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REAL-VALUED DUALS OF H-CONES

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Abstract.

We prove that an H-cone is always specifically solid in its second real-valued dual. Moreover if an H-cone possesses a special unit then it is even solid and increasingly dense in its second real-valued dual.

Introduction.

The dual of an H-cone is the set of extended real-valued additive, left order continuous functions finite on a dense set. The concept of an H-cone was introduced in 1970. Since then it has been unknown how well an H-cone can be embedded into its second dual. Moreover it has been an open question if an H-cone is solid in its second dual with respect to initial or specific order. We prove the surprising result that an H-cone is always specifically solid in the real-valued second dual and under some nice conditions even solid and dense with respect to the initial order. The same results also hold for duals.

We do not assume that the reader is familiar with the theory of H-cones. The notion of an H-cone is closely connected to the theory of vector lattices (Riesz spaces). In fact, if a vector lattice E is Dedekind complete (conditionally complete) then E^+ is an H-cone. Roughly speaking an H-cone is formed from a certain type of vector space in which we consider two partial orderings called initial and specific order. A potential-theoretic model of an H-cone is the cone of positive superharmonic functions on a harmonic space.

1. Preliminaries.

First we recall the concept of an H-cone. Second we present an equivalent definition used subsequently.

Let $(S, +, \leqq)$ be a partially ordered abelian semigroup with a neutral element 0 and having the properties

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$$u \geqq 0, \tag{1.1}$$

$$u \leqq v \Leftrightarrow u + w \leqq v + w, \tag{1.2}$$

for all u and v in S. A partial order called *specific order* denoted by $\preccurlyeq$ is defined in S by

$$s \preccurlyeq t \Leftrightarrow t = s + s' \quad \text{for some } s' \in S.$$

The least upper bound and the greatest lower bound of a subset E of S is denoted by $\vee E$ and $\wedge E$, respectively.

A structure $(S, +, \leqq)$ satisfying (1.1) and (1.2) is called an *ordered convex cone* if it admits an operation of multiplication by positive real numbers such that

$$\alpha(x + y) = \alpha x + \alpha y, \qquad (\alpha + \beta)x = \alpha x + \beta x,$$

$$(\alpha\beta)x = \alpha(\beta x), \qquad 1x = x, \qquad x \leqq y \Rightarrow \alpha x \leqq \alpha y,$$

for all $\alpha, \beta \in \mathrm{R}^+$ and $x, y \in S$.

DEFINITION 1.1. An ordered convex cone $(S, +, \leqq)$ is an *H-cone* if the following axioms hold:

(H_1) any nonempty upward directed family $F \subset S$ has a least upper bound satisfying $\bigvee(x + F) = x + \bigvee F$ for all $x \in S$,

(H_2) any nonempty family $F \subset S$ has a greatest lower bound satisfying $\bigwedge(x + F) = x + \bigwedge F$ for all $x \in S$,

(H_3) (Dominated decomposition property) for any $u, v_1, v_2 \in S$ with $u \leqq v_1 + v_2$ there exist $u_1, u_2 \in S$ such that $u = u_1 + u_2$, $u_1 \leqq v_1$ and $u_2 \leqq v_2$.

The concept of an H-cone was introduced by Boboc and Cornea in [2, 3]. It is further developed by Boboc, Bucur and Cornea (see [4] and other references there).

DEFINITION 1.2. Let E be an ordered vector space and S be a convex subcone of E such that $S \subset E^+$ and $E = S - S$. The cone S is called an *H-cone* in E if it possesses the following properties:

(A_1) any nonempty upward directed and dominated subset F of S has a least upper bound in E and $\vee F \in S$,

(A_2) any nonempty subset F of S has a greatest lower bound in E,

(A_3) for any s and t in S the greatest lower bound of the set $\{u \in S \mid s - t \leqq u\}$ denoted by $R(s - t)$ satisfies $R(s - t) \in S$ and $s - R(s - t) \in S$.

THEOREM 1.3. *Definitions* 1.1 *and* 1.2 *are equialent.*

PROOF. Assume that S is an H-cone in the sense of Definition 1.1. Define in $S \times S$ an equivalence relation $\sim$ by $(u, v) \sim (s, t) \Leftrightarrow u + t = v + s$. Setting $u - v$

equal to the equivalence class generated by (u, v) the structure $S - S$ is an ordered vector space with respect to the partial ordering $\leqq$ defined by

$$u - v \leqq s - t \Leftrightarrow u + t \leqq s + v.$$

Moreover $S - S$ is an H-cone, since clearly (A_1) and (A_2) hold in $S - S$ and by [4, Proposition 2.1.2.], property (A_3) is valid in $S - S$.

Conversely, assume that S is an H-cone in an ordered vector space E in the sense of Definition 1.2. Then obviously (H_1) holds in S. Assume that $u \leqq v_1 + v_2$ for $u, v_1, v_2 \in S$. Then (A_1) leads to $v_1 \geqq R(s - v_2)$. As $R(u - v_2) \geqq u - v_2$ we have $v_2 \geqq u - R(u - v_2)$. Hence picking $u_1 = R(u - v_2)$ and $u_2 = u - R(u - v_2)$ we obtain (A_1). Let now F be a nonempty subset of S and w be a greatest lower bound of F in E. Then Rw belongs to S and $Rw \leqq x$ forall $x \in F$. Hence Rw is a lower bound of F, and so from $Rw \geqq w$ it follows that $Rw = w$. Thus the set F has in S the greatest lower bound which is the same as the greatest lower bound in E. Consequently the greatest lower bound is translation invariant completing the proof.

Any pair of elements in an H-cone has mixed envelopes introduced by Arsove and Leutwiler in algebraic potential theory ([1] and other references there).

THEOREM 1.4. *Let S be an H-cone. Then for any elements s and t in S there exist a mixed lower envelope*

$$s \curlywedge t = \max\{x \in S \mid x \preccurlyeq s, \quad x \leqq t\}$$

and a mixed upper envelope

$$s \curlyvee t = \min\{x \in s \mid x \succcurlyeq s, \quad x \geqq t\}$$

satisfying the equality

$$s \curlywedge t + t \curlyvee s = s + t.$$

PROOF. See [1, Theorem 2.5].

The least upper bound of two elements s and t in an H-cone has a useful formula

$$s \vee t = R(s + t - s \wedge t) \tag{1.3}$$

proved in [4, Corollary 2.1.3].

Let $(S, \leqq)$ be an ordered convex cone. A subset T of S is called *solid* if for all $s \in T$ the condition $t \leqq s$ for some $t \in S$ implies $t \in T$. A subset T of S is called *order dense* in S if the condition

$$s = \bigvee\{t \in T \mid t \leqq s\}$$

holds for all s in S. A subset T of S is *increasingly dense*, if T is order dense in S and directed upwards.

Let T be a subspace of a vector lattice $(E, \leqq)$. Then T is called solid in E if T^+ and T^- are solid in E^+. In addition T is order dense in E if T^+ is order dense in E^+.

Recall also that an ordered set is Dedekind complete (in other words conditionally complete) if any bounded set has a least upper bound. It is relatively easy to see that an H-cone is Dedekind complete with respect to the partial order $\leqq$. Moreover the following result holds with respect to the specific order [4, Theorem 2.1.5.].

THEOREM 1.5. *Let S be an H-cone in a vector lattice $E = S - S$. Then E is Dedekind complete with respect to the specific order.*

Obviously the positive cone of Dedekind complete vector lattice is an H-cone in which the initial order and the specific order coincide.

DEFINITION 1.6. Let $(S, \leqq)$ be an ordered convex cone. A function φ from S into $\mathrm{R}^+ \cup \{\infty\}$ is called *left order continuous*, if $\varphi(\vee F) = \sup_{f \in F} \varphi(f)$ for any non-empty upward directed set $F \subset S$ possessing a least upper bound.

The set of left order continuous additive maps finite on a dense subset of S is called a *dual* of S. The set of left order continuous and additive real-valued functions on S is called sa *real-valued dual* of S and denoted by $S^\times$. If E is a vector lattice then we set $E^\times = (E^+)^\times - (E^+)^\times$. A partial ordering in $E^\times$ (resp. in $S^\times$) is defined by $\mu \leqq \psi$ if $\mu(s) \leqq \psi(s)$ for all $s \in E^+$ (resp. $s \in S$).

The evaluation map $x \mapsto \hat{x}$ is defined by

$$\hat{x}(\mu) = \mu(x)$$

for $x \in S$ and $\mu \in S^\times$(or $\mu \in E^\times$).

In this work we consider only real-valued duals of H-cones. Duals of H-cones are considered in a subsequent paper.

Note that a left order continuous mapping φ is increasing; that is, the condition $s \leqq t$ implies $\varphi(s) \leqq \varphi(t)$. A left order continuous additive map φ is also positively homogeneous: $\varphi(\alpha x) = \alpha\varphi(x)$ for all $\alpha \in \mathrm{R}^+$. Moreover, a real additive map φ from a positive cone E^+ of an ordered vector space E to R^+ is left order continuous if and only if

$$\inf_{f \in F} \varphi(f) = 0$$

for any non-empty downward directed subset F of E with $\wedge F = 0$.

Order continuous maps play an important role in the theory of vector lattices. Fremlin has proved the following useful result in [6, p. 85–86]. We point out that our notation $E^\times$ is the same as Fremlin's notation in [6] if E is a vector lattice. The notation $E^\times$ in [6] is the same as $L^\times(E, \mathsf{R})$.

THEOREM 1.7. *Let E be a vector lattice. If E is Dedekind complete, the image of the evaluation map is order dense and solid in* $E^{\times\times}$.

Applying Theorem 1.5 we obtain the next corollary.

COROLLARY 1.8. *Let S be an H-cone in an ordered vector space E. Then the assertions of the preceding theorem hold for E with respect to the specific order.*

We need the following version of the Hahn-Banach theorem proved by Fuchsteiner and Lusky in [7, Theorem 1.3.2].

THEOREM 1.9. *Let S be an ordered convex cone and G be a subcone of S. Assume that* $\mu\colon G \to \mathsf{R}$ *is increasing positively homogeneous additive and* $p\colon S \to \mathsf{R}$ *is increasing positively homogeneous subadditive with* $p \geqq \mu$ *on G. Then* μ *can be extended to an increasing positively homogeneous additive mapping* $\tilde{\mu}$ *on S if and only if*

$$\mu(g_1) \leqq \mu(g_2) + p(s) \tag{1.4}$$

for all g_1, g_2 *in G and* $s \in S$ *satisfying* $g_1 \leqq g_2 + s$.

2. Embedding theorems.

We are mainly interested to solve the question how well an H-cone S can be embedded in its second real-valued dual $S^{\times\times}$. We shall prove that an H-cone S is always specifically solid in $S^{\times\times}$. Moreover under some relatively weak assumptions an H-cone is even solid in $S^{\times\times}$. The crucial idea is to utilize the fact that an H-cone is a subcone of a Dedekind complete vector lattice with respect to the specific order.

We need some results of the dual theory of H-cones studied by Boboc, Bucur and Cornea (see [4] and other references there) stated next.

LEMMA 2.1. *Let S be an H-cone. Then* $S^\times$ *is an H-cone. Moreover for all f and g in* $S^\times$ *we have*

$$R(f - g)(x) = \sup\{f(y) - g(y) \mid y \leqq x\}. \tag{2.1}$$

In addition the evaluation map $x \mapsto \hat{x}$ *from S to* $S^{\times\times}$ *possesses the properties*

(a) $\hat{s} + \hat{t} = \widehat{s + t}$ *and* $\alpha\hat{s} = \widehat{\alpha s}$,
(b) $R(\hat{s} - \hat{t}) = \widehat{R(s - t)}$,
(c) $\hat{s} \wedge \hat{t} = \widehat{s \wedge t}$,
(d) $\hat{s} \vee \hat{t} = \widehat{s \vee t}$,

for all s and t in S and $\alpha \in \mathsf{R}^+$.

PROOF. The equation (2.1) follows from [4, Proposition 2.2.5], since R^+ is an H-cone. The property (a) is obvious. The property (b) is proved in [3, Proposition 3] and (d) is obtained from (b) by (1.3). From [4, Theorem 2.3.7] it follows (c). Lastly $S^\times$ is an H-cone by [4, Theorem 2.2.6].

The greatest lower bound of two additive left order continuous mappings is given by the following proposition proved in [5, Theorem 1.4].

PROPOSITION 2.2. *Let S be an H-cone and μ_1, μ_2 be additive left order continuous mappings from S into R^+. Then the greatest lower bound of μ_1 and μ_2 in the set of additive left order continuous mappings is given by the formula*

$$\mu_1 \wedge \mu_2(s) = \inf\{\mu_1(s_1) + \mu_2(s_2) \mid s_1 + s_2 = s, s_1, s_2 \in S\}$$

for all $s \in S$. Moreover, there exist elements s_1 and s_2 in S such that $s = s_1 + s_2$ and $\mu_1 \wedge \mu_2(s_i) = \mu_i(s_i)$ for $i = 1, 2$.

We are ready to state that the image of the evaluation map from an H-cone to $S^{\times\times}$ is specifically solid. This result is a generalized version of Theorem 1.7.

THEOREM 2.3. *Let S be an H-cone and the map $x \mapsto \hat{x}$ be an evaluation map from S to $S^{\times\times}$. Then the set $\{\hat{s} \mid s \in S\}$ is specifically solid in $S^{\times\times}$.*

PROOF. Let E be a vector lattice and S an H-cone in E. By Theorem 1.5 the vector lattice E is Dedekind complete with respect to the specific order and $(E, \preccurlyeq)^+ = S$. In this proof we consider only specific order in E and use the notation E_s for $(E, \preccurlyeq)$. Then the notation $E_s^\times$ stands for the set $E^\times$ with respect to the specific order. Hence $(E_s^+)^\times = (S, \preccurlyeq)^\times$ and $E_s^\times = (S, \preccurlyeq)^\times - (S, \preccurlyeq)^\times$.

Let $\mu \in S^{\times\times}$ and assume that there exists an element x in S such that $\mu \preccurlyeq \hat{x}$. We extend μ to the set $S^\times - S^\times$ by $\mu(f - g) = \mu(f) - \mu(g)$. Note that $S^\times - S^\times$ is a subspace of $E_s^\times$. Then we have $\hat{x} \geqq \mu$ on $(S^\times - S^\times)^+$, since from $\hat{x} \succcurlyeq \mu$ it follows that

$$(\hat{x} - \mu)(f) \geqq (\hat{x} - \mu)(g) \Leftrightarrow \hat{x}(f - g) \geqq \mu(f - g)$$

for all f and g in $S^\times$ with $f \geqq g$.

In order to apply the version of the Hahn-Banach theorem stated in Theorem 1.9 we have to check the condition (1.4). Let f and g belong to $(S^\times - S^\times)^+$ and assume that $h \in (E_s^\times)^+$ satisfy $f \leqq g + h$ in $S = E_s^+$. From Proposition 2.2 it follows that $f \wedge g \in (S^\times - S^\times)^+$ and $f - f \wedge g \leqq h$. Hence we have

$$\mu(f) = \mu(f \wedge g) + \mu(f - f \wedge g) \leqq \mu(g) + \hat{x}(f - f \wedge g) \leqq \mu(g) + \hat{x}(h).$$

Consequently there exists an increasing, additive function $\mu_0\colon (E_s^\times)^+ \to \mathsf{R}^+$ such that

$$\mu_0 = \mu \text{ on } (S^\times - S^\times)^+, \quad \mu_0 \leqq \hat{x} \text{ on } (E_s^\times)^+.$$

The mapping μ_0 may be extended to $E_s^\times$ naturally. Denote this extension also by μ_0. Then $0 \leqq \mu_0 \leqq \hat{x}$ on $E_s^\times$ and applying Corollary 1.8 in $E_s^{\times\times}$ we observe that $\mu_0 = \hat{z}$ on $E_s^\times$ for some $z \in E_s$. Therefore there exists u and v in E_s such that $\mu = \hat{u} - \hat{v}$ on $S^\times$. Thus we have

$$\mu = R(\hat{u} - \hat{v}) = R(\widehat{u - v})$$

by Lemma 2.1 (b).

DEFINITION 2.4. Let S be an H-cone. An element $u \in S$ is called a *generator* if $s = \bigvee_{n \in \mathrm{N}} (nu \quad \setminus s)$ for all $s \in S$.

LEMMA 2.5. *If S is an H-cone and $u \in S$ is a generator then $R(\hat{u} + \psi - \mu) \neq \hat{u}$ for all distinct mappings μ and ψ in $S^{\times\times}$ satisfying $\mu \geqq \psi$.*

PROOF. Let u be a generator in an H-cone S. Assume that $\mu \in S^{\times\times}$ and $\psi \in S^{\times\times}$ such that $\mu \geqq \psi$ and $\mu \neq \psi$. On the contrary assume that $R(\hat{u} + \psi - \mu) = u$. Applying (2.1) we have

$$\sup\{g(u) + \psi(g) - \mu(g) \mid g \leqq f, \quad g \in S^\times\} = \hat{u}(f)$$

for all $f \in S^\times$. Let $\varepsilon > 0$ and $f \in S^\times$. Then there exists $g_n \in S^\times$ for all $n \in \mathrm{N}$ such that $g_n \leqq f$ and

$$f(u) - g_n(u) + \mu(g_n) - \psi(g_n) < \frac{\varepsilon}{2^n}. \tag{2.2}$$

Hence

$$\sup_{n \in \mathrm{N}} g_n(u) = f(u).$$

Using (2.2) we infer that

$$\sup_{n \in \mathrm{N}} g_n(s) = f(s)$$

for all s in S with $s \preccurlyeq u$. Since u is a generator and f is left order continuous, the equality

$$\sup_{n \in \mathrm{N}} g_n(s) = f(s)$$

holds in fact for all $s \in S$. Hence we have $f = \bigvee_{n \in \mathrm{N}} g_n$. Since $\bigvee_{k=1}^n g_n \preccurlyeq \sum_{k=1}^n g_n$ we notice from (2.2) that

$$(\mu - \psi)\left(\bigvee_{k=1}^{n} g_k\right) \leqq \sum_{k=1}^{n} \mu(g_n) - \psi(g_n) \leqq \varepsilon$$

and therefore $(\mu - \psi)(f) \leqq \varepsilon$. Since ε was arbitrary we conclude $\mu - \psi = 0$, which is a contradiction.

LEMMA 2.6. *Let S be an H-cone. Let D be a specifically solid subset of S. If for all distinct elements s and t in S such that $s \geqq t$ there exists an element $u \in D$ with $R(u + t - s) \neq u$, then D is increasingly dense in S.*

PROOF. Let E be a vector lattice and S be an H-cone in E. Assume that $s \in S$. Set

$$F = \{y \in D - S \mid 0 < y \leqq s\}.$$

Then F is non-empty. Indeed, the element $R(u - s)$ is nonzero for some $u \in D$ and $u - R(u - s) \leqq s$. Denote by $\bigvee_E$ the least upper bound in E. Assume that $\bigvee_E F \neq s$. Then there exists an element $x \ngeqq s$ in E such that $x \geqq y$ for all $y \in F$. Since $s - s \wedge x > 0$ there exists an element u in D satisfying $R(u + s \wedge x - s) \neq u$. Setting $y = u - R(u + s \wedge x - s)$ we infer that $y \in F$ and $y \leqq s - s \wedge x$. Thus we have

$$y + f \leqq y + s \wedge x \leqq s$$

for all $f \in F$. Inductively we obtain $ny \leqq s$ for all $n \in \mathbf{N}$. Since S is Archimedean, we infer $y = 0$, which is impossible. Hence $s = \bigvee_E F$. As D is specifically solid we see that $R(u - t) \in D$ for all $u - t \in F$ and by (1.3) $t \vee v \in F$ for all $t, v \in D \cap F$, Consequently, $s = \bigvee F \cap D$ and $F \cap D$ is directed upwards, completing the proof.

THEOREM 2.7. *Let S be an H-cone possessing a generator. Then the set $\{\hat{s} \mid s \in S\}$ is solid and increasingly dense in $S^{\times\times}$.*

PROOF. Let u be a generator in an H-cone S. According to Lemma 2.5 we have $R(\hat{u} + \psi - \mu) \neq \hat{u}$ for all distinct mappings μ and ψ in $S^{\times\times}$ with $\mu \geqq \psi$. In addition Theorem 2.3 states that the set $\{\hat{s} \mid s \in S\}$ is specifically solid in $S^{\times\times}$. Now by the preceding lemma $\{\hat{s} \mid s \in S\}$ is increasingly dense and therefore also solid in $S^{\times\times}$.

An element u of an H-cone S is called a *weak unit*, if $s = \bigvee_{n \in \mathbf{N}} (nu \wedge s)$ for all $s \in S$. Note that a Dedekind complete vector lattice E can be extended to a vector lattice F possessing a weak unit such that E^+ is solid and dense in F^+ (see [8, p. 142]). Then F^+ is an H-cone possessing a generator. Hence Theorem 2.7 is a generalization of Theorem 1.7.

A large class of H-cones possesses a generator as stated next.

THEOREM 2.8. *Let S be an H-cone possessing a weak unit. If S admits a countable dense set then it has a generator.*

PROOF. Assume that an H-cone S possesses a weak unit u and a countable dense set $D = \{s_n \mid n \in \mathbf{N}\}$. Put $D_u = \{s_n \in D \mid s_n \leqq u\}$. We prove that the element

$$v = \sum_{s_n \in D_u} \frac{1}{2^n} s_n = \bigvee_k \left(\sum_{\substack{s_n \in D_n \\ n \leqq k}} \frac{1}{2^n} s_n \right)$$

is a generator. Assume first that $x \in S$ and $x \leqq nu$ for some $n \in \mathbf{N}$. Denote by I the set of indexes k for which $s_k \leqq x$ and $s_k \in D_u$. Then $x = \bigvee_{k \in I} s_k$ and therefore we have $x = \bigvee_{k \in I} (ns_k \curlywedge x)$. Since $ns_k \preccurlyeq n2^k v$ we obtain $x = \bigvee_{k \in \mathbf{N}} (kv \curlywedge x)$. If now $x \in S$ is arbitrary element of S then $x = \bigvee_{k \in \mathbf{N}} (ki \wedge x)$ which renders that $x = \bigvee_{n,k \in \mathbf{R}} (nu \curlywedge (ku \wedge x))$. Consequently the relation $x = \bigvee_{n \in \mathbf{N}} (nu \curlywedge x)$ holds for all $x \in S$.

Recall that an H-cone S is called a *standard H-cone* ([4]) if it possesses a weak unit u and there exists a coutable dense set of elements f satisfying for all $\varepsilon > 0$ the condition:

$$f = \vee F \quad \text{and } F \text{ directed upwards} \Rightarrow f \leqq g + eu \quad \text{for some } g \in F.$$

COROLLARY 2.9. *A standard H-cone S possesses a generator and therefore the image of the evaluation map from S into $S^{\times\times}$ is solid and increasingly dense in $S^{\times\times}$.*

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