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APPLICATIONS OF ULTRAPRODUCTS TO INFINITE DIMENSIONAL HOLOMORPHY

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1. Introduction.

The purpose of this paper is to apply ultraproduct techniques to some problems in Infinite Dimensional Holomorphy. The central problem we consider is the following: Given a continuous polynomial P, or more generally, a holomorphic function, f, defined on a Banach space X, can we extend P or f to a larger space containing X? Questions such as these were first tackled by Aron and Berner [AB]. They showed how polynomials, and certain holomorphic functions, can be extended to the bidual X^{**}. From this, they were able to construct extensions for other spaces containing X. However, some questions were left open. For example, it was not known whether the Aron-Berner extension of a continuous polynomial P from X to X^{**} had the same norm as P. This question was recently answered in the affirmative by Davie and Gamelin [DG].

We present a new approach to this extension problem. Our approach is to work with an ultrapower $(X)_{\mathscr{U}}$ of the Banach space X rather than the bidual of X. There is a canonical embedding of X into $(X)_{\mathscr{U}}$, and it is relatively simple to construct extensions of polynomials and holomorphic functions from X into $(X)_{\mathscr{U}}$. For certain special ultrapowers of X we have roughly speaking, $X \subset X^{**} \subset (X)_{\mathscr{U}}$, and so we obtain extensions from X to its bidual as byproduct of our extension process. There is not one, but several ultrapower extension processes. One of these processes is modelled on the Aron-Berner method, and in this case we extend the scope of the result of Davie and Gamelin mentioned above. The other extension process which we discuss is more adaptable for dealing with holomorphic functions.

Our methods yield new results concerning the *polarization constants* of a Banach space. The polarization constants of X are a sequence of real numbers $K_n(X)$ which contain information about the geometric structure of X. The number $K_n(X)$ arises when one compares the norm of a homogeneous poly-

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nomial of degree n on X with the norm of the symmetric n-linear function which generates the polynomial. We show that the bidual X^{**} has the same polarization constants as X, at least when the bidual has the metric approximation property.

We begin with some notation and terminology, and we recall some of the pertinent facts concerning ultraproducts which will be required.

The spaces considered will be Banach spaces over either the real or the complex field. The notation $B_r(x)$ represents the open ball with centre x and radius r.

We now look briefly at some of the basic properties of ultraproducts. We refer the reader to [He] and [Si] for further details. Let $\mathscr{U}$ be an ultrafilter on a set I. If the limit with respect to $\mathscr{U}$ of a family $\{a_i\colon i \in I\}$ exists, we denote it by $\lim_{i,\mathscr{U}} a_i$, or simply by $\lim_{\mathscr{U}} a_i$ if the role of the index i is unambiguous. Let $\{X_i\colon i \in I\}$ be a family of Banach spaces indexed by I. The ultraproduct $(X_i)_{\mathscr{U}}$ consists of elements of the form $x = (x_i)_{\mathscr{U}}$, with $x_i \in X_i$, for each $i \in I$, where the norms of x_i are bounded, and where $(x_i)_{\mathscr{U}} = (y_i)_{\mathscr{U}}$ if $\lim_{\mathscr{U}} \|x_i - y_i\| = 0$. The norm of $(X_i)_{\mathscr{U}}$ is given by

$$\|(x_i)_{\mathscr{U}}\| = \lim_{\mathscr{U}} \|x_i\|,$$

We may consider X as a subspce of the ultrapower $(X)_{\mathscr{U}}$ by means of the canonical embedding $x \mapsto (x_i)_{\mathscr{U}}$ where $x_i = x$ for every i.

There is one particular construction of an ultrapower of X which will be important for our purposes, and so we shall describe it in some detail. First, we recall the Principle of Local Reflexivity [De]: Let M and N be finite dimensional subspaces of X^{**} and X^* respectively, and let $\varepsilon > 0$. Then there exists a linear mapping $T\colon M \to X$ such that:

(i) T is an ε-isometry; that is, $(1-\varepsilon)\|x^{**}\| \leqq \|T(x^{**})\| \leqq (1+\varepsilon)\|x^{**}\|$ for every $x^{**} \in M$;

(ii) $T(x^{**}) = x^{**}$ for every $x^{**} \in M \cap X$;

(iii) $x^*(T(x^{**})) = x^{**}(x^*)$ for every $x^{**} \in M$, $x^* \in N$.

Now let the indexing set I consist of all triples $i = (M_i, N_i, \varepsilon_i)$, where M_i and N_i are finite dimensional subspaces of X^{**} and X^* respectivity and $\varepsilon_i > 0$. For each i choose an ε_i-isometry $T_i\colon M_i \to X$ in accordance with Local Reflexivity, so that $x^*(T_i(x^{**})) = x^{**}(x^*)$ for every $x^{**} \in M_i, x^* \in N_i$, and $T_i(x^{**}) = x^{**}$ for every $x^{**} \in M_i \cap X$. We define an ordering on I by setting $i < j$ if $M_i \subset M_j$, $N_i \subset N_j$ and $\varepsilon_i > \varepsilon_j$. Then the collection of sets of the form $B_i = \{j \in I\colon i \leqq j\}$ is a filterbase. Let $\mathscr{U}$ be an ultrafilter on I which contains this filterbase. Then the canonical embedding of X into the ultrapower $(X)_{\mathscr{U}}$ extends to a canonical embedding $J\colon X^{**} \to (X)_{\mathscr{U}}$; furthermore, $J(X^{**})$ is the range of a norm one projection on $(X)_{\mathscr{U}}$. The embedding J is defined by

$$J(x^{**}) = (x_i)_u, \quad \text{where} \quad x_i = \begin{cases} T_i(x^{**}) & \text{if} \quad x^{**} \in M_i, \\ 0 & \text{if} \quad x^{**} \notin M_i. \end{cases}$$

The projection pr: $(X)_u \to J(X^{**})$ is given by $\mathrm{pr}((x_i)_u) = J(w^* - \lim_u x_i)$, where the limit taken here is the weak*-limit in X^{**} of the family (x_i). We shall refer to a filter of the type constructed here as a *local ultrafilter for* X, and we shall call $(X)_u$ a *local ultrapower of* X.

We shall also make use of the Local Duality of Ultraproducts. To state this principle, let us first recall that there is a canonical embedding J of $(X^*)_u$ into $(X)_u^*$, given by

$$J((x_i^*)_u)(x_i)_u = \lim_u x_i^*(x_i).$$

Now let M and N be finite dimensional subspaces of $(X)_u^*$ and $(X)_u$ respectively, and let $\varepsilon > 0$. Then there exists an ε-isometry T: $M \to (X_u^*)_u$ such that:
(i) $JT(\varphi)x = \varphi(x)$ for every $\varphi \in M$, $x \in N$;
(ii) $JT(\varphi) = \varphi$ for every $\varphi \in M \cap J((X^*)_u)$.

Next, we look at polynomials and holomorphic functions on Banach spaces. We refer to [Ch] or [Di] for a full account. Let X be a Banach space over the real or the complex field. A scalar-valued function P on X is a continuous n-homogeneous polynomial if there exists a continuous symmetric n-linear function A on X such that $P(x) = A(x, \ldots, x)$ for every $x \in X$. The function A is uniquely determined by P; indeed, we have the *Polarization Identity* which gives A in terms of P:

$$A(x_1, \ldots, x_n) = \frac{1}{2^n n!} \sum_{\varepsilon_1, \ldots, \varepsilon_n = \pm 1} \varepsilon_1 \ldots \varepsilon_n P(\varepsilon_1 x_1 + \ldots + \varepsilon_n x_n).$$

Let $\mathscr{L}(^nX)$ denote the space of all continuous n-linear functions from X^n into the scalar field normed by $A \mapsto \sup\{|A(x_1, \ldots, x_n)|: \|x_k\| \leqq 1, 1 \leqq k \leqq n\}$. The Banach space of continuous n-homogeneous polynomials on X is denoted by $\mathscr{P}(^nX)$. The norm on $\mathscr{P}(^nX)$ is given by $\|P\| = \sup\{|P(x)|: \|x\| \leqq 1\}$. Let $\mathscr{P}_f(^nX)$ denote the space of continuous n-homogeneous polynomials of finite-type on X, that is, polynomials of the form $x \mapsto \sum_{i=1}^k (x_i^*)^n$, where $x_i^* \in X^*$, $1 \leqq i \leqq k$.

Finally, we mention holomorphic functions. Let X be a complex Banach space and let Ω be an open subset of X. A function f: $\Omega \to \mathbf{C}$ is *holomorphic* if f has a Taylor series expansion at each point; in other words, for each point z of Ω there exists a sequence of polynomials $P_n \in \mathscr{P}(^nX)$ such that the series $\sum_{n=0}^{\infty} P_n(x - z)$ converges to $f(x)$ uniformly for x in some neighbourhood of z. This is equivalent to the existence of a Fréchet derivative for f at each point of Ω. If f is a holomorphic function from X into $\mathbf{C}$, and $\sum_{n=0}^{\infty} P_n$ is the Taylor series of f at the origin then $\sum_{n=0}^{\infty} P_n(x)$ converges to $f(x)$ for every $x \in X$. However, if X is infinite dimensional, then this series need not converge uniformly on every ball. This is related to the fact that f need not be bounded on the ball $B_r(0)$ for large r. Denoting by R_z the radius of uniform convergence of a power series $\sum_{n=0}^{\infty} P_n(x-z)$ about the point $z \in X$, we have

$$R_z = (\limsup_n \|P_n\|^{\frac{1}{n}})^{-1}.$$

The space of holomorphic functions from Ω into $\mathbf{C}$ is denoted $\mathscr{H}(\Omega)$. The space $\mathscr{H}_b(\Omega)$ consists of the holomorphic functions from Ω into $\mathbf{C}$ of *bounded type*, that is, functions which are bounded on Ω-bounded sets. If X is infinite dimensional then $\mathscr{H}_b(X)$ is a proper subset of $\mathscr{H}(X)$.

2. Extension of Polynomials and Holomorphic Functions.

In this section we investigate different ways of extending polynomials and holomorphic functions from a Banach space into an ultrapower. We consider two approaches. One is a generalization of the Aron-Berner process. The other is more convenient for dealing with extensions of holomorphic functions.

Let $(X)_u$ be an ultrapower of a Banach space X. For $A \in L(^nX)$ we define an n-linear function $\hat{A}$ on $(X)_u$ by

$$\hat{A}(x_1,\dots,x_n) = \lim_{i_1,u} \dots \lim_{i_n,u} A(x_{i_1}^{(1)},\dots,x_{i_n}^{(n)})$$

for $x_k = (x_i^{(k)})_u \in (X)_u$. It is easy to see that $\hat{A}$ is well-defined, that $\hat{A}$ is an extension of A and that $\|\hat{A}\| = \|A\|$. However, if A is symmetric, it does not necessary follow that $\hat{A}$ is symmetric. Taking the iterated limits in a different order may lead to different values for the extension.

The Aron-Berner extension of a continuous n-linear function A on X is obtained by extending A to an n-linear function $\bar{A}$ on X^{**} by weak-star continuity, one variable at a time, from last to first.

Now let $P \in \mathscr{P}(^nX)$, and let A be the continuous symmetric n-linear function that generates P. We obtain an extension, $\hat{P}$, of P to $(X)_u$ by defining

$$\hat{P}((x_i)_u) = \hat{A}((x_i)_u)^n = \lim_{i_1,u} \dots \lim_{i_n,u} A(x_{i_1}^{(1)},\dots,x_{i_n}^{(n)})$$

for every $(x_i)_u \in (X)_u$.

Proposition 2.1. (a) *Let P be a continuous n-homogeneous polynomial on X. Then the extension, $\hat{P}$, of P to $(X)_u$ has the property that $\|P\| = \|\hat{P}\|$.*

(b) *If $(X)_u$ is a local ultrapower of X, then the restriction of $\hat{P}$ to the canonical image of X^{**} in $(X)_u$ coincides with the Aron-Berner extension of P to X^{**}.*

Proof. Let $z = (x_i)_u \in (X)_u$ with $\|z\| \leqq 1$, choosing x_i so that $\|x_i\| \leqq 1$ for every

$i \in I$. Let $\varepsilon > 0$. Then, proceeding exactly as in the proof of Theorem 1 in [DG], we obtain $N \in \mathbb{N}$ and $i_1, \ldots, i_N \in I$ such that

$$|\hat{P}(z) - P(x)| < \varepsilon,$$

where $x = (x_{i_1} + \ldots + x_{i_N})/N$. It follows that $\|\hat{P}\| = \|P\|$. This concludes the proof of part (a).

Now, let $(X)_{\mathcal{U}}$ be a local ultrapower of X. For simplicity we consider the case $n = 2$.

Let $x^{**}, y^{**} \in X^{**}$ and let $x^{**} = (x_i)_{\mathcal{U}}$ and $y^{**} = (y_i)_{\mathcal{U}}$ be the canonical images of x^{**} and y^{**} in $(X)_{\mathcal{U}}$, where x_i, y_i are chosen as described in the introduction. Let A be the continuous symmetric bilinear function associated with P and let $\bar{A}$ be the Aron-Berner extension of A. Then

$$\hat{A}(x^{**}, y^{**}) = \lim_{i,\mathcal{U}} \lim_{j,\mathcal{U}} A(x_i, y_j).$$

First, fixing x_i,

$$\lim_{j,\mathcal{U}} A(x_i, y_j) = \lim_{j,\mathcal{U}} A(x_i, T_j y^{**}).$$

If the functional $y^*\colon x \mapsto A(x_i, x)$ belongs to N_j, then we have $y^*(T_j y^{**}) = y^{**}(y^*)$, that is,

$$A(x_i, T_j y^{**}) = \bar{A}(x_i, y^{**}),$$

and it follows that

$$\lim_{j,\mathcal{U}} A(x_i^{\,i}, T_j y^{**}) = \bar{A}(x_i, y^{**}).$$

Hence,

$$\hat{A}(x^{**}, y^{**}) = \lim_{i,\mathcal{U}} \bar{A}(x_i, y^{**}),$$

and it follows in the same way that

$$\lim_{i,\mathcal{U}} \bar{A}(x_i, y^{**}) = \bar{A}(x^{**}, y^{**}).$$

This proves (b).

The disadvantage of this extension is that symmetry might be lost; if A is a continuous symmetric n-linear function, there is no guarantee that the extended function $\hat{A}$ is also symmetric. Aron, Cole and Gamelin [ACG] point out that if symmetry is preserved in the case of bilinear functions, then it is preserved for every n-linear function. Furthermore, they show [ACG, Theorem 8.3] that the Aron-Berner extension of every continuous symmetric bilinear function is symmetric if and only if every continuous symmetric linear operator from X into X^* is weakly compact. We have an analogous result for the ultrapower extension.

PROPOSITION 2.2. *For every ultrafilter $\mathcal{U}$ and every continuous symmetric bilinear function A on X, the ultrapower extension $\hat{A}$ on $(X)_{\mathcal{U}}$ is symmetric if and only if every continuous symmetric linear operator from X into X^* is weakly compact.*

PROOF. If $\hat{A}$ is always symmetric, then, taking $\mathcal{U}$ to be the local ultrafilter for X, it follows from the proof of Proposition 2.1 that the Aron-Berner extension preserves symmetry and hence by the result quoted above, every continuous symmetric linear operator from X into X^* is weakly compact. Conversely, suppose that every continuous symmetric linear operator from X into X^* is weakly compact. Let A be a continuous symmetric bilinear function on X, and let $(X)_{\mathcal{U}}$ be an ultrapower of X. Let $(x_i)_{\mathcal{U}}$, $(y_i)_{\mathcal{U}}$ be two elements of $(X)_{\mathcal{U}}$. Since the operator $x \in X \mapsto A(x, \cdot) \in X^*$ is weakly compact, it follows that the family $(A(x_i, \cdot))$ has a weak limit point with respect to $\mathcal{U}$. So we may let

$$y^* = w\text{-}\lim_{i,\mathcal{U}} A(x_i, \cdot).$$

Therefore

$$\lim_{j,\mathcal{U}} \lim_{i,\mathcal{U}} A(x_i, y_j) = \lim_{j,\mathcal{U}} y^*(y_j) = y^{**}(y^*),$$

where y^{**} is the weak*-limit with respect to $\mathcal{U}$ of the family (y_i). Taking the iterated limits in the reverse order, we have

$$\lim_{i,\mathcal{U}} \lim_{j,\mathcal{U}} A(x_i, y_j) = \lim_{i,\mathcal{U}} y^{**}(A(x_i, \cdot)) = y^{**}(y^*).$$

Thus the iterated limits have the same value.

If we modify the extension process slightly, we obtain an extension which preserves the symmetry for every space. We define

$$\tilde{A}(x_1, \ldots, x_n) = \lim_{\mathcal{U}} A(x_i^{(1)}, \ldots, x_i^{(n)}),$$

that is, we take a joint limit instead of an iterated limit. It is easy to see that $\tilde{A}$ is a continuous symmetric n-linear function on $(X)_{\mathcal{U}}$ which extends A and that $\|\tilde{A}\| = \|A\|$. Let P be the polynomial generated by A. If we denote by $\tilde{P}$ the polynomial generated by $\tilde{A}$, then

$$\tilde{P}((x_i)_{\mathcal{U}}) = \lim_{\mathcal{U}} P(x_i).$$

$\tilde{P}$ is an extension on P and it is trivial that $\|\tilde{P}\| = \|P\|$. We summarize this in the following proposition.

PROPOSITION 2.3. (a) *Let A be a continuous symmetric n-linear function on X. Then A has a continuous symmetric n-linear extension $\tilde{A}$, to $(X)_{\mathscr{U}}$ with the same norm, defined by*

$$\tilde{A}(x_1,\ldots,x_n) = \lim_{\mathscr{U}} A(x_i^{(1)},\ldots,x_i^{(n)})$$

for every $x_k = (x_i^{(k)})_{\mathscr{U}} \in (X)_{\mathscr{U}}$, $1 \leqq k \leqq n$.

(b) *Let P be a continuous n-homogeneous polynomial on X. Then P extends to a continuous n-homogeneous polynomial $\tilde{P}$ on $(X)_{\mathscr{U}}$ satisfying*

$$\tilde{P}((x_i)_{\mathscr{U}}) = \lim_{\mathscr{U}} P(x_i)$$

and we have $\|P\| = \|\tilde{P}\|$.

The following example (cf. [ACG]) shows that the iterated extension and the symmetric extension can give different polynomials on the ultrapower.

EXAMPLE. Let $A: l_1 \times l_1 \to \mathbf{C}$ be given by

$$A(x,y) = [x_1y_2 + (x_1 + x_2 + x_3)y_4 + \ldots] + [y_1x_2 + (y_1 + y_2 + y_3)x_4 + \ldots].$$

Then A is a continuous, symmetric bilinear function on l_1 such that

$$\lim_{n,\mathscr{U}} \lim_{m,\mathscr{U}} A(e_{2n}, e_{2m}) = 1 \quad \text{and} \quad \lim_{m,\mathscr{U}} A(e_{2m}, e_{2m}) = 0,$$

where $\mathscr{U}$ is any countably incomplete ultrafilter on $\mathbf{N}$. Now let $\mathscr{U}$ be any countably incomplete ultrafilter on a set I. (For example, it is easy to see that local ultrafilters are countably incomplete.) Thus, $\mathscr{U}$ contains a decreasing sequence of elements $U_1, U_2, \ldots$, whose intersection is empty. Now we define the family (x_i) as follows: $x_i = 0$ if i is not in U_1, and $x_i = e_{2n}$ if i is in U_n but not in U_{n+1}. Let x denote the element of the ultrapower $(l_1)_{\mathscr{U}}$ corresponding to this family. Then

$$\lim_{i,\mathscr{U}} \lim_{j,\mathscr{U}} A(x_i, x_j) = 1 \quad \text{but} \quad \lim_{i,\mathscr{U}} A(x_i, x_i) = 0,$$

and so $\hat{P}(x)$ is not equal to $\tilde{P}(x)$.

We now look at the interaction between differentiation and extension. First we note that there is a canonical embedding J of $(\mathscr{P}(^kX))_{\mathscr{U}}$ into $\mathscr{P}(^k(X)_{\mathscr{U}})$ given by

$$J((P_i)_{\mathscr{U}})(x_i)_{\mathscr{U}} = \lim_{\mathscr{U}} P_i(x_i).$$

Thus we may identify $(\mathscr{P}(^kX))_{\mathscr{U}}$ with a subspace of $\mathscr{P}(^k(X)_{\mathscr{U}})$.

Let $P \in \mathscr{P}(^nX)$ and let A be the continuous symmetric n-linear function that generates P. We wish to compute the kth derivative of $\tilde{P}$ at a point $(x_i)_{\mathscr{U}}$ in $(X)_{\mathscr{U}}$. We have

$$d^k\tilde{P}((x_i)_u)((y_i)_u) = \frac{n!}{(n-k)!}\tilde{A}((x_i)_u)^{n-k}((y_i)_u)^k$$

$$= \lim_u \frac{n!}{(n-k)!} A(x_i)^{n-k}(y_i)^k = \lim_u d^kP(x_i)(y_i).$$

Using the above-mentioned identification, we may state this result as follows

PROPOSITION 2.4. *Let $P \in \mathscr{P}(^nX)$ and let $(x_i) \in (X)_u$. Then*

$$d^k\tilde{P}((x_i)_u) = (d^kP(x_i))_u \quad \textit{for every} \quad k, 1 \leqq k \leqq n.$$

Next, we focus our attention on extending holomorphic functions. The problem is to find a holomorphic Hahn-Banach theorem. It is known that this is not possible in general; even in the case $X = c_0$ and $X^{**} = l^\infty$ there exists a holomorphic function $f\colon X \to \mathbf{C}$ which cannot be holomorphically extended to X^{**}. In [AB] Aron and Berner have extended holomorphic functions on a Banach space X to an open subset of the bidual containing X. The Aron-Berner method is first to extend polynomials and then to use local Taylor series representations to extend holomorphic functions locally. They had to show that their extensions were coherent in the "overlaps" by examining how their extension process for polynomials interacted with the process of taking derivatives. We have a much simpler way of extending f, because we can get around the coherence problem by the fact that the extension $\tilde{f}$ satisfies $\tilde{f}((x_i)_u) = \lim_u f(x_i)$ at every point in its domain. Therefore $\tilde{f}$ is uniquely defined.

PROPOSITION 2.5. *Let $f \in \mathscr{H}(X)$. Then there exists a connected open subset $\mathcal{O}_f$ of $(X)_u$ containing X, and a holomorphic function $\tilde{f}$ on $\mathcal{O}_f$ which is an extension of f, and satisfies*

$$\tilde{f}((x_i)_u) = \lim_u f(x_i) \quad \textit{for every} \quad (x_i)_u \in \mathcal{O}_f.$$

PROOF. Let $z \in X$ and let R_z denote the radius of uniform convergence of the Taylor series $\sum_{n=0}^\infty P_n$ of f at z. Then f is bounded on any closed ball $\bar{B}_r(z)$ of radius $r < R_z$. Since $\|P_n\| = \|\tilde{P}_n\|$, the series $\sum_{n=0}^\infty \tilde{P}_n$ on $(X)_u$ has the same radius of uniform convergence as the series $\sum_{n=0}^\infty P_n$. The function $\tilde{f}_z\colon B_{R_z}(z) \to \mathbf{C}$, defined by $\tilde{f}_z((x_i)_u) = \sum_{n=0}^\infty \tilde{P}_n((x_i)_u - z)$, is an element of $\mathscr{H}_b(B_{R_z}(z))$.

Let $r < R_z$ be fixed. Now, if $x = (x_i)_u \in (X)_u$ and $\|x - z\| \leqq r$, then we may assume that $\|x_i - z\| \leqq r$ for every $i \in I$. Hence $\lim_u f(x_i)$ exists, since f is bounded on $\bar{B}_r(z)$. Further, for given $\varepsilon > 0$, there exists an $N \in \mathbf{N}$ such that

$$\left| f(x_i) - \sum_{n=0}^m P_n(x_i - z) \right| < \varepsilon \quad \text{for every} \quad m > N \quad \text{and every} \quad i \in I.$$

Thus, $\lim_{\mathscr{U}} f(x_i) = \sum_{n=0}^{\infty} \tilde{P}_n((x_i)_{\mathscr{U}} - z)$ uniformly in $\bar{B}_r(z)$, and it follows that $\tilde{f}_z((x_i)_{\mathscr{U}}) = \lim_{\mathscr{U}} f(x_i)$ for every $(x_i)_{\mathscr{U}} \in B_{R_z}(z)$.

Now, let $\mathcal{O}_f = \cup_{z \in X} B_{R_z}(z)$. Then $\mathcal{O}_f$ is a connected open subset of $(X)_{\mathscr{U}}$ containing X and it is clear from the above that we may define a holomorphic function $\tilde{f}\colon \mathcal{O}_f \to \mathrm{C}$, by $\tilde{f}|_{B_{R_z}(z)} = \tilde{f}_z$, and that $\tilde{f}$ is an extension of f.

If we now apply this result to the case where $\mathscr{U}$ is a local ultrafilter for X, we get an alternative to the Aron-Berner extension.

COROLLARY 2.6. *Let $f \in \mathscr{H}(X)$. Then there exists a connected open subset Ω of X^{**} containing X, and a holomorphic function $\tilde{f}$ on Ω which is an extension of f.*

For holomorphic functions of bounded type, the situation concerning extensions is particularly straightforward:

PROPOSITION 2.7. *Let $f \in \mathscr{H}_b(X)$. Then there exists a holomorphic function of bounded type $\tilde{f}$ on $(X)_{\mathscr{U}}$ which is an extension of f, and satisfies*

$$\tilde{f}((x_i)_{\mathscr{U}}) = \lim_{\mathscr{U}} f(x_i) \quad \textit{for every} \quad (x_i)_{\mathscr{U}} \in (X)_{\mathscr{U}}.$$

This follows from the proof of Proposition 2.5, since the radius of uniform convergence at the origin is infinite.

For derivatives of functions of bounded type, we have a result similar to Proposition 2.4.

PROPOSITION 2.8. *Let $f \in \mathscr{H}_b(X)$ and let $(x_i)_{\mathscr{U}} \in (X)_{\mathscr{U}}$. Then*

$$\hat{d}^k \tilde{f}((x_i)_{\mathscr{U}}) = (\hat{d}^k f(x_i))_{\mathscr{U}} \quad \textit{for every} \quad k \geqq 1.$$

PROOF. Let $\sum_{n=0}^{\infty} P_n$ be the Taylor series expansion of f at the origin. Then the Taylor series expansion of the function $\hat{d}^k \tilde{f}$ is $\sum_{n=k}^{\infty} \hat{d}^k \tilde{P}_n$, and so

$$\hat{d}^k \tilde{f}((x_i)_{\mathscr{U}})((y_i)_{\mathscr{U}}) = \sum_{n=k}^{\infty} \hat{d}^k \tilde{P}_n((x_i)_{\mathscr{U}})((y_i)_{\mathscr{U}})$$

$$= \sum_{n=k}^{\infty} \lim_{\mathscr{U}} \hat{d}^k P_n(x_i)(y_i) = \lim_{\mathscr{U}} \sum_{n=k}^{\infty} \hat{d}^k P_n(x_i)(y_i) = \lim_{\mathscr{U}} \hat{d}^k f(x_i)(y_i),$$

since the series $\sum_{n=k}^{\infty} |\hat{d}^k P_n(x_i)(y_i)|$ converges uniformly in $i \in I$.

REMARK. Let f be a holomorphic function on X, not necessarily of bounded type. Then, by applying a similar argument to the above to the Taylor series expansion of f at each point of X, it follows that there exists a connected open subset Ω_f of $\mathcal{O}_f$ containing X, such that

$$\hat{d}^k \tilde{f}((x_i)_{\mathscr{U}}) = (\hat{d}^k f(x_i))_{\mathscr{U}}$$

for every $(x_i)_{\mathscr{U}} \in \Omega_f$ and every $k \geqq 1$.

While writing this paper it has came to our attention that S. Dineen and R. Timoney [DT] have proved results similar to some of the results in this section, using an ultrapower method.

3. Polarization Constants of a Banach space.

Let X be a Banach space over the real or complex field. Let P be a continuous n-homogeneous polynomial on X, and let A be the continuous symmetric n-linear function on X which generates P. It follows from the Polarization Identity that

$$\|P\| \leqq \|A\| \leqq \frac{n^n}{n!}\|P\|.$$

The constant $n^n/n!$ which appears here is the best possible in general. In the case $X = l_1$ this constant is achieved for every n. On the other hand, when X is a Hilbert space, we have $\|A\| = \|P\|$ for every P.

We define the *Polarization Constants* of a Banach space X to be the smallest numbers $K_n(X)$ satisfying

$$\|P\| \leqq \|A\| \leqq K_n(X)\|P\|,$$

for every $P \in \mathscr{P}(^nX)$. Thus we have

$$1 \leqq K_n(X) \leqq \frac{n^n}{n!} \quad \text{for every } n.$$

In view of the Polarization Identity, we have

$$K_n(X) = \sup\left\{\frac{1}{2^n n!}\left|\sum_{\varepsilon_1,\ldots,\varepsilon_n = \pm 1} \varepsilon_1 \ldots \varepsilon_n P(\varepsilon_1 x_1 + \ldots + \varepsilon_n x_n)\right| : \right.$$
$$\left. P \in \mathscr{P}(^nX), \|P\| \leqq 1, x_i \in X, \|x_i\| \leqq 1, 1 \leqq i \leqq n\right\}$$

There are many open questions concerning these constants. We refer to [Ha, Sa1, Sa2] for further details.

Our main results in this section are that, subject to a suitable approximation property of X, the ultrapower $(X)_u$ and the bidual X^{**} have the same polarization constants as X.

We recall that a Banach space X has the *metric approximation property* if, for every $\varepsilon > 0$ and every compact set $K \subset X$, there is a finite rank operator $S: X \to X$ such that $\|S\| \leqq 1$ and $\|Sx - x\|_K < \varepsilon$.

The following lemma is well-known, but we give a proof for the sake of completeness.

LEMMA 3.1. *Let X have the metric approximation property, and let $P \in \mathscr{P}({}^nX)$. Then for every compact subset K of X and every $\varepsilon > 0$, there exists a $Q \in \mathscr{P}_f({}^nX)$ such that*

$$\|P - Q\|_K < \varepsilon \quad \textit{and} \quad \|Q\| \leqq \|P\|.$$

PROOF. Since P is uniformly continuous on K, there exists a $\delta > 0$ such that $|P(x) - P(y)| < \varepsilon$ for all $x \in K$ and $y \in X$ with $\|x - y\| < \delta$. Since X has the metric approximation property, there is a finite rank operator S: $X \to X$ with $\|S\| \leqq 1$ such that $\|Sx - x\| < \delta$ for each $x \in K$. Then $Q = P \circ S \in \mathscr{P}_f({}^nX)$ and $\|Q\| \leqq \|P\|$, and if $x \in K$, it follows that $|P(x) - Q(x)| < \varepsilon$.

A Banach space X has the $1+$ *uniform approximation property* if the following holds: For each natural number n there is an $m(n)$ such that, given an n-dimensional subspace $M \subset X$ and $\varepsilon > 0$, there exists an operator $T \in \mathscr{L}(X, X)$ such that $T|_M = \mathrm{id}_M$, $\operatorname{rank}(T) \leqq m(n)$ and $\|T\| \leqq 1 + \varepsilon$. If each of the Banach spaces X_i, $i \in I$, has the $1+$ uniform approximation property, and $\mathscr{U}$ is an ultrafilter on I, then the ultraproduct $(X_i)_{\mathscr{U}}$ has the metric approximation property [He].

Since every continuous n-homogeneous polynomial on X extends to a continuous n-homogeneous polynomial on $(X)_{\mathscr{U}}$ with the same norm, it follows that

$$K_n(X) \leqq K_n((X)_{\mathscr{U}}) \quad \text{for every } n.$$

Our aim now is to establish the reverse inequality.

THEOREM 3.2. *If each of the Banach spaces X_i has the $1+$ uniform approximation property, then the polarization constants of the ultraproduct $(X_i)_{\mathscr{U}}$ are given by:*

$$K_n((X_i)_{\mathscr{U}}) = \lim_{\mathscr{U}} K_n(X_i) \quad \textit{for every } n.$$

PROOF. Let $\eta > 0$. Choose $P \in \mathscr{P}({}^n(X_i)_{\mathscr{U}})$, $\|P\| = 1$, $x_k = (x_i^{(k)})_{\mathscr{U}} \in (X_i)_{\mathscr{U}}$, $\|x_k\| \leqq 1$ for $k = 1, \dots, n$, such that

$$K_n((X_i)_{\mathscr{U}}) \leqq \frac{1}{2^n n!}\left|\sum_{\varepsilon_1, \dots, \varepsilon_n = \pm 1} \varepsilon_1 \dots \varepsilon_n P(\varepsilon_1 x_1 + \dots + \varepsilon_n x_n)\right| + \eta.$$

Since $(X_i)_{\mathscr{U}}$ has the metric approximation property we get by Lemma 3.1 that there is a $Q \in \mathscr{P}_f({}^n(X_i)_{\mathscr{U}})$ such that $\|Q\| \leqq \|P\| = 1$, for which

$$\frac{1}{2^n n!}\left|\sum_{\varepsilon_1, \dots, \varepsilon_n = \pm 1} \varepsilon_1 \dots \varepsilon_n P(\varepsilon_1 x_1 + \dots + \varepsilon_n x_n)\right|$$

$$\leqq \frac{1}{2^n n!}\left|\sum_{\varepsilon_1, \dots, \varepsilon_n = \pm 1} \varepsilon_1 \dots \varepsilon_n Q(\varepsilon_1 x_1 + \dots + \varepsilon_n x_n)\right| + \eta$$

(i.e. we take $K = \{\varepsilon_1 x_1 + \ldots + \varepsilon_n x_n : \varepsilon_1, \ldots, \varepsilon_n = \pm 1\}$). We now have,

$$K_n((X_i)_u) \leqq \frac{1}{2^n n!} \left| \sum_{\varepsilon_1, \ldots, \varepsilon_n = \pm 1} \varepsilon_1 \ldots \varepsilon_n Q(\varepsilon_1 x_1 + \ldots + \varepsilon_n x_n) \right| + 2\eta,$$

where $Q = \sum_{j=1}^m \Psi_j^n$ for some $\Psi_1, \ldots, \Psi_m \in (X_i)_u^*$ and $\|Q\| \leqq 1$.

Let now $M = \operatorname{span}\{\Psi_1, \ldots, \Psi_m\} \subset (X_i)_u^*$, and $N = \operatorname{span}\{x_1, \ldots, x_n\} \subset (X_i)_u$. Then by the local duality of ultraproducts, given $\varepsilon > 0$, there is an ε-isometry $T: M \to (X_i^*)_u$ such that

$$JT(\Psi)x = \Psi(x) \quad \text{for all} \quad \Psi \in M \quad \text{and} \quad x \in N,$$

where J is the canonical embedding of $(X_i^*)_u$ into $(X_i)_u^*$. Let $x_j^* = JT(\Psi_j) \in (X_i)_u^*$, and let $\bar{Q} \in \mathscr{P}_f(^n(X_i)_u)$ be given by

$$\bar{Q}(x) = \sum_{j=1}^m x_j^*(x)^n,$$

i.e. $\bar{Q} = \sum_{j=1}^m (x_j^*)^n$. Then $\bar{Q}(\varepsilon_1 x_1 + \ldots + \varepsilon_n x_n) = Q(\varepsilon_1 x_1 + \ldots + \varepsilon_n x_n)$.

We now estimate the norm $\|\bar{Q}\|_{\mathscr{P}(^n(X_i)_u)}$: Let $x = (x_i)_u \in (X_i)_u$ with $\|x\| \leqq 1$. Then $|\bar{Q}(x)| = |\sum_{j=1}^m JT(\Psi_j)(x)^n| = |\sum_{j=1}^m (JT)^t \hat{x}(\Psi_j)^n|$, where $\hat{}:(X_i)_u \to (X_i)_u^{**}$ is the canonical mapping. Since $\|(JT)^t\| = \|JT\| \leqq 1 + \varepsilon$ and $M^* = (X_i)_u^{**}/M^\perp$, we can choose $z^{**} \in (X_i)_u^{**}$ with $\|z^{**}\| \leqq 1 + 2\varepsilon$, such that $|\sum_{j=1}^m (JT)^t \hat{x}(\Psi_j)^n| = |\sum_{j=1}^m z^{**}(\Psi_j)^n|$. By Goldstine's theorem there exists a net $z_\alpha \in (X_i)_u$ such that $z_\alpha \to z^{**}$ in $\sigma((X_i)_u^{**}, (X_i)_u^*)$ and $\|z_\alpha\| = \|z^{**}\|$ for each α. Hence we get that $|\bar{Q}(x)| \leqq (1 + 2\varepsilon)^n$.

To summarize, we have

$$K_n((X_i)_u) \leqq \frac{1}{2^n n!} \left| \sum_{\varepsilon_1, \ldots, \varepsilon_n = \pm 1} \varepsilon_1 \ldots \varepsilon_n \bar{Q}(\varepsilon_1 x_1 + \ldots + \varepsilon_n x_n) \right| + 2\eta,$$

where $\bar{Q} = \sum_{j=1}^m (x_j^*)^n$, and $\|\bar{Q}\|_{\mathscr{P}(^n(X_i)_u)} \leqq (1 + 2\varepsilon)^n$.

Since $x_j^* \in J((X_i^*)_u \mid c(X_i)_u^*$, we get with $x_j^* = (x_{ji}^*)_u$, that $x_j^*(x_i)_u = \lim_{i,u} x_{ji}^*(x_i)$. Hence

$$\bar{Q}(x) = \lim_u \bar{Q}_i(x_i),$$

where $\bar{Q}_i = \sum_{j=1}^m (x_{ji}^*)^n \in \mathscr{P}_f(^n X_i)$.

Let us now turn to the norm $\|\bar{Q}_i\|_{\mathscr{P}(^n X_i)}$: For each $i \in I$ choose $x_i \in X_i$, $\|x_i\| \leqq 1$, such that $\|\bar{Q}_i\| - \varepsilon \leqq |\bar{Q}_i(x_i)|$. Let $x = (x_i)_u$. Then $\|x\| \leqq 1$, and $\|\bar{Q}\| \geqq |\bar{Q}(x)| = \lim_u |\bar{Q}_i(x_i)| \geqq \lim_u \|\bar{Q}_i\| - \varepsilon$, so $\lim_u \|\bar{Q}_i\| \leqq \|\bar{Q}\| + \varepsilon$.

Thus,

$$K_n((X_i)_u) \leqq \lim_u \frac{1}{2^n n!} \left| \sum_{\varepsilon_1, \ldots, \varepsilon_n = \pm 1} \varepsilon_1 \ldots \varepsilon_n \bar{Q}_i(\varepsilon_1 x_i^{(1)} + \ldots + \varepsilon_n x_i^{(n)}) \right| + 2\eta,$$

where $x_k = (x_i^{(k)})_{\mathscr{U}}$, $\|x_k\| \leqq 1$ for $k = 1, \ldots, n$, and $\lim_{\mathscr{U}} \|\bar{Q}_i\| \leqq \|\bar{Q}\| + \varepsilon$.
Therefore,

$$K_n((X_i)_{\mathscr{U}}) \leqq ((1 + 2\varepsilon)^n + \varepsilon) \lim_{\mathscr{U}} K_n(X_i) + 2\eta,$$

and since $\varepsilon > 0$ and $\eta > 0$ are arbitrarily small, the required inequality follows.

COROLLARY 3.3. *If X has the $1+$ uniform approximation property, then*

$$K_n((X)_{\mathscr{U}}) = K_n(X) \quad \textit{for every } n.$$

If we apply the above theorem to the case where $\mathscr{U}$ is a local ultrafilter on X, we can show that the bidual X^{**} has the same polarization constants as X, when X has the $1+$ uniform approximation property. However, by modifying the proof of Theorem 3.2, we can prove this result about the bidual for a larger class of spaces.

THEOREM 3.4. *Let X be a Banach space for which X^{**} has the metric approximation property. Then*

$$K_n(X^{**}) = K_n(X) \quad \textit{for every } n.$$

PROOF. Almost exactly as in the proof of Theorem 3.2 one can show that, for given $\eta > 0$ and $\varepsilon > 0$,

$$K_n(X^{**}) \leqq \frac{1}{2^n n!} \left| \sum_{\varepsilon_1, \ldots, \varepsilon_n = \pm 1} \varepsilon_1 \ldots \varepsilon_n \bar{Q}(\varepsilon_1 x_1 + \ldots + \varepsilon_n x_n) \right| + 2\eta,$$

where $\bar{Q} = \sum_{j=1}^m (x_j^*)^n \in \mathscr{P}(^n X^{**})$, and $\|\bar{Q}\|_{\mathscr{P}(^n X^{**})} \leqq (1 + 2\varepsilon)^n$. Instead of the local duality of ultraproducts one uses the principle of local reflexivity applied to X^*.

Now $\bar{Q}$ restricts to a polynomial on X, and an application of Goldstine's theorem shows that

$$\|\bar{Q}\|_{\mathscr{P}(^n X)} = \|\bar{Q}\|_{\mathscr{P}(^n X^{**})}.$$

We now apply the principle of local reflexivity again, this time to X. Let $M = \operatorname{span}\{x_1^{**}, \ldots, x_n^{**}\} \subset X^{**}$, where $\|x_k^{**}\| \leqq 1$ for $k = 1, \ldots, n$, and $N = \operatorname{span}\{x_1^*, \ldots, x_m^*\} \subset X^*$. Then there is an ε-isometry S: $M \to X$ such that

$$x^*(Sx^{**}) = x^{**}(x) \quad \text{for all} \quad x^{**} \in M \quad \text{and} \quad x^* \in N.$$

Let $x_j = Sx_j^{**} \in X$. Then $\|x_j\| \leqq 1 + \varepsilon$ and $\bar{Q}(\varepsilon_1 x_1^{**} + \ldots + \varepsilon_n x_n^{**}) = \bar{Q}(\varepsilon_1 x_1 + \ldots + \varepsilon_n x_n)$.
Thus,

$$K_n(X^{**}) \leqq \frac{1}{2^n n!} \left| \sum_{\varepsilon_1, \ldots, \varepsilon_n = \pm 1} \varepsilon_1 \ldots \varepsilon_n \bar{Q}_i(\varepsilon_1 x_1 + \ldots + \varepsilon_n x_n) \right| + 2\eta,$$

where $\|\bar{Q}\|_{\mathscr{P}(^nX)} \leqq (1 + 2\varepsilon)^n$ and $\|x_j\| \leqq 1 + \varepsilon$ for $j = 1,\ldots,n$.

Hence,

$$K_n(X^{**}) \leqq (1 + 2\varepsilon)^n(1 + \varepsilon)^n K_n(X) + 2\eta,$$

and the required inequality follows.

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