MATH SCAND. 71 (1992), 161–166

APPROXIMATION BY NEAREST INTEGER CONTINUED FRACTIONS

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Abstract.

Let x be a real number, $x = [\varepsilon_0 b_0; \varepsilon_1 b_1, \ldots, \varepsilon_n b_n, \ldots]$ be its expansion in nearest integer continued fraction. Let A_n/B_n be its nth convergent and $\Theta_n = B_n^2 |x - A_n/B_n|$. In this note we prove that $\min(\Theta_{n-1}, \Theta_n, \ldots, \Theta_{n+k}) < 2/(3 + \sqrt{5} - 2\alpha_k)$, where $\alpha_1 = 2/5$, $\alpha_i = 1/(3 - \alpha_{i-1})$. The result is best possible.

1. Introduction.

A hundred years ago, Hurwitz [2] proved a fundamental theorem on the approximation of irrational numbers by rational numbers.

THEOREM 1. *For each irrational number x, there are infinitely many rational numbers p/q such that $|x - p/q| < 1/(\sqrt{5}\, q^2)$.*

To find these rational numbers p/q, Borel [1] considered simple continued fractions. Let $x = [a_0; a_1, \ldots, a_i, \ldots]$ be the expansion of x in simple continued fraction, where a_0 is an integer and a_i $(i \geqq 1)$ are positive integers. Let $p_i/q_i = [a_0; a_1, \ldots, a_i]$ be its ith convergent and $\theta_i = q_i^2 |x - p_i/q_i|$. Borel proved the following theorem.

THEOREM 2. $\min(\theta_{n-1}, \theta_n, \theta_{n+1}) < 1/\sqrt{5} = 0.4472\ldots$

Theorem 2 asserts that at least one of any three consecutive convergents satisfies Hurwitz' theorem. For more information on Borel's theorem, see [9], [10], [11].

Besides simple continued fraction, there is another tool of approximation. It is the nearest integer continued fraction, which expands each real number x as $x = [\varepsilon_0 b_0; \varepsilon_1 b_1, \ldots, \varepsilon_n b_n, \ldots]$ (finite or infinite), where $\varepsilon_n \in \{-1, 1\}$ and b_n are positive integers for $n = 1, 2, \ldots$. Let $A_n/B_n = [\varepsilon_0 b_0; \varepsilon_1 b_1, \ldots, \varepsilon_n b_n]$ *be its* nth convergent and $\Theta_n = B_n^2 |x - A_n/B_n|$. Since nearest integer continued fraction

Received October 8, 1991.

generally gives faster approximation than simple continued fraction, one may naturally expect that the Borel's theorem also holds. But supprisingly Jager and Kraaikamp [6] proved the following curious result.

THEOREM 3. $\min(\Theta_{n-1}, \Theta_n, \Theta_{n+1}) < 5(5\sqrt{5} - 11)/2 = 0.4508\ldots$. *The result is best possible.*

There is a gap of 0.0036 between the approximations by the two kinds of continued fractions. In this paper we will show that this gap can never be filled up by taking more consecutive Θ's. We prove a generalization of Theorem 3: $\min(\Theta_{n-1}, \Theta_n, \ldots, \Theta_{n+k}) < 2/(3 + \sqrt{5} - 2\alpha_k)$, where $\alpha_1 = 2/5, \alpha_i = 1/(3 - \alpha_{i-1})$.

2. Preliminaries.

We first recall some basic facts about nearest integer continued fraction.

Let x be the real number and $[x]$ be the greatest integer less than or equal to x. Then $[x] \leqq x < [x] + 1$. Let $\{x\} = [x + 1/2]$. Then $\{x\}$ is the nearest integer to x and $\{x\} - 1/2 \leqq x < \{x\} + 1/2$.

The simple continued fraction expansion of x is the following expression (finite or infinite)

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots\cdots}} = [a_0; a_1, a_2, \ldots],$$

where $a_n (n = 0, 1, 2, \ldots)$ are obtained by the iteration: $x_0 = x$, $a_0 = [x_0]$; $x_n = (x_{n-1} - a_{n-1})^{-1}$, $a_n = [x_n]$.

The nearest integer continued fraction expansion of x is the following expression (finite or infinite):

$$x = \varepsilon_0 b_0 + \cfrac{\varepsilon_1}{b_1 + \cfrac{\varepsilon_2}{b_2 + \cdots\cdots}} = [\varepsilon_0 b_0; \varepsilon_1 b_1, \varepsilon_2 b_2, \ldots],$$

where $\varepsilon_n \in \{-1, 1\}$ and $b_n > 0 (n = 0, 1, 2, \ldots)$ are obtained by the iteration: $x_0 = x$, $\varepsilon_0 b_0 = \{x_0\}$; $x_n = (x_{n-1} - b_{n-1})^{-1}$, $\varepsilon_n b_b = \{x_n\}$, ε_n is the sign of $\{x_n\}$.

Nearest integer continued fraction was first introduced by Minnigerode ([7], 1873). For details, see Perron [8]. In the following we quote some known facts in [8].

(i) $b_n \geqq 2$, $b_n + \varepsilon_{n+1} \geqq 2$ for $n = 1, 2, \ldots$;

(ii) Let $A_n/B_n = [\varepsilon_0 b_0;\ \varepsilon_1 b_1, \ldots, \varepsilon_n b_n]$ be the nth convergent.

Then

$$A_{-1} = 1,\ A_0 = 0,\ A_n = b_n A_{n-1} + \varepsilon_n A_{n-2}\ (n > 1);$$

$$B_{-1} = 0,\ B_0 = 1,\ B_n = b_n B_{n-1} + \varepsilon_n B_{n-2}\ (n > 1);$$

(iii) $$B_{n-1}/B_n < (\sqrt{5} - 1)/2 \text{ for } n = 1, 2 \ldots;$$

$$B_{n-1}/B_n < (3 - \sqrt{5})/2 \text{ for } n = 2, 3, \ldots, \text{if } b_n \geqq 3;$$

(iv) Let $s^n x = [0; \varepsilon_{n+1} b_{n+1}, \varepsilon_{n+2} b_{n+2}, \ldots]$. Then

$$x - \frac{A_n}{B_n} = \frac{(-1)^n \varepsilon_1 \ldots \varepsilon_n s^n x}{B_n (B_n + B_{n-1} s^n x)}.$$

Property (iii) plays an important role in our proof. It was found by Hurwitz ([3], 1889). See also [4], [5], [8, Satz 5.18 (B)].

Let $\Theta_n = B_n^2 |x - A_n/B_n|$. By (iv) we have

$$\Theta_n = \frac{\varepsilon_{n+1} s^n x}{1 + \dfrac{B_{n-1}}{B_n}} = \frac{1}{\dfrac{1}{\varepsilon_{n+1} s^n x} + \varepsilon_{n+1} \dfrac{B_{n-1}}{B_n}}.$$

Since $\dfrac{1}{\varepsilon_{n+1} s^n x} = b_{n+1} + \cfrac{\varepsilon_{n+2}}{b_{n+2} + \cfrac{\varepsilon_{n+3}}{b_{n+3} + \cdots\cdots}}$,

letting $P = b_{n+2} + \dfrac{\varepsilon_{n+3}}{b_{n+3} + \cdots\cdots} = [b_{n+2}; \varepsilon_{n+3} b_{n+3}, \ldots]$ and $Q = B_n/B_{n-1} = [b_n; \varepsilon_n b_{n-1}, \varepsilon_{n-1} b_{n-2}, \ldots, \varepsilon_2 b_1]$ we have

(1) $$\Theta_n^{-1} = b_{n+1} + \varepsilon_{n+2} P^{-1} + \varepsilon_{n+1} Q^{-1}.$$

The following two equalities are easily checked.

(2) $$\Theta_{n+1}^{-1} = P + \frac{\varepsilon_{n+2}}{\Theta_n^{-1} - \varepsilon_{n+2} P^{-1}},$$

(3) $$\Theta_{n-1}^{-1} = Q + \frac{\varepsilon_{n+1}}{\Theta_n^{-1} - \varepsilon_{n+1} Q^{-1}}.$$

By property (i), (iii), we have $P \geqq 2$, $Q^{-1} < (\sqrt{5} - 1)/2$. If $b_n \geqq 3$, then $Q^{-1} < (3 - \sqrt{5})/2$.

3. Main result.

THEOREM 4. *Let X be a real number, $x = [\varepsilon_0 b_0; \varepsilon_1 b_1, \ldots, \varepsilon_n b_n, \ldots]$ be its expansion in nearest integer continued fraction. Let A_n/B_n be its* nth *convergent and*

$\Theta_n = B_n^2|x - A_n/B_n|$. *Then for any* $k \geqq 1$, $\min(\Theta_{n-1}, \Theta_n, \ldots, \Theta_{n+k}) < 2/(3 + \sqrt{5} - 2\alpha_k)$, *where* α_k *can be obtained by the relation* $\alpha_1 = 2/5$, $\alpha_i = 1/(3 - \alpha_{i-1})$. *The constant* $2/(3 + \sqrt{5} - 2\alpha_k)$ *cannot be replaced by a smaller number.*

PROOF. We discuss all possible cases on ε_{n+1} and ε_{n+2}.

Case 1. $\varepsilon_{n+1} = \varepsilon_{n+2} = 1$.

From (1) we have $\Theta_n^{-1} = b_{n+1} + P^{-1} + Q^{-1} \geqq 2 + P^{-1} + Q^{-1}$. If $\Theta_n > 1/\sqrt{5}$, then $\sqrt{5} - 2 > P^{-1} + Q^{-1}$. Hence at least one of P^{-1}, Q^{-1} must be less than $(\sqrt{5} - 2)/2$, or equivalently, at least one of P, Q exceeds $2(\sqrt{5} + 2) > \sqrt{5}$. By (2), (3), we know that at least one of Θ_{n+1}^{-1}, Θ_{n-1}^{-1} exceeds $\sqrt{5}$, hence $\min(\Theta_{n-1}, \Theta_n, \Theta_{n+1}) < 1/\sqrt{5}$.

Case 2. $\varepsilon_{n+1} = -1$, $\varepsilon_{n+2} = 1$.

By properties (i), (iii), we have $b_n \geqq 3$ and $Q^{-1} < (3 - \sqrt{5})/2$.

If $\Theta_n > 1/\sqrt{5}$, then $\sqrt{5} > \Theta_n^{-1} = b_{n+1} + P^{-1} - Q^{-1} > 2 + P^{-1} - (3 - \sqrt{5})/2 = (1 + \sqrt{5})/2 + P^{-1}$, hence $P > (\sqrt{5} + 1)/2$. By (2) we have $\Theta_{n+1}^{-1} = P + \dfrac{1}{\Theta_n^{-1} - P^{-1}} > P + \dfrac{1}{\sqrt{5} - P^{-1}}$. It is easily seen that $P + \dfrac{1}{\sqrt{5} - P^{-1}}$ is an increasing function in P by finding its derivative, hence we have

$$\Theta_{n+1}^{-1} > \frac{\sqrt{5}+1}{2} + \frac{1}{\sqrt{5} - \left(\dfrac{\sqrt{5}+1}{2}\right) - 1} = \sqrt{5}, \text{ and } \Theta_{n+1} > 1/\sqrt{5}.$$

Case 3. $\varepsilon_{n+1} = 1$, $\varepsilon_{n+2} = -1$.

By property (iii), we have $Q^{-1} < (\sqrt{5} - 1)/2$, or $Q > (\sqrt{5} + 1)/2$.

If $\Theta_n > 1/\sqrt{5}$, then $\Theta_n^{-1} < \sqrt{5}$. By (3) $\Theta_{n-1}^{-1} = Q + \dfrac{1}{\Theta_n^{-1} - Q^{-1}} > Q + \dfrac{1}{\sqrt{5} - Q^{-1}}$. Since $Q + \dfrac{1}{\sqrt{5} - Q^{-1}}$ is an increasing function in Q and $Q > (\sqrt{5} + 1)/2$, we have $\Theta_{n-1}^{-1} > \sqrt{5}$, or $\Theta_{n-1} < 1/\sqrt{5}$.

Case 4. $\varepsilon_{n+1} = \varepsilon_{n+2} = -1$.

There are two possibilities.

Case 4a. There is a positive integer s satisfying $3 \leqq s \leqq k$, $\varepsilon_{n+1} = \varepsilon_{n+2} = \ldots = \varepsilon_{n+s-1} = -1$, $\varepsilon_{n+s} = 1$.

In case 2, consider $\varepsilon_{n+s-1} = -1$, $\varepsilon_{n+s} = 1$ instead of $\varepsilon_{n+1} = -1$, $\varepsilon_{n+2} = 1$, we have $\min(\Theta_{n+s-2}, \Theta_{n+s-1}) < 1/\sqrt{5}$.

Case 4b. $\varepsilon_{n+1} = \varepsilon_{n+2} = \ldots = \varepsilon_{n+k} = -1$.

By properties (i), (iii), we have $b_{n+2} \geqq 3, \ldots, b_{n+k-1} \geqq 3$ and $Q^{-1} < (3-\sqrt{5})/2$. Therefore $P = [b_{n+2}; \varepsilon_{n+3}b_{n+3}, \ldots] > [3, -3, \ldots, -3, 2]$ where there are k consecutive -3s. Let $\alpha_{k-1} = -[0; -3, \ldots, -3, 2]$ with $k-1$ consecutive -3s. Then $\alpha_1 = 1/(3-1/2) = 2/5$, $\alpha_2 = 1/(3-2/5) = 5/13, \ldots, \alpha_i = 1/(3-\alpha_{i-1})$ $(2 \leqq i \leqq k-1)$. It is easily seen that α_i approaches to $(3-\sqrt{5})/2$ increasingly and $P^{-1} > \alpha_k$. Hence $\Theta_{n+1}^{-1} > b_{n+2} - P^{-1} - Q^{-1} > 3 - \alpha_k - (3-\sqrt{5})/2 = (3+\sqrt{5}-2\alpha_k)/2$.

From the discussions above, we know that

$$\min(\Theta_{n-1}, \Theta_n, \ldots, \Theta_{n+k}) < 2/(3+\sqrt{5}-2\alpha_k).$$

To show that the numbers $2/(3+\sqrt{5}-2\alpha_k)$ are best possible, for any $\varepsilon > 0$ and any fixed integer $k > 0$, we consider the real number $x_m = [0; \overline{-3, \ldots, -3, 2}]$. If m is large enough, we have $\min(\Theta_{r(m+1)-k-1}, \Theta_{r(m+1)-k}, \ldots, \Theta_{r(m+1)}) > 2/(3+\sqrt{5}-2\alpha_k) - \varepsilon$ for any positive integer r. Therefore, the constants on the right side of the previous inequality cannot be replaced by smaller numbers.

COROLLARY 1. $\min(\Theta_{n-1}, \Theta_n, \ldots, \Theta_{n+k}) < 1/\sqrt{5}$ *is false for general irrational number* x.

COROLLARY 2. $\min(\Theta_{n-1}, \Theta_n, \Theta_{n+1}) < 5(5\sqrt{5}-11)/2 = 0.4508\ldots.$

$$\min(\Theta_{n-1}, \Theta_n, \Theta_{n+1}, \Theta_{n+2}) < 13(13\sqrt{5}-29)/2 = 0.4477\ldots.$$

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