(p_{n_1})\pi(a_1)\pi(p_{n_1}) = \pi(a_1) = \pi(\varepsilon_1).$$

By 2.1 and 2.2, there is a projection $q_1 \leqq p_{n_1}$ such that

$$\|q_1 - \varepsilon_1\| < \delta/16.$$

$\pi(q_1) = \pi(\varepsilon_1)$ and a unitary $w_1 \in \tilde{A}$ such that

$$w_1^* q_1 w_1 = \varepsilon_1, \ \|w_1 - 1\| < \delta/8.$$

and $\pi(w_1) = 1$. Set $v_1 = w_1^* q_1$. Since

$$\|\varepsilon_1 - q_1\| < \delta/16$$

and

$$\|(1 - p_{n_1})\varepsilon_1\| < \delta/64,$$

there is an integer m_1 such that

$$\|(\varepsilon_{m_1} - \varepsilon_1)(p_{n_1} - q_1)(\varepsilon_{m_1} - \varepsilon_1) - (p_{n_1} - q_1)\| < \delta/8.$$

Since $b_{m_1} \in S_1$, we may assume that

$$\varepsilon_{m_1} b_{n_1} = b_{n_1} \varepsilon_{m_1} = b_{m_1}.$$

Therefore

$$\pi[(\varepsilon_{m_1} - \varepsilon_1)(p_{n_1} - q_1)(\varepsilon_{m_1} - \varepsilon_1)] = \pi(p_{n_1} - q_1).$$

By applying 2.1 and 2.2, we obtain a projection $q_2' \leqq \varepsilon_{m_1} - \varepsilon_1$ and a unitary $w_1' \in [(1 - \varepsilon_1)A(1 - \varepsilon_1)]$ such that

$$\|q_2' - (p_{n_1} - q_1)\| < \delta/4,$$

$$\pi(q_2') = \pi(p_{n_1} - q_1),$$

$$(w_1')^*(p_{n_1} - q_1)w_1' = q_2',$$

$$\|w_1' - (1 - \varepsilon_1)\| < \delta/2$$

and $\pi(w_1') = 1 - \varepsilon_1$. Set $v_2 = (w_1')^*(p_{n_1} - q_1)$, $q_1' = \varepsilon_1$ and $q_2 = p_{n_1} - q_1$. Then

$$q_1 + q_2 = p_{n_1},\ q_1' + q_2' = \varepsilon_{m_1},$$

$$v_i^* v_i = q_i,\ v_i v_i^* = q_i',$$

$$\pi(v_i) = \pi(q_1) = \pi(q_i'),$$

and

$$\|v_i - q_i'\| < \delta/2^i,$$

$i = 1, 2.$

By repeating these argument and induction we construct sequences of projections $\{q_k\}$ and $\{q_k'\}$ such that $\sum_{m=1}^{2k} q_m = p_{n_k}$, $\sum_{m=1}^{2k} q_m' = \varepsilon_{m_k}$ and $\pi(q_k) = \pi(q_k')$, and a sequence of partial isometries $\{v_k\}$ such that

$$v_k^* v_k = q_k,\ v_k v_k^* = q_k',$$

$$\pi(v_k) = \pi(q_k) = \pi(q_k')$$

and

$$\|v_k - q'_k\| < \delta/2^k.$$

It is easy to check that $\sum_{k=1}^{\infty} v_k$ is a unitary in $M(A)$ satisfying

$$\|v - 1\| < \delta, \pi(v) = 1$$

and $v^* S v = S_1$.

Now take $u = u_0 v$. Then

$$\pi(u) = \pi(u_0)\pi(v) = \pi(u_0) = \bar{u},$$

$$u^* S u = \bigcup_{n=1}^{\infty} e_n A e_n$$

and

$$\|u - 1\| = \|\bar{u} - 1\| + \delta < 1.$$

This completes the proof.

Recall that a projection p in A^{**} is open if it is in $(A_+)^m$ (see [Pe 3, 3.11.9, 3.11.10]) and p is closed if $1 - p$ is open. If $p \in A^{**}$, then $\bar{p}$ denotes the smallest closed projection majorizing p. If p is open, we denote by $\mathrm{Her}(p)$ the hereditary C^*-subalgebra $pA^{**}p \cap A$.

THEOREM 2.4. *A σ-unital C^*-algebra is singly supported if it satisfies:*

(1) *there is a sequence of closed ideals $\{I_k\}$ of A, if p_k is the central open projection corresponding to I_k, then $p_k \geqq \bar{p}_{k+1} \geqq p_{k+1}$;*

(2) *there is an approximate identity $\{e_n\}$ for A such that if $\pi_k : A \to A/I_k$ denotes the quotient map, then $\pi(e_n)$ is a projection for $n \leqq m_k$ for every k and some m_k;*

(3) *for any $a \in A$,*

$$\lim_{k\to\infty} \|a p_k\| = 0.$$

Moreover, for each support algebra S of A, there is a unitary $u \in M(A)$ with $\|u - 1\| < 1$ such that

$$u^* S u = \bigcup_{n=1}^{\infty} e_n A e_n.$$

PROOF. For any $a \in A_+$ and $\varepsilon > 0$ there is n such that

$$\|a p_m\| < \varepsilon \text{ for } m \geqq n.$$

Let f_k be the continuous function defined in 0.2 with $k > 1/\varepsilon$,

$$f_k(a p_n) = 0.$$

Since p_n commutes with a, we have

$$f_k(a)p_n = 0.$$

Since $f_k(a)a \to a$ in norm, it follows that

$$A_0 = \{a \in A : ap_n = 0 \quad \text{for some } n\}$$

is dense in A. Set $B_k = \mathrm{Her}(1 - \bar{p}_k)$. By (1), $\bigcup_{k=1}^{\infty} B_k = A_0$.

Set $F_k = \{f : f(p_k) = 0,\ f \in Q\}$, where Q is the quasi-state space of A. Let $F = \bigcup_{k=1}^{\infty} F_k$. We claim that F is weak *-dense in Q.

Suppose that $\varphi_0 \in Q$, $\varepsilon > 0$ and $a_1, a_2, \ldots, a_m \in A$ and

$$G = \{\varphi \in Q : |\varphi_0(a_i) - \varphi(a_i)| < \varepsilon,\ i = 1, 2, \ldots, m\}.$$

Let $\varphi \in G$ such that

$$|\varphi_0(a_i) - \varphi(a_i)| < \varepsilon/2, \quad i = 1, 2, \ldots, m.$$

There is n such that

$$\| f_n(e_n)a_i - a_i \| < \varepsilon/2$$

for $i = 1, 2, \ldots, m$. Set $\varphi_1 = \varphi(f_n(e_n)\cdot)$. Then

$$\begin{aligned} &|\varphi_0(a_i) - \varphi_1(a_i)| \\ &\leqq |\varphi_p(a_i) - \varphi(a_i)| + |\varphi(a_i) - \varphi_1(a_i)| < \varepsilon. \end{aligned}$$

So $\varphi_1 \in G$. Since

$$\|e_n p_k\| \to 0 \quad \text{as} \quad k \to \infty,$$

by taking a subsequence if necessary, we may assume (as in the first part of the proof) that

$$f_n(e_n)p_n = 0.$$

Therefore $\varphi_1(p_n) = 0$. So $\varphi_1 \in F$. This proves the claim.

We may assume that $\pi_k(e_m)$ are projections in A/I_k for all $k \leqq m$. Let $S = S(a)$, where a is a strictly positive element, and a_n be as in 0.2. So $S = \bigcup_{n=1}^{\infty} a_n A a_n$ and $\pi_1(S)$ is a support algebra of A/I_1. Since $\{\pi_1(e_n)\}$ forms an approximate identity for A/I_1, there is, by Theorem 1.2, a unitary $u_1 \in M(A/I_1)$ such that

$$u_1^*(\pi_1(S))u_1 = \bigcup_{n=1}^{\infty} \pi(e_n)A/I_1\pi(e_n)$$

and

$$\|u_1 - 1\| < 1 - \delta$$

for some $\delta > 0$.

By Lemma 2.3 and induction, we have a sequence $\{u_k\}$ such that

(i) u_k is a unitary in $M(A/I_k)$;

(ii) $u_k^*\pi_k(S)u_k = \bigcup_{n=1}^{\infty} \pi_k(e_n)A/I_k(e_n)$;

(iii) $\pi_k(u_k) = u_l$ for all $n \geqq k$;

(iv) $\|u_k - 1\| < 1 - \delta$; $k = 1, 2, \ldots$.

For any k, if $b \in B_k$ then

$$(1 - \bar{p}_k)u_m\pi_m(b) = u_m(1 - \bar{p}_k)\pi_m(b) = u_m\pi_m(b) = u_m\pi_m(b)(1 - \bar{p}_k)$$

for $m \geqq k$. Therefore $u_m\pi_m(b) \in \pi_m(B_k)$ for $m \geqq k$. Since $\pi_m|_{B_k}$ is an isomorphism, there are unique $u(b)$ and $(b)u$ in B_k such that for all m,

$$\pi_m(u(\mathrm{b})) = \mathrm{u}_m\pi_m(b) \quad \text{and}$$
$$\pi_m((b)u) = \pi_m(b)u_m.$$

Since $\bigcup_{k=1}^{\infty} B_k$ is dense in A and

$$\|u(b)\| \leqq \|b\| \quad \text{and} \quad \|(b)u\| \leqq \|b\|,$$

we conclude that there is a u in $M(A)$ such that for all $b \in \bigcup_{k=1}^{\infty} B_k$,

$$\pi_m(u(b)) = u_m\pi_m(b) \quad \text{and}$$
$$\pi_m((b)u) = \pi_m(b)u_m.$$

Therefore $\pi_k(u)_k$ for each k, u is a unitary and $\|u - 1\| \leqq 1 - \delta < 1$. Since $\|ap_k\| \to 0$, as $k \to \infty$, we may assume that $a_n = f_n(a) \in B_n$, by passing a subsequence. By (ii)

$$u_n^*\pi_n(a_n)u_n \leqq \pi(e_{m_n})$$

for some m_n. Therefore

$$u^*a_nu(1 - p_n) \leqq e_{m_n}(1 - p_n) \leqq e_{m_n}.$$

Since $u^*a_nu = u^*a_nu(1 - p_n)$, we conclude that $u^*a_nu \leqq e_{m_n}$. Similarly for each m, there is m_k such that

$$ue_mu^* \leqq a_{m_k}.$$

Therefore,

$$u^*Su = \bigcup_{n=1}^{\infty} e_nAe_n.$$

This completes the proof.

COROLLARY 2.5. *Let X be a locally compact, σ-compact Hausdorff space and let A be a C^*-algebra with an approximate identity consisting of projections. Then*

$C_0(X) \otimes A$ is singly supported. Moreover, if S_1 and S_2 are two support algebras of $C_0(X) \otimes A$, there is a unitary u in $M(C_0(X) \otimes A))$ such that

$$u^* S_1 u = S_2 \quad \text{and} \quad \|u - 1\| < 1.$$

PROOF: It is enough to show that $B = C_0(X) \otimes A$ satisfies the conditions (1), (2) and (3) in 2.4. There are compact subsets X_n and open subsets G_n of X satisfying

$$X_n \subset G_n \subset X_{n+1}, \quad n = 1, 2, \ldots.$$

Identify $C_0(X) \otimes A$ with $C_0(X, A)$. Let

$$I_k = \{f \in C_0(X, A) : f(x) = 0 \text{ if } x \in X_k\},$$

and p_k be the open projection corresponding to I_k. Then (1) and (3) are satisfied. Since X is normal, there are functions h_k on X such that $0 \leqq h_k \leqq 1$, $h_k(x) = 1$ if $x \in X_k$ and $h_k(x) = 0$ if $x \in X \backslash G_k$. Let $\{p_n\}$ be an approximate identity for A consisting of projections. Set

$$e_k = h_k p_k, \quad k = 1, 2, \ldots.$$

Then $\{e_k\}$ forms an approximate identity for B. Moreover,

$$\pi_k(e_n) \quad \text{are projections}$$

in A/I_k for $n \geqq k$. So the condition (3) is also satisfied.

REMARK 2.6. It should be noted if X is not compact but X is connected, then $C_0(X) \otimes A$ has no nonzero projections.

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