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# HEREDITARY CS-MODULES

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In [13], [14], B. L. Osofsky proved that a ring $R$ with the property that all its cyclic right modules are injective is semiprime Artinian. Consequently, any right hereditary right self-injective ring is semiprime Artinian [13, Corollary]. This interesting fact is quoted by several authors (see, for example, [1], [2]). In [15, p. 46 Proposition 2.24], Osofsky gives a second proof that right hereditary right self-injective rings are semiprime Artinian.

The purpose of this note is to extend Osofsky's result on hereditary self-injective rings to modules. We prove that, for any ring $R$, any right $R$-module which is an hereditary CS-module is a direct sum of Noetherian modules. Moreover, as an application of this result, we show that, for any ring $R$ which is either commutative or semiprime, any hereditary continuous right $R$-module is semisimple. Several other applications are given. For example, any right continuous ring with all right ideals countably generated is semiperfect. It follows that, for any ring $R$, any countable continuous right $R$-module is a direct sum of uniform submodules.

These results generalise [12] and the proofs given substantially simplify the corresponding proofs in [12].

## 1. The main theorem.

Throughout this note, $R$ will denote an associative ring with identity and all modules considered will be unitary right $R$-modules. Following [2], a module $M$ is called a CS-*module* provided every submodule of $M$ is essential in a direct summand of $M$. Moreover, $M$ is called *continuous* if $M$ is a CS-module such that every submodule of $M$ isomorphic to a direct summand of $M$ is itself a direct summand of $M$. For basic properties of CS-modules and continuous modules we refer to [11]. A module $M$ is called an *hereditary* module if every submodule of $M$ is projective (see [4] or [6]). In particular, it is well known that if $R$ is a right hereditary ring then any projective right $R$-module is hereditary.

To prove our main theorem, we require some lemmas.

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LEMMA 1. *Let $R$ be any ring and $M$ a right $R$-module such that $M$ is a projective* CS-*module. Then there exists an index set $I$ such that $M$ is a direct sum $\oplus_I M_i$ of submodules $M_i$ $(i \in I)$ of $M$ such that each submodule $M_i$ contains a finitely generated essential submodule.*

PROOF. By Kaplansky's Theorem (see, for example [3, p. 120]), the module $M$ is a direct sum of countably generated submodules. By [11, Proposition 2.7], without loss of generality, we may suppose that $M$ is countably generated. There exists a countable set of elements $m_1, m_2, m_3, \ldots$ in $M$ such that $M = \sum_i m_i R$. By hypothesis, there exist submodules $M_1$, $N_1$ of $M$ such that $M = M_1 \oplus N_1$ and $m_1 R$ is essential in $M_1$. Suppose that $n_i$ is the projection of $m_i$ in $N_i$ for all $i \geqq 2$. By [11, Proposition 2.7] again, there exists a direct summand $M_2$ of $N_1$ which contains $n_2 R$ as an essential submodule. Continuing in this manner we obtain a direct sum $M_1 \oplus M_2 \oplus M_3 \oplus \ldots$ of submodules in the module $M$ such that

$$m_1 R + \ldots + m_k R \subseteq M_1 \oplus \ldots \oplus M_k,$$

for all positive integers $k$. It follows that $M = \oplus_i M_i$. Moreover, by construction, each submodule $M_i$ contains a finitely generated (in fact cyclic) essential submodule.

Let $R$ be a ring and $M$ a right $R$-module. For any element $m$ in $M$, let $r(m)$ denote the annihilator of $m$ in $R$, i.e.

$$r(m) = \{r \in R: mr = 0\}.$$

Recall that a right $R$-module $M$ is called *nonsingular* if $r(m)$ is not an essential right ideal of $R$ for any non-zero element $m$ of $M$. The next lemma is due essentially to Sandomierski [16].

LEMMA 2. *Let $R$ be any ring and $P$ a nonsingular projective right $R$-module such that $P$ contains a finitely generated essential submodule. Then $P$ is finitely generated.*

PROOF. See [1, Proposition 8.24].

The next result is immediate by Lemmas 1 and 2.

COROLLARY 3. *Let $R$ be any ring. If a right $R$-module $M$ is a nonsingular projective* CS-*module then $M$ is a direct sum of finitely generated modules.*

We shall require the following key lemma of Osofsky [14].

LEMMA 4. *Let $\{e_i\colon i \in I\}$ be an infinite set of orthogonal idempotents in a ring $R$. Suppose that for each non-empty subset $P$ of $I$ there exists an idempotent $f$ in $R$ such that $e_i = fe_i$ for all $i$ in $P$ and $e_j f = 0$ for all $j$ in $I \backslash P$. Let $K = \{r \in R\colon e_i r = 0\ (i \in I)\}$.*

*Let $M$ be any right $R$-module containing $R$ as a submodule. Then the right $R$-module $M/(K + \sum_I e_i R)$ is not injective.*

The next lemma is well known but we do not know a convenient reference for it.

LEMMA 5. *Let $R$ be a ring and $M$ a right $R$-module such that the endomorphism ring $S = \mathrm{End}(M_R)$ of $M$ does not contain an infinite set of orthogonal idempotents. Then $M$ is a finite direct sum of indecomposable submodules.*

PROOF. Suppose that the result is false. Then $M$ is not indecomposable, so that $M = N_1 \oplus K_1$, for some non-zero submodules $N_1$, $K_1$. Either $N_1$ or $K_1$ is not indecomposable, so that we can suppose without loss of generality that $K_1 = N_2 \oplus K_2$ for some non-zero submodules $N_2$ and $K_2$. Repeating this argument we obtain an infinite properly ascending chain $N_1 \subseteq N_2 \subseteq N_3 \subseteq \dots$ of direct summands of $M$. For each $k \geqq 1$ there exists an idempotent $e_k$ in $S$ such that $N_k = e_k M$. It is easy to check that $e_1 S \subseteq e_2 S \subseteq e_3 S \subseteq \dots$ is a properly ascending chain of direct summands of $S_S$ (see, for example, [11, Lemma 3.1]). Since the ring $S$ does not contain an infinite set of orthogonal idempotents, it is well known that the right $S$-module $S$ has ACC on direct summands, a contradiction. Thus $M$ is a finite direct sum of indecomposable submodules.

We are now in a position to prove our main theorem.

THEOREM 6. *Let $R$ be any ring and $M$ a right $R$-module such that $M$ is an hereditary* CS*-module. Then $M$ is a direct sum of Noetherian uniform modules.*

PROOF. We show first that $M$ is nonsingular. Let $0 \neq m \in M$. Note that $R/r(m) \cong mR$, and $mR$ is projective. Hence $r(m) = eR$ for some idempotent $e$ in $R$. Clearly, $r(m)$ is not essential. It follows that $M$ is nonsingular. By Corollary 3, $M$ is a direct sum of finitely generated modules. By [11, Proposition 2.7], we may suppose, without loss of generality, that $M$ is finitely generated, and prove that $M$ is both Noetherian and a direct sum of uniform modules.

Let $S = \mathrm{End}(M_R)$. Since $M$ is a finitely generated hereditary module it follows that $S$ is a right hereditary ring (see [4, Corollary 2] or [6, Theorem 2.5]). Suppose that $S$ contains an infinite collection $\{e_i\colon i \in I\}$ of orthogonal idempotents. Let $P$ be any non-empty subset of $I$. Let $N$ denote the submodule $\oplus_P e_i M$ of $M$. Since $M$ is a CS-module, there exist submodules $L$, $L'$ of $M$ such that $M = L \oplus L'$ and $N$ is an essential submodule of $L$.

Let $f\colon M \to L$ denote the canonical projection. Clearly $f$ is an idempotent in $S$ and $e_i = fe_i$ for all $i \in P$. Let $j \in I \backslash P$. Let $x \in M$. Then $f(x) \in L$, and, by [1, Lemma 1.1], there exists an essential right ideal $H$ of $R$ such that $f(x)H \subseteq N$. Thus $e_j f(x)H = 0$. Because $M$ is nonsingular, it follows that $e_j f(x) = 0$. Therefore $e_j f(x) = 0$ for all $x \in M$. We have now proved that $e_j f = 0$ for all $j \in I \backslash P$.

Let $E$ denote the injective hull of the right $S$-module $S$. Let $K = \{s \in S\colon e_i s = 0 \ (i \in I)\}$. By Lemma 4, $E/(K + \sum_I e_i S)$ is not injective. However, because $S$ is right hereditary, every quotient of $E$ is an injective right $S$-module, a contradiction. This contradiction shows that $S$ cannot contain an infinite set of orthogonal idempotents.

By Lemma 5, the right $R$-module $M$ is a finite direct sum of indecomposable submodules. By [11, Proposition 2.7] each of these indecomposable summands is a CS-module and hence is uniform. Thus $M$ is a finite direct sum of uniform submodules. Finally, $M$ is Noetherian by Lemma 2.

A ring $R$ is called a *right* CS-*ring* if the right $R$-module $R$ is a CS-module. As an immediate consequence of the above theorem we obtain [12, Theorem 3.1]:

COROLLARY 7. *Any right hereditary right* CS-*ring is right Noetherian.*

In [7, Proposition 2], Jøndrup proved that if $R$ is a commutative ring then any hereditary module with Krull dimension has Krull dimension at most 1. In view of this fact, Theorem 6 has another immediate corollary, as follows.

COROLLARY 8. *Let $M$ be an hereditary* CS-*module over a commutative ring $R$. Then $M$ is a direct sum of Noetherian uniform modules, each with Krull dimension at most* 1.

## 2. Continuous modules.

In this section, our concern is with continuous modules. First of all, we consider hereditary continuous modules.

PROPOSITION 9. *Let $R$ be any ring and $M$ an hereditary continuous right $R$-module. Then $M$ is a direct sum of Noetherian uniform submodules, each with endomorphism ring a division ring.*

PROOF. By Theorem 6, $M$ is a direct sum of Noetherian uniform submodules. Let $N$ denote any uniform direct summand of $M$. By the proof of Theorem 6, $N$ is nonsingular. Let $f \in \mathrm{End}(N_R)$. Suppose that $f$ has non-zero kernel $K$. Let $x \in N$. There exists an essential right ideal $E$ of $R$ such that $xE \subseteq K$, and hence $f(x)E = 0$, giving $f(x) = 0$. It follows that $f = 0$. Hence any non-zero endomphism $f$ of $N$ has zero kernel, and hence is an automorphism by [11, Proposition 2.7]. Thus $\mathrm{End}(N_R)$ is a division ring.

LEMMA 10. *Let $R$ be any ring and $M$ a finitely generated quasi-projective right $R$-module such that* $\mathrm{Hom}_R(M, N) \neq 0$ *for every non-zero submodule $N$ of $M$. Then $M$ is semisimple if and only if the endomorphism ring* $\mathrm{End}(M_R)$ *of $M$ is semiprime Artinian.*

PROOF. See [5, p. 102 Remarks].

THEOREM 11. *Let R be a ring which is either commutative or semiprime. Then any hereditary continuous right R-module is semisimple.*

PROOF. Let $M$ be an hereditary continuous module. By Proposition 9, we can suppose, without loss of generality, that $M$ is Noetherian and $\mathrm{End}(M_R)$ is a division ring. If $R$ is a commutative ring, then it is well known that $M$ is a self-generator, i.e. $M$ generates all its submodules, and in particular $\mathrm{Hom}_R(M, N) \neq 0$ for every non-zero submodule $N$ of $M$. On the other hand, if $R$ is semiprime then Zelmanowitz [17, Proposition 1.2] has shown that $\mathrm{Hom}_R(M, N) \neq 0$ for every non-zero submodule $N$ of $M$. In any case, $M$ is semisimple by Lemma 10.

Let $R$ be a ring. In view of Theorem 11 (and Osofsky's Theorem mentioned in the introduction), it is natural to ask whether every hereditary injective (or even continuous) right $R$-module is semisimple. That this is not the case can be seen by the following example of Miller and Turnidge [10].

EXAMPLE 12. Let $D$ be a division ring, and $A$ the ring of countably infinite column-finite matrices over the ring $D$. Let $R$ denote the subring of $A$ consisting of all upper triangular matrices with entries from $D$. For all $1 \leqq i, j < \infty$, let $e_{ij}$ denote the matrix unit of $R$ having 1 as $(i,j)$th entry and all other entries 0. Let $M = e_{11}R$. It is shown in [10] that $M$ is a Noetherian injective right ideal of $R$. All $R$-submodules of $M$ appear in the chain $M = e_{11}R \supset e_{12}R \supset e_{13}R \supset \ldots \supset 0$. It is easy to check that, for each $j \geqq 1$, $r(e_{1j})$ is a direct summand of $R_R$, and hence $e_{1j}R$ is projective. Thus $M$ is an hereditary injective right $R$-module which is not semisimple.

Let $R$ be any ring. A right $R$-module $M$ will be called *regular* if every cyclic submodule of $M$ is a direct summand of $M$. Clearly, any regular uniform module is simple. Thus Theorem 6 gives the following fact at once: *for any ring R, every regular hereditary* CS-*module over R is semisimple.* In particular, if $R$ is a right hereditary ring then a right $R$-module $M$ is semisimple whenever $M$ is a regular projective CS-module. The next result gives a necessary and sufficient condition for a regular projective CS-module over a general ring to be semisimple.

PROPOSITION 13. *Let R be any ring and let M be a right R-module such that M is a regular projective* CS-*module. Then M is semisimple if and only if every submodule of M is a direct sum of countably generated modules.*

PROOF. The necessity is clear. Conversely, suppose that every submodule is a direct sum of countably generated submodules. A standard argument shows that every countably generated submodule of $M$ is a direct sum of cyclic submodules and hence is projective. Thus every submodule of $M$ is projective. Apply Theorem 6.

COROLLARY 14. *Let $R$ be a right continuous ring such that every right ideal of $R$ is countably generated. Then $R$ is a semiperfect ring. In particular, any countable right continuous ring is semiperfect.*

PROOF. Let $J$ denote the Jacobson radical of $R$. Then the ring $R/J$ is a von Neumann regular right continuous ring and idempotents can be lifted (see, for example, [11, Corollary 3.9 and Theorem 3.11]). Clearly, every right ideal of the ring $R/J$ can be countably generated. Thus $R/J$ is semiprime Artinian by Proposition 13. The last part is immediate.

In [8], Lawrence proved that a countable right self-injective ring satisfies ACC on right annihilators, and thus is a QF-ring (see, for example [3, Theorem 24.20]). Using similar methods, Megibben [9] proved that if $R$ is an arbitrary ring and $M$ a countable injective $R$-module then $R$ satisfies ACC on annihilators of subsets of $M$, and hence $M$ is $\Sigma$-injective (see, for example, [3, Proposition 20.3A]). We now give an example of a countable continuous commutative ring which does not satisfy ACC on annihilators. Note that this ring is semiperfect (Corollary 14) but is not semiprimary.

EXAMPLE 15. Let $p$ be any rational prime. Let $F = \mathbb{Z}/\mathbb{Z}p$, the field of $p$ elements and $G$ the Prufer $p$-group $C(p^\infty)$. Then the group algebra $R = F[G]$ is a countable continuous commutative ring which does satisfy ACC on annihilators.

PROOF. Clearly the ring $R$ is commutative. Suppose that $G$ has the presentation

$$G = \langle x_1, x_2, x_3, \dots \mid x_1^p = 1, x_{i+1}^p = x_i \quad (i \geqq 1)\rangle$$

For each $i \geqq 1$, let $G_i$ denote the subgroup $\langle x_i \rangle$. It is well known that $G$ is the union of its finite subgroups $G_i$ ($i \geqq 1$), and hence $R$ is the union of its finite subrings $F[G_i]$ ($i \geqq 1$). Thus $R$ is a countable ring. Let $\alpha$, $\beta$ be any non-zero elements of $R$. Then there exists $j \geqq 1$ such that $\alpha$ and $\beta$ both belong to the subring $S = F[G_j]$ of $R$. But $G_j$ is a cyclic $p$-group, so that $S$ is an Artinian principal ideal ring and hence $S\alpha \subseteq S\beta$ or $S\beta \subseteq S\alpha$. It follows that $R$ is a chain ring (i.e. the ideals of $R$ are totally ordered). In particular, the $R$-module $R$ is uniform, and hence $R$ is a CS-ring.

It is well known that the augmentation ideal $J$ of $R$ is a nil ideal (just note that, for each $i \geqq 1$, $(x_i - 1)^q = x_i^q - 1 = 0$, where $q = p^i$). But $R/J \cong F$, so that $R$ is a local ring. In particular, the only direct summands of $R$ are 0, $R$. Let $\varphi$: $R \to R$ be any monomorphism. Suppose that $\varphi(R) \subseteq J$. Let $\lambda = \varphi(1)$, and let $n$ denote the index of nilpotency of $\lambda$. Then $\varphi(\lambda^{n-1}) = \lambda^{n-1}\varphi(1) = \lambda^n = 0$, so that $\lambda^{n-1} = 0$, a contradiction. Thus $\varphi(R) \nsubseteq J$, and hence $\varphi(R) = R$. Thus $R$ is a continuous ring.

Next we show that $R$ does not satisfy ACC on annihilators. For each $i \geqq 1$, let $J_i$ denote the augmentation ideal of the subring $F[G_i]$. Then $J_1R \subseteq J_2R \subseteq J_3R \subseteq \ldots$ is an infinite ascending chain of annihilators in $R$. Moreover, the ring $R$ is not semiprimary since the Jacobson radical of $R$ is $J$ which is an idempotent ideal, because, for each $i \geqq 1$,

$$x_{i+1} - 1 = (x_i - 1)^p \in J^p \subseteq J^2.$$

Any $\Sigma$-injective module is a direct sum of uniform modules (see, for example, [3, Corollary 20.6A]). Hence, by [9], any countable injective module is a direct sum of uniform modules. Using Corollary 14, we now show that a similar result holds for countable continuous modules.

THEOREM 15. *Let $R$ be any ring. Then any countable continuous right $R$-module is a direct sum of uniform modules.*

PROOF. Let $M$ be a countable continuous right $R$-module. By the proof of Lemma 1, $M$ is a direct sum of submodules each containing a finitely generated essential submodule. Without loss of generality, we can suppose that $M$ contains a finitely generated essential submodule $N$. Suppose that $N = x_1R + \ldots + x_nR$, for some positive integer $n$. Let $S = \mathrm{End}(M_R)$. Define an Abelian group homomorphism $\pi\colon S \to M^n$ by

$$\pi(f) = (f(x_1), \ldots, f(x_n)) \quad (f \in S)$$

Clearly the kernel $K$ of $\pi$ is given by

$$K = \{f \in S \colon f(N) = 0\}.$$

Let $J$ denote the Jacobson radical of the ring $S$. By [11, Proposition 3.5 and Theorem 3.11], the ring $S/J$ is von Neumann regular and right continuous, where

$$J = \{f \in S \colon \ker f \text{ is essential in } M\}.$$

Clearly $K \subseteq J$. But $M$ is countable, and hence so too is $M^n$, and also $S/J$. By Corollary 14, $S/J$ is semiprime Artinian. Thus $S$ does not contain an infinite set of orthogonal idempotents. By Lemma 5, $M$ is a finite direct sum of indecomposable submodules. Now apply [11, Proposition 2.7] and note that every indecomposable continuous module is uniform.

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