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BICROSSPRODUCT KAC ALGEBRAS, BICROSSPRODUCT GROUPS AND VON NEUMANN ALGEBRAS OF TAKESAKI'S TYPE

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§ 0. Introduction

A series of Majid's works [M1, 2, 3] proved that a notion of a matched pair of groups was important in the theory of Hopf algebras as well as in the theory of Kac algebras. In these papers, Majid exhibited a lot of examples of matched pairs of groups, relating them to the classical Yang-Baxter equations. He then showed that every matched pair yields two Hopf algebras in the purely algebraic case and two Hopf-von Neumann algebras in the operator algebraic setting. In each case, his construction, referred to as bicrossproduct construction, produces non-commutative and non-cocommutative algebras. It should be noted also that, in the von Neumann algebraic situation, the additional condition that a matched pair is modular in the sense of Majid guarantees that the bicrossproduct algebras are in fact Kac algebras. All of these suggest that it would be worth while to carry out a further detailed investigation of matched pairs and their associated bicrossproduct algebras.

When a matched pair $(G_1, G_2, \alpha, \beta)$ is given, one may construct another group $G_{1\alpha} \bowtie_\beta G_2$ out of it, called the bicrossproduct group. It is obtained by "taking the semidirect product simultaneously" of the groups G_1 and G_2. (See § 1 for the details). Majid illuminated in [M1] a connection among the given groups G_1, G_2 and the bicrossproduct group in the Hopf algebraic level, by introducing a double crossproduct of a matched pair of Hopf algebras. However, no connection has been given so far among the bicrossproduct Kac algebras and the bicrossproduct group. The purpose of this paper is to describe one connection among them in the operator algebraic level. It is phrased in terms of von Neumann algebras considered by Takesaki in his work [T1].

The organization of the paper is as follows. In § 1, we first recall the notion of a matched pair and Majid's construction of the bicrossproduct Kac algebras associated with it. We then review a certain kind of von Neumann algebras considered by Takesaki in [T1]. We call such algebras von Neumann algebras of

Takesaki's type. Section 2 is devoted to establishing a connection among the bicrossproduct group. It is shown that every bicrossproduct Kac algebra is a von Neumann algebra of Takesaki's type. It is of great interest to determine what kind of conditions would in general ensure that a von Neumann algebra of Takesaki's type is a Kac algebra.

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§ 1. Preliminaries.

In this section, we first recall the notion of a matched pair of locally compact groups and their actions, due to Takeuchi [Ta]. We shall also mention Majid's construction of the bicrossproduct involutive Hopf-von Neumann (or Kac) algebra associated with a given (modular) matched pair. Secondly, we review some type of von Neumann algebras considered by Takesaki in [T1].

To explain the concept of a matched pair, let us begin by considering two locally compact groups G_1 and G_2 with their left Haar measures μ_1 and μ_2, respectively. We assume that G_1 acts on, and is at the same time acted on by, the set G_2 continuously and nonsingularly. By nonsingularity of a group action, we mean that the measure class in question of the measure space is preserved by the action. We denote by α (resp. β) the action of G_1 (resp. G_2). We shall keep using the letters α, β for the induced actions of G_1 and G_2 on algebras $L^\infty(G_2)$ and $L^\infty(G_1)$, respectively. Namely, we have

$$\alpha_g(k)(s) = k(\alpha_{g^{-1}}(s)), \quad \beta_s(f)(g) = f(\beta_{s^{-1}}(g)),$$

where $k \in L^\infty(G_2)$, $(f \in L^\infty(G_1))$, $g \in G_1$ and $s \in G_2$. By assumption, it makes sense to consider the Radon-Nikodym derivatives

$$\chi(g,s) = \frac{d\mu_2 \circ \alpha_g}{d\mu_2}(s), \quad \Psi(s,g) = \frac{d\mu_1 \circ \beta_s}{d\mu_1}(g) \quad (g \in G_1, s \in G_2).$$

The functions χ and Ψ are cocycles on $G_1 \times G_2$, and are assumed to be jointly continuous. We further assume that the actions α and β satisfy the following compatibility conditions:

$$(1.1) \qquad \begin{cases} \alpha_g(e) = e & \beta_s(e) = e \\ \alpha_g(st) = \alpha_{\beta_t(g)}(s)\alpha_g(t), & \beta_s(gh) = \beta_{\alpha_h(s)}(g)\beta_s(h), \end{cases}$$

where $g, h \in G_1$ and $s, t \in G_2$. In this case, we say that the system $(G_1, G_2, \alpha, \beta)$ is a matched pair. We refer readers to Lemma 2.2 of [M3] for the properties that χ and Ψ enjoy in case of $(G_1, G_2, \alpha, \beta)$ being a matched pair. A matched pair $(G_1, G_2, \alpha, \beta)$ is said to be modular [M3] if

$$\frac{\chi(g,s)}{\chi(g,e)} = \frac{\Psi(s,g)}{\Psi(s,e)} = 1, \quad \frac{\delta_2(\alpha_g(s))}{\delta_2(s)} = \frac{\delta_1(\beta_s(g))}{\delta_1(g)}$$

for all $g \in G_1$ and $s \in G_2$, where δ_i $(i = 1, 2)$ indicates the modular functions of G_i. In [M3], Majid gave abundunt examples of (modular) matched pairs of Lie groups and their actions. He also showed that, if $(G_1, G_2, \alpha, \beta)$ is a matched pair, the crossed products $L^\infty(G_2) \times {}_\alpha G_1$ and $L^\infty(G_1) \times {}_\beta G_2$ can be equipped with a structure of an involutive Hopf-von Neumann algebra (see [E&S] for the definition of an involutive Hopf-von Neumann algebra). He called the crossed products the bicrossproduct Hopf-von Neumann algebras. In particular, with the additional condition that the matched pair is modular, the crossed products become Kac algebras (see [E&S]), dual to each other. He then called the algebras the bicrossproduct Kac algebras. We remark that these involutive Hopf-von Neumann algebras are not commutative or cocommutative except in the trivial case.

Next we briefly review a certain type of von Neumann algebras considered by Takeṣaki in [T1]. These algebras were investigated to study commutation relation for the regular representation of a locally compact group. Now let G be a locally compact group. Denote by λ (resp. ρ) the left (resp. right) regular representation of G. Suppose that H is a closed subgroup of G. Then we consider the following von Neumann subalgebras of $L^\infty(G)$:

$$L^\infty(G) \cap \lambda(H)', \quad L^\infty(G) \cap \rho(H)'.$$

These are the fixed point subalgebras of $L^\infty(G)$ with respect to the automorphism groups $\{\mathrm{Ad}\lambda(h) : h \in H\}$ and $\{\mathrm{Ad}\rho(h) : h \in H\}$. We denote them by $L^\infty(H\backslash G)$ and $L^\infty(G/H)$, respectively, since they can be naturally identified with the set of all essentially bounded functions on the coset spaces $H\backslash G$ and H/G, respectively, with appropriate measures. Following [T1], we set

$$M(H\backslash G, \rho(H)) = L^\infty(H\backslash G) \vee \rho(H)'',$$

$$M(G/H, \lambda(H)) = L^\infty(G/H) \vee \lambda(H)''.$$

By Theorem 3 of [T1], they are commutant to each other. We say that a von Neumann algebra $\mathscr{P}$ is of Takesaki's type if $\mathscr{P}$ is $*$-isomorphic to one of the algebras introduced above.

§ 2. Main results.

This section is devoted to proving the main theorem in the paper which states that the bicrossproduct Kac algebras associated with a matched pair can be realized as von Neumann algebras of Takesaki's type. To attain this goal, we need to find a group that serves as G in the preceding section. Such a group can be obtained as a bicrossproduct group [M1, 2, 3].

Let $(G_1, G_2, \alpha, \beta)$ be a modular matched pair. It will be fixed throughout this section. We shall retain notations introduced in the previous section. We now turn the set $G_1 \times G_2$ into a group by defining group operations as follows:

$$(g,s)\cdot(h,t) = (\beta_{t^{-1}}(g)h, s\alpha_g(t^{-1})^{-1}), \quad ((g,s),(h,t)\in G_1 \times G_2),$$
$$(g,s)^{-1} = (\beta_s(g^{-1}), \alpha_{g^{-1}}(s)^{-1}).$$

It is easily verified, using the compatibility condition (1.1), that the above operations indeed make $G_1 \times G_2$ a group. Moreover, under the product topology, the product and inverse operations are clearly continuous. Thus $G_1 \times G_2$ becomes a locally compact group. We denote it by $G_{1\beta} \bowtie_\alpha G_2$, or simply by $G_1 \bowtie G_2$ if there is no danger of confusion. It is called the bicrossproduct group associated with the matched pair $(G_1, G_2, \alpha, \beta)$. We put

$$\tilde{G}_1 = \{(g,e)\in G_1 \bowtie G_2 : g\in G_1\}, \quad \tilde{G}_2 = \{(e,s)\in G_1 \bowtie G_2 : s\in G_2\}.$$

Since

$$(g,e)\cdot(h,e) = (gh,e), \quad (g,e)^{-1} = (g^{-1},e),$$
$$(e,s)\cdot(e,t) = (e,st), \quad (e,s)^{-1} = (e,s^{-1}),$$

it follows that $\tilde{G}_1$ and $\tilde{G}_2$ are closed subgroups of $G_1 \bowtie G_2$. Moreover, G_1 (resp. G_2) is topologically isomorphic to $\tilde{G}_1$ (resp. $\tilde{G}_2$).

REMARK 2.1. In this paper, we adopted Definition 2.1 of [M3] for a matched pair of groups and their actions. However, Majid employed another (but "equivalent") formulation for a matched pair in [M1, 2] (see also [Ta]), where actions $\tilde{\alpha}$ and $\tilde{\beta}$ of G_1 and G_2, respectively, satisfy the following conditions:

$$\tilde{\alpha}_g(e) = e, \qquad \tilde{\beta}_s(e) = e, \quad (g, h\in G_1, s, t\in G_2),$$
$$\tilde{\alpha}_g(st) = \tilde{\alpha}_g(s)\tilde{\alpha}_{\tilde{\beta}_{s^{-1}}(g^{-1})^{-1}}(t), \quad \tilde{\beta}_s(gh) = \tilde{\beta}_s(g)\tilde{\beta}_{\tilde{\alpha}_{g^{-1}}(s^{-1})^{-1}}(h).$$

For such a "matched pair", Majid constructed the bicrossproduct group (see Theorem 2.3 of [M1]). For $\tilde{\alpha}$ and $\tilde{\beta}$ as above, we can obtain a matched pair in our sense by defining α and β to be

$$a_g(s) = \tilde{\alpha}_g(s^{-1})^{-1}, \quad \beta_s(g) = \tilde{\beta}_s(g^{-1})^{-1} \quad (g\in G_1, s\in G_2).$$

Thus, through this correspondence, we may identify the above mentioned definition of a matched pair with ours. Under the identification $(\alpha, \beta) \leftrightarrow (\tilde{\alpha}, \tilde{\beta})$, the bicrossproduct group constructed in Theorem 2.3 of [M1] corresponds to our group defined above.

We denote by $\mathcal{K}(G_1 \bowtie G_2)$ the set of all continuous functions on $G_1 \bowtie G_2$ with compact support. Since we equipped $G_1 \bowtie G_2$ with the product topology, the measure $\mu = \mu_1 \times \mu_2$ is a Borel measure on $G_1 \bowtie G_2$. Our first aim is to locate a left Haar measure on the bicrossproduct group.

LEMMA 2.2. *The Borel measure $\chi(\cdot,\cdot)^{-1}d(\mu_1 \times \mu_2)$ is a left Haar measure on $G_1 \bowtie G_2$. Moreover, the function*

$$\delta(g,s) = \delta_1(g)\delta_2(s)\chi(g,e)^{-1}\Psi(s,e)^{-1} \quad ((g,s) \in G_1 \bowtie G_2)$$

is the modular function with respect to the above left Haar measure.

PROOF. First we note that, by modularity of the matched pair, the cocycles χ and Ψ satisfy

$$\chi(g,s) = \chi(g,e), \quad \Psi(s,g) = \Psi(s,e)$$

for all $g \in G_1$ and $s \in G_2$. Thus cocycle identity implies that they are group homomorphisms in the first argument.

Let F be in $\mathcal{K}(G_1 \bowtie G_2)$. Then, thanks to Fubini's theorem, we have

$$\tag{2.3} \begin{aligned} &\int F((g,s)(h,t))\chi(h,e)^{-1}d(\mu_1 \times \mu_2)(h,t) \\ &= \iint F(\beta_{t^{-1}}(g)h, s\alpha_g(t^{-1})^{-1})\chi(h,e)^{-1}d\mu_1(h)d\mu_2(t) \\ &= \iint F(h, s\alpha_g(t^{-1})^{-1})\chi(\beta_{t^{-1}}(g)^{-1}h,e)^{-1}d\mu_1(h)d\mu_2(t) \\ &= \iint F(h, s\alpha_g(t)^{-1})\chi(\beta_t(g)^{-1}h,e)^{-1}\delta_2(t)^{-1}d\mu_2(t)d\mu_1(h). \end{aligned}$$

From homomorphism property of χ and the second identity of Lemma 2.2 in [M3], it follows that

$$\begin{aligned} \chi(\beta_t(g)^{-1}h,e)^{-1} &= \chi(\beta_t(g),e)\chi(h,e)^{-1} \\ &= \delta_2(t)\delta_2(\alpha_g(t))^{-1}\chi(g,e)\chi(h,e)^{-1}. \end{aligned}$$

Thus we may continue calculation (2.3) as follows:

$$(2.3) = \iint F(h, s\alpha_g(t)^{-1})\delta_2(\alpha_g(t))^{-1}\chi(h,e)^{-1}d(\mu_2 \circ \alpha_g)(t)d\mu_1(h)$$

$$= \iint F(h, st^{-1})\delta_2(t)^{-1}\chi(h,e)^{-1}d\mu_2(t)d\mu_1(h)$$

$$= \iint F(h, st)\chi(h,e)^{-1}d\mu_2(t)d\mu_1(h)$$

$$= \iint F(h, t)\chi(h,e)^{-1}d\mu_2(t)d\mu_1(h)$$

This computation shows that the measure $\chi(\cdot,\cdot)^{-1}\mathrm{d}(\mu_1 \times \mu_2)$ defines a left invariant functional on $\mathscr{K}(G_1 \bowtie G_2)$. Therefore it is a left Haar measure on $G_1 \bowtie G_2$. To find the modular function, we compute the following with F as above.

$$(2.4) \quad \iint F(\beta_{t^{-1}}(g)h, s\alpha_g(t^{-1})^{-1})\chi(g,e)^{-1}d\mu_1(g)d\mu_2(s)$$

$$= \iint F(\beta_{t^{-1}}(g)h, s)\chi(g,e)^{-1}\delta_2(\alpha_g(t^{-1}))d\mu_2(s)d\mu_1(g)$$

$$= \iint F(gh, s)\chi(\beta_t(g), e)^{-1}\delta_2(\alpha_{\beta_t(g)}(t^{-1}))d(\mu_1 \circ \beta_t)(g)d\mu_2(s)$$

$$= \iint F(gh, s)\chi(\beta_t(g), e)^{-1}\delta_2(\alpha_{\beta_t(g)}(t^{-1}))\Psi(t,e)d\mu_1(g)d\mu_2(s)$$

$$= \iint F(g, s)\chi(\beta_t(gh^{-1}, e)^{-1}\delta_2(\alpha_{gh^{-1}}(t))^{-1}\Psi(t,e)\delta_1(h)^{-1}d\mu_1(g)d\mu_2(s).$$

The last equality is due to the identity: $e = \alpha_{gh^{-1}}(t^{-1}t) = \alpha_{\beta_t(gh^{-1})}(t^{-1})\alpha_{gh^{-1}}(t)$. By the second identity of Lemma 2.2 in [M3] again, we have

$$\delta_2(t)\chi(gh^{-1}, e) = \delta_2(\alpha_{gh^{-1}}(t))\chi(\beta_t(gh^{-1}), e).$$

Hence (2.4) continues as follows:

$$(2.4) = \iint F(g, s)\chi(gh^{-1}, e)^{-1}\delta_2(t)^{-1}\Psi(t,e)\delta_1(h)^{-1}d\mu_1(g)d\mu_2(s)$$

$$= \delta_1(h)^{-1}\delta_2(t)^{-1}\chi(h,e)\Psi(t,e)\iint F(g,s)\chi(g,e)^{-1}d\mu_1(g)d\mu_2(s).$$

It is now easy to see that the modular function has the desired form.

We put $\mathscr{H} = L^2(G_1) \otimes L^2(G_2)$. Bicrossproduct Kac algebras $\mathscr{M}$ and $\hat{\mathscr{M}}$, where

$\mathcal{M} = L^\infty(G_2) \times_\alpha G_1$ and $\hat{\mathcal{M}} = L^\infty(G_1) \times_\beta G_2$, are both acting on $\mathcal{H}$ in a standard form. We shall retain these notations in what follows.

THEOREM 2.5. *The Kac algebras $\hat{\mathcal{M}}$ and $\mathcal{M}\hat{\mathcal{M}}'$ are both von Neumann algebras of Takesaki's type. In fact, we have*

$$\hat{\mathcal{M}} \cong M(G_1 \bowtie G_2/\tilde{G}_2, \lambda(\tilde{G}_2)),$$

$$\hat{\mathcal{M}}' \cong M(\tilde{G}_2 \setminus G_1 \bowtie G_2, \rho(\tilde{G}_2)),$$

where λ (resp. ρ) is the left (resp. right) regular representation of $G_1 \bowtie G_2$. These isomorphisms are spatial isomorphisms.

PROOF. By Theorem 3 of [T1], it suffices to show the second isomorphism. First we note that the commutant $\hat{\mathcal{M}}'$ is engendered by $L^\infty(G_1) \otimes \mathbf{C}$ and $\{v(t) \otimes \rho_2(t) : t \in G_2\}''$, where ρ_2 is the right regular representation of G_2 and $v(\cdot)$ is the canonical implementation of the action β given by $\{v(t)f\}(g) = \Psi(t^{-1}, g)^{1/2} f(\beta_{t^{-1}}(g))$ $(f \in L^2(G_1))$. Let us denote by $L^2(G_1 \bowtie G_2)$ the Hilbert space of square-integrable functions on $G_1 \bowtie G_2$ with respect to the left Haar measure obtained in Lemma 2.2. Then it is easy to see that the equation

$$\{K\xi\}(g, s) = \xi(g, \alpha_{g^{-1}}(s)) \quad (\xi \in \mathcal{H})$$

defines a unitary operator from $\mathcal{H}$ onto $L^2(G_1 \bowtie G_2)$. For any $\eta \in L^2(G_1 \bowtie G_2)$, we have

$$\{K(f \otimes 1)K^*\eta\}(g, s) = f(g)\eta(g, s).$$

We write $\tilde{f}$ $(f \in L^\infty(G_1))$ for the function on $G_1 \bowtie G_2$ given by $\tilde{f}(g, s) = f(g)$. Then the above calculations means that $K(L^\infty(G_1) \otimes \mathbf{C})K^* = \{\tilde{f} : f \in L^\infty(G_1)\}$. Meanwhile, for η as above, we have

$$
\begin{aligned}
(2.6) \qquad & \{K(v(t) \otimes \rho_2(t))K^*\eta\}(g, s) \\
&= \{(v(t) \otimes \rho_2(t))K^*\eta\}(g, \alpha_{g^{-1}}(s)) \\
&= \delta_2(t)^{1/2} \Psi(t^{-1}, e)^{1/2} \{K^*\eta\}(\beta_{t^{-1}}(g), \alpha_{g^{-1}}(s)t) \\
&= \delta_2(t)^{1/2} \Psi(t, e)^{-1/2} \eta(\beta_{t^{-1}}(g), \alpha_{\beta_{t^{-1}}(g)}(\alpha_{g^{-1}}(s)t)).
\end{aligned}
$$

The matched-pair relations (1.1) yield

$$
\begin{aligned}
\alpha_{\beta_{t^{-1}}(g)}(\alpha_{g^{-1}}(s)t) &= \alpha_{\beta_t(\beta_{t^{-1}}(g))}\alpha_{g^{-1}}(s))\alpha_{\beta_{t^{-1}}(g)}(t) \\
&= s\alpha_{\beta_{t^{-1}}(g)}(t) \\
&= s\alpha_g(t^{-1})^{-1}.
\end{aligned}
$$

The last step is guaranteed by the identity: $e = \alpha_g(tt^{-1}) = \alpha_{\beta_{t^{-1}}(g)}(t)\alpha_g(t^{-1})$. Hence we can further compute (2.6) in the following way.

$$\begin{aligned}(2.6) &= \delta_2(t)^{1/2}\Psi(t,e)^{-1/2}\eta(\beta_{t^{-1}}(g), s\alpha_g(t^{-1})^{-1})\\ &= \delta(e,t)^{1/2}\eta((g,s)(e,t))\\ &= \{\rho(e,t)\eta\}(g,s).\end{aligned}$$

Here δ is the modular function of $G_1 \bowtie G_2$ that appeared in Lemma 2.2. Consequently, we get

$$K\{v(t) \otimes \rho_2(t) : t \in G_2\}'' K^* = \rho(\tilde{G}_2)''.$$

Therefore it is enough to show that the set $\{\tilde{f} : f \in L^\infty(G_1)\}$ exactly coincides with the algebra $L^\infty(\tilde{G}_2 \setminus G_1 \bowtie G_2)$. For this, we introduce a unitary U on $\mathscr{H}$ given by

$$\{U\xi\}(g,s) = \chi(g^{-1}, s)^{1/2}\xi(g, \alpha_{g^{-1}}(s)) \quad (\xi \in \mathscr{H}).$$

Then we have that $\{UK^*\eta\}(g,s) = \chi(g,e)^{-1/2}\eta(g,s)$ for any $\eta \in L^2(G_1 \bowtie G_2)$. From this, it easily follows that $UK^*\lambda(e,t)KU^* = 1 \otimes \lambda_2(t)$, where λ_2, of course, stands for the left regular representation of G_2. Obviously, Ad UK^* maps the algebra $L^\infty(G_1 \bowtie G_2)$ onto $L^\infty(G_1) \bar{\otimes} L^\infty(G_2)$.From ergodicity of the action Ad λ_2 of G_2 on $L^\infty(G_2)$, it results that

$$UK^*L^\infty(\tilde{G}_2 \setminus G_1 \bowtie G_2)KU^* = L^\infty(G_1) \otimes \mathbf{C}.$$

Since $U^*(L^\infty(G_1) \otimes \mathbf{C})U = L^\infty(G_1) \otimes \mathbf{C}$, we see that

$$\begin{aligned}L^\infty(\tilde{G}_2 \setminus G_1 \bowtie G_2) &= KU^*(L^\infty(G_1) \otimes \mathbf{C})UK^*\\ &= K(L^\infty(G_1) \otimes \mathbf{C})K^*\\ &= \{\tilde{f} : f \in L^\infty(G_1)\}.\end{aligned}$$

This completes the proof.

In order to prove that the Kac algebras $\mathscr{M}$ and $\mathscr{M}'$ are also von Neumann algebras of Takesaki's type, we consider the set $G_1 \times G_2$ with the product topology which is in turn equipped with the following group structure:

$$\begin{aligned}(g,s)\cdot(h,t) &= (g\beta_s(h^{-1})^{-1}, \alpha_{h^{-1}}(s)t),\\ (g,s)^{-1} &= (\beta_{s^{-1}}(g)^{-1}, \alpha_g(s^{-1})).\end{aligned}$$

Let us denote by $G_1 \odot G_2$ the locally compact space $G_1 \times G_2$ with this group structure. Set

$$\bar{G}_1 = \{(g,e) : g \in G_1\}, \quad \bar{G}_2 = \{(e,s) : s \in G_2\}.$$

Then $\bar{G}_1$ and $\bar{G}_2$ are closed subgroups of $G_1 \odot G_2$ that are topologically isomorphic to G_1 and G_2, respectively. By a method similar to that in Lemma 2.2, one can prove that the Borel measure $\Psi(\cdot,\cdot)^{-1}d(\mu_1 \times \mu_2)$ is a left Haar measure on $G_1 \odot G_2$ with δ as its modular function again. Then, by arguing as we did in the proof of Theorem 2.5, we may conclude that the von Neumann algebra $\mathscr{M}'$ is spatially isomorphic to $M(\bar{G}_1 \setminus G_1 \odot G_2, \bar{\rho}(\bar{G}_1))$, where $\bar{\rho}$ is the right regular representation of $G_1 \odot G_2$. To sum up, we obtain

THEOREM 2.7. *The Kac algebras $\mathscr{M}$ and $\mathscr{M}'$ are also von Neumann algebras of Takesaki's type. Indeed, we have*

$$\mathscr{M} \cong M(G_1 \odot G_2/\bar{G}_1, \bar{\lambda}(\bar{G}_1)),$$

$$\mathscr{M}' \cong M(\bar{G}_1 \setminus G_1 \odot G_2, \bar{\rho}(G_1)).$$

Here $\bar{\lambda}$ is the left regular representation of $G_1 \odot G_2$. The above isomorphisms are spatial isomorphisms.

REMARK 2.8. In [DeC], were proved our Theorems 2.5 and 2.7, by introducing what he called an action of a locally compact group on a Kac algebra, in the case where $\alpha = \mathrm{id}$. (So β acts on G_1 by automorphisms. In this case, the bicrossproduct group is the ordinary semidirect product). Hence our theorems contain his results.

Theorems 2.5 and 2.7 assert that a von Neumann algebra of Takesaki's type can be a Kac algebra in a certain situation. It is an interesting problem to determine under what conditions it comes equipped with a Kac algebra structure.

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