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THE EXISTENCE OF NONTRIVIAL SOLUTIONS FOR A CLASS OF VOLTERRA EQUATIONS WITH SMOOTH KERNELS

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Abstract.

The paper is devoted to the study of the equation $u = k * g(u)$ with a smooth k and a monotonous g. Some necessary and sufficient condition for the existence of nontrivial solutions u of this equation is given.

1. Introduction.

The following equation

$$u(x) = \int_0^x k(x-s)g(u(s))\,ds \quad (0 \leqq x), \tag{1.1}$$

where $k \geqq 0$ is locally integrable and g is continuous, nondecreasing with $g(0) = 0$ can be used for comparison purposes in the study of integral Volterra equations by monotonicity methods (see [2]). From this point of view the question of uniqueness of the trivial solution $u \equiv 0$ of (1.1) is very important.

This problem in the case of $k(x) = x^{\alpha-1}$, $\alpha > 0$ was solved satisfactorily by G. Gripenberg in [2] (see also [4], [6]). Unfortunately, this results cannot be used to the investigation of the problem of uniqueness of the trivial solution of a large class of equations (1.1) with smooth kernels $k \in C^\infty([0,1])$ such that

$$k(0) = 0 \text{ and } k^{(n)}(0) = 0, \quad n = 1, 2, \ldots. \tag{1.2}$$

In this paper we try to discuss such a case. Namely, we will show that if $k(x) = \beta x^{-\beta-1}\exp(-x^{-\beta})$, $\beta > 0$, then under some assumptions on g equation (1.1) has a nontrivial continuous solution if and only if

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(1.3)
$$\int_0^\delta \frac{ds}{s[-\ln(s/g(s))]^{1+1/\beta}} < \infty \quad (\delta > 0).$$

Then we generalize this result, which allows us to consider the case of kernels of the form $k(x) = \exp\{-x^{-\beta-1}h(x)\}$, $\beta > 0$.

2. Assumptions and auxiliary theorems.

Throughout this paper we assume

(2.1) $g: [0, \infty) \to [0, \infty)$ is a continuous, nondecreasing function such that $g(0) = 0$, $g(x)/x$ is nonincreasing for $x > 0$ and $g(x)/x \to \infty$ as $x \to 0+$;

(2.2) $h: [0, \infty] \to [0, \infty)$ is a continuous, monotonuous function such that $h(\frac{x}{2})/h(x) \to 1$, as $x \to 0+$;

(2.3) $-x^{-\beta}h(x)$ is concave on $(0, 1)$;

Observe that the question of uniqueness or nonuniqueness of the trivial solution depends only on the values of k and g in a neighbourhood of zero. Therefore the assumptions above could be reformulated to take this fact into account.

As a consequence of (2.2) and (2.3) we can get

COROLLARY 2.1. *For any $\alpha, a > 0$*

$$h(ax)/h(x) \to 1 \text{ and } x^{-\alpha}h(x) \to \infty, \text{ as } x \to 0+.$$

From monotonicity and continuity properties of convex functions (see [7]), we obtain

COROLLARY 2.2. *The function $x^{-\beta}h(x)$ is nonincreasing, absolutely continuous on (δ_1, δ) for every sufficiently small $\delta_1, \delta > 0$.*

Set $K(x) = \int_0^x k(s)ds$ and denote by K^{-1} its inverse.

For any continuous $w: [0, \delta] \to (0, \infty)$ we define

$$T(w)(x) = \int_0^x k(x-s)g(w(s))ds, \quad 0 \leqq x \leqq \delta.$$

In this paper ε, $\delta > 0$ always denotes small constants. We permit them to change their values from paragraph to paragraph.

The essential facts concerning the existence of nontrivial solutions of (1.1) are collected in the following theorems (see [1], [5], [6]):

THEOREM 2.1. *Let* ln$k$ *be concave on* $(0, \delta)$ *and* (2.1) *be satisfied. Then the condition*

$$\int_0^\delta (K^{-1})'\left(\frac{s}{g(s)}\right)\frac{ds}{g(s)} < \infty \quad (\delta > 0). \tag{2.4}$$

is necessary for the existence of a nontrivial continuous solution of (1.1).

THEOREM 2.2. *Let* $k \geqq 0$ *be locally integrable and* (2.1) *be satisfied. If there exists a continuous, nonegative, nontrivial function* F *on* $[0, \delta]$ *such that*

$$F(x) \leqq \int_0^x k(x-s)g(F(s))ds \quad (0 \leqq x \leqq \delta), \tag{2.5}$$

then equation (1.1) *has a unique continuous solution* u *such that* $u(x) > 0$ *for* $0 < x \leqq \delta$. *Moreover,* u *is a nondecreasing function.*

3. The main result.

In this section we are going to give necessary and sufficient conditions for the existence of nontrivial solutions of (1.1) for some class of kernels $k(x) = \exp(-x^{-\beta}h(x))$. First we consider the case $k(x) = \beta x^{-\beta-1}\exp(-x^{-\beta})$, $\beta > 0$, because it is much simpler and it helps to motivate the general results.

THEOREM 3.1. *Let* $k(x) = \beta x^{-\beta-1}\exp((-x^{-\beta}), \beta > 0$ *and let* (2.1) *be satisfied. Then* (1.1) *has a nontrivial continuous solution if and only if* (1.3) *is satisfied.*

PROOF. First we note that $K(x) = \exp(-x^{-\beta})$, $K^{-1}(x) = (-\ln x)^{-1/\beta}$ and $(K^{-1})'(x) = (\beta x)^{-1}(-\ln x)^{-1-1/\beta}$. Since ln$k$ is concave on $(0, \delta)$ the necessity of (1.3) follows from Theorem 2.1 immediately.

Now we are going to prove that if (1.3) holds then equation (1.1) has a nontrivial solution.

Define

$$F_c^{-1}(x) = c\int_0^x (K^{-1})'\left(\frac{s}{g(s)}\right)\frac{ds}{g(s)} \text{ and } \phi(x) = x\left(\frac{x}{g(x)}\right)^{1/2} \quad (0 < x < \delta)$$

Our aim will be to find c such that the inverse function F to F_c^{-1} satisfies (2.5). Since integrating by parts and then taking $s = F_c^{-1}(\tau)$ we obtain

$$T(F(x)) = \int_0^{F(x)} K(x - F_c^{-1}(\tau))dg(\tau),$$

it remains to prove only the following:

LEMMA 3.1. *There exists a constant c such that*

$$\int_0^y K(F_c^{-1}(y) - F_c^{-1}(\tau))dg(\tau) \geqq y$$

PROOF. Since by (2.1) $\phi(y) \leqq y$ on $(0, \delta)$, we get

$$\int_0^y K(F_c^{-1}(y) - F_c^{-1}(\tau))dg(\tau) \geqq K(F_c^{-1}(y) - F_c^{-1}(\phi(y)))g(\phi(y)). \tag{3.2}$$

Note that $z(K^{-1})'(z)$ is nondecreasing on $(0, \delta)$. Therefore we can obtain

$$F_c^{-1}(y) - F_c^{-1}(\phi(y)) = c\int_{\phi(y)}^{y} (K^{-1})'\left(\frac{s}{g(s)}\right)\frac{ds}{g(s)} \geqq \tag{3.3}$$

$$c\int_{\phi(y)}^{y} (K^{-1})'\left(\frac{s}{g(y)}\right)\frac{ds}{g(y)} = c\left(K^{-1}\left(\frac{y}{g(y)}\right) - K^{-1}\left(\frac{\phi(y)}{g(y)}\right)\right) \quad (0 < y < \delta).$$

Since in our case

$$K^{-1}(z^2) - K^{-1}(z^3) = (2^{-1/\beta} - 3^{-1/\beta})K^{-1}(z) \quad (0 < z < \delta),$$

from (3.3) it follows that if $c(2^{-1/\beta} - 3^{-1/\beta}) \geqq 1$, then we have

$$K(F_c^{-1}(y) - F_c^{-1}(\phi(y))) \geqq \left(\frac{y}{g(y)}\right)^{1/2}. \tag{3.4}$$

Now, observe that by (2.1) we get

$$g(\phi(y)) \geqq \phi(y)\frac{g(y)}{y} = y^{1/2}g(y)^{1/2}.$$

Therefore our assertion follows from (3.2) and (3.4).

Now we can pass to more general case ($k(x) = \exp(-x^{-\beta}h(x))$. In our considerations we will need the following facts.

LEMMA 3.2. *Let* (2.2), (2.3) *be satisfied. Then*

$$\exp(-(1+\varepsilon)x^{-\beta}h(x)) \leqq K(x) \leqq \exp(-x^{-\beta}h(x)) \quad (0 < x < \delta_\varepsilon).$$

PROOF. By Corollary 2.2 $k(x)$ is nondecreasing on $(0, \delta)$. Hence we easily get the right inequality.

We also have

$$K(x) \geqq (\varepsilon x)k((1-\varepsilon)x) \quad (0 < x < \delta).$$

Hence $\ln K(x) = \ln(\varepsilon x) - (1-\varepsilon)^{-\beta}x^{-\beta}h((1-\varepsilon)x)$, $0 < x < \delta$ and in view of Corollary 2.1 we obtain

$$\ln K(x) \geqq -(1+\varepsilon)x^{-\beta}h(x) \quad (0 < x < \delta_\varepsilon),$$

which completes our proof.

LEMMA 3.3. *Let* (2.2), (2.3) *be satisfied. Then for any $n > 0$ we have*

$$K\left(\frac{x}{n^{1/\beta}+\varepsilon}\right) \leqq K^n(x) \leqq K\left(\frac{x}{n^{1/\beta}-\varepsilon}\right) \quad (0 < x < \delta_\varepsilon). \tag{3.5}$$

PROOF. By Corollary 2.1 it follows that for any $a > 0$

$$\left(\frac{x}{a-\varepsilon}\right)^{-\beta} h\left(\frac{x}{a-\varepsilon}\right) \leqq a^\beta x^{-\beta}h(x) \leqq \left(\frac{x}{a+\varepsilon}\right)^{-\beta} h\left(\frac{x}{a+\varepsilon}\right) \quad (0 < x < \delta).$$

Hence our assertion follows from the estimates given in Lemma 3.2.

Taking $y = K(x)$ and applying (3.5) we easily verify the following:

LEMMA 3.4. *Let* (2.2), (2.3) *be satisfied. Then there exists a constant c such that*

$$K^{-1}(y^2) - K^{-1}(y^3) \geqq cK^{-1}(y) \quad (0 < y < \delta).$$

From the concavity of $\ln k$ it follows that $1/(K^{-1})'$ is concave. Hence we obtain

LEMMA 3.5. The function

$$z(K^{-1})'(z) \text{ is nondecreasing on } (0, \delta).$$

Now we are ready to consider equation (1.1) with $k(x) = \exp(-x^{-\beta}h(x))$. In this case we can state the following:

THEOREM 3.2. *Let* (2.1)–(2.3) *be satisfied. Then equation* (1.1) *has a nontrivial solution if and only if* (2.4) *is satisfied.*

PROOF. The necessity of (2.4) is a consequence of Theorem 2.1. In view of Lemmas 3.4 and 3.5 the proof of the sufficiency of (2.4) goes in the same way as that in Theorem 3.1.

Some kernels, for which (2.2), (2.3) hold, are presented below.

EXAMPLE 3.1. LET $k(x) = \exp(-x^{-\beta}), \beta > 0, 0 < x < \delta$. Then $K(x) \sim \beta^{-1}x^{\beta+1} \exp(-x^{-\beta})$ and $(K^{-1})'(x) \sim \beta^{-1}x^{-1}(-\ln x)^{-1-1/\beta}$.

EXAMPLE 3.2. Let $k(x) = \exp(x^{-\beta}\ln x)$, $\beta > 0$, $0 < x < \delta$. Then $K(x) \sim \beta^{-1}x^{\beta+1}(\ln x)^{-1}\exp(x^{-\beta}\ln x)$ and $(K^{-1})' \sim \beta^{-1-1/\beta}x^{-1}(-\ln x)^{-1-1/\beta} (\ln(-\ln x))^{1/\beta}$.

REFERENCES

1. P. J. Bushell, W. Okrasinski, *Uniqueness of solutions for a class of nonlinear Volterra integral equations with convolution kernel*, Math. Proc. Cambridge Philos. Soc. 106 (1989), 547–552.
2. G. Gripenberg, *Unique solutions of some Volterra integral equations*, Math. Scand. 48 (1981), 59–67.
3. G. Gripenberg, *On the uniqueness of solutions of Volterra equations*, J. Integral Equations 2 (1990), 421–430.
4. W. Mydlarczyk, *The existence of nontrivial solutions of Volterra equations*, Math. Scand. 68 (1991), 83–89.
5. W. Mydlarczyk, *Remarks on a nonlinear Volterra equation*, Ann. Polon. Math. LIII (1991), 227–232.
6. W. Okrasinski, *Nontrivial solutions to nonlinear Volterra equations*, SIAM J. Math. Anal., to appear.
7. R. T. Rockafellar, *Convex Analysis*, Princeton University Press, New Jersey, 1970

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