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HANKEL OPERATORS ON BERGMAN SPACES WITH CHANGE OF WEIGHT

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Abstract.

We consider big Hankel operators on a weighted Bergman space and show (under weak hypotheses on the weights) that boundedness and S_p-properties are preserved if the weight is replaced by another such that the quotient of the weights is bounded above and below outside a compact set.

This result fails for small Hankel operators.

1. Introduction and result.

Let A be the weighted Bergman space of analytic functions in $L^2(\Omega, \mu)$, where Ω is an open subset of $\mathbf{C}^n$ and μ is a suitable measure on Ω. The big Hankel operator with symbol f, where f is a measurable function on Ω, is then defined by

$$H_f(g) = (I - P)(fg), \quad g \in A \tag{1.1}$$

where P is the orthogonal projection $L^2 \to A$. More precisely, the domain of H_f is $\mathscr{D}(H_f) = \{g \in A : fg \in L^2\}$, and we say that H_f is a bounded operator if $\mathscr{D}(H_f)$ is dense in A and $H_f : \mathscr{D}(H_f) \to L^2(\mu)$ is bounded. H_f then extends to a bounded linear operator $A \to A^\perp \subset L^2(\mu)$, and one may further ask whether this operator (or, equivalently, $H_f P : L^2(\mu) \to L^2(\mu)$) is compact, belongs to the Schatten ideal S_p $(0 < p < \infty)$, etc.

Boundedness, compactness and S_p-properties of big Hankel operators have been studied by many authors, see for example Axler (1986), Arazy, Fisher and Peetre (1988) and the book by Zhu (1990), which also contains many further references.

We will here show that these properties are preserved if we replace the measure μ with an equivalent measure ν such that the Radon-Nikodym derivatives $d\mu/d\nu$ and $d\nu/d\mu$ are bounded. This is perhaps not too surprising, since then $L^2(\Omega, \mu = L^2(\Omega, \nu)$ with equivalent norms, but it is not obvious because the Hankel operators for the two measures differ, being defined using different

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projections onto A. In fact, the corresponding statement for the small Hankel operators $\bar{P}(fg)$ is false, see Section 4.

Our result is actually a little bit more general. First, the bounded equivalence of μ and ν is only required close to the boundary; on a compact subset the measure may be changed arbitrarily (as long as it remains finite there). Note that such a change of measure preserves the space A, so that the two Hankel operators in question are defined on the same space (with two different but equivalent norms), but the spaces $L^2(\mu)$ and $L^2(\nu)$ may differ so that the situation for the ranges is more complicated.

Secondly, with an application to the results of Peng, Rochberg and Wu (1991) in view, see Example 4, we will allow A to contain non-analytic functions, and also allow for the possibility to study the Hankel operators restricted to a subspace of A. Hence we make the following assumptions for the remainder of this paper.

Ω is a set (equipped with a σ-field that we will ignore to mention further), μ is a σ-finite positive measure on Ω, A is a closed subspace of $L^2(\Omega, \mu)$, A_0 is a closed subspace of A, and $P = P_\mu$ is the orthogonal projection onto A. We define the Hankel operator $H_f = H_f^\mu$ as above, i.e.

$$H_f^\mu(g) = (I - P_\mu)(fg), \quad g \in \mathscr{D}(H_f^\mu) = \{g \in A : fg \in L^2(\mu)\} \tag{1.2}$$

and may regard H_f^μ as an operator $A_0 \to L^2(\mu)$ whenever $\mathscr{D}(H_f^\mu)$ is dense in A_0 and H_f^μ is bounded there.

In our first result, ν is another measure on Ω such that $L^2(\mu) = L^2(\nu)$ (with equivalent norms). Hence A is also a closed subspace of $L^2(\nu)$, and we may define the Hankel operator H_f^ν as in (1.2).

We use $\| \|_{S_p(\mu)}$ to denote the norm in $S_p(L^2(\mu))$ when $0 < p < \infty$, and the operator norm in $B(L^2(\mu))$ when $p = \infty$.

THEOREM 1. *Suppose that Ω, μ, A, A_0 are as above and that ν is a measure on Ω such that $\nu = \varphi\mu$ for a positive function φ such that both φ and φ^{-1} are bounded. Let f be a measurable function on Ω. Then H_f^μ is bounded (compact, S_p) on A_0 if and only if H_f^ν is, and for $0 < p \leqq \infty$,*

$$C^{-1} \|H_f^\mu\|_{S_p(\mu)} \leqq \|H_f^\nu\|_{S_p(\nu)} \leqq C \|H_f^\mu\|_{S_p(\mu)}, \tag{1.3}$$

where $C = (\|\varphi\|_{L^\infty} \|\varphi^{-1}\|_{L^\infty})^{1/2}$.

For the second result we assume, in addition to the assumptions above, that Ω is a topological space and that the elements of A are given as functions defined everywhere on Ω (and not just μ-a.e.). A function is locally bounded if it is bounded on every compact set.

THEOREM 2. *Suppose that Ω, μ, A, A_0 are as above and that ν is another measure on Ω such that*

(i) *μ and ν are finite on compact subsets of Ω.*
(ii) *there exists a compact set $K_0 \subset \Omega$ such that $\nu = \varphi\mu$ on $\Omega \backslash K_0$, where φ and φ^{-1} are bounded on $\Omega \backslash K_0$;*
(iii) *for every compact $K \subset \Omega$, there exists a compact $K_1 \subset \Omega$, a positive function ψ in $L^2(K, \mu + \nu)$ and a constant $C < \infty$ such that*

$$\sup_{x \in K} |g(x)|/\psi(x) \leqq C\left(\int_{K_1 \backslash K} |g(x)|^2 \, d\mu\right)^{1/2}, \quad g \in A. \tag{1.4}$$

Then A is a closed subspace of $L^2(\nu)$, and if f is a locally bounded, measurable function on Ω, then H_f^μ is bounded (compact, S_p) on A_0 if and only if H_f^ν is.

REMARK 1. There is no norm estimate similar to (1.3) for Theorem 2. In fact, taking f with compact support, f may be 0 ν-a.e., and thus $H_f^\nu = 0$, while $H_f^\mu \neq 0$. The proof below leads to an estimate

$$\|H_f^\mu\|_{S_p(\mu)} \leqq C_1 \|H_f^\nu\|_{S_p(\nu)} + C_2 \sup_{K_1} |f| \tag{1.5}$$

(and the same with μ and ν interchanged) for some constants C_1 and C_2 (depending on p) and a fixed compact $K_1 \subset \Omega$.

REMARK 2. The proofs can also be used to obtain estimates for the singular numbers individually. For Theorem 1 we obtain

$$s_n(H_f^\mu) \leqq C s_n(H_f^\nu) \tag{1.6}$$

and for Theorem 2 for example

$$s_{2n}(H_f^\mu) \leqq C_1 s_n(H_f^\nu) + \sup_{K_1} |f| a_n, \quad n \geqq 0, \tag{1.7}$$

for a fixed sequence $(a_n)_0^\infty \in \cap_{p>0} \ell^p$. (The sequence (a_n) is obtained as a constant times the singular numbers for the restriction operator $g \to \chi_{K_1} g$ acting on A. In typical cases these decrease exponentially.)

REMARK 3. Theorem 2 has really nothing to do with topology and compact sets; we may let Ω be any set such that some subsets of Ω are defined to be 'bounded', assuming only that the union of two bounded sets is bounded, and replace 'compact' with 'bounded'.

REMARK 4. We assumed that the functions in A are defined everywhere so that they are well-defined as elements of $L^2(\nu)$. It actually suffices that they are defined $(\mu + \nu)$-a.e. and that (1.4) holds with the left hand side replaced by $\|g/\psi\|_{L^\infty(K, \mu+\nu)}$.

REMARK 5. When A is the set of analytic functions in $L^2(\Omega, \mu)$ and f is conjugate analytic, it is shown by Arazy, Fisher, Janson and Peetre (1990) (under much more general conditions on μ and ν than here) that the Hilbert-Schmidt norms $\|H_f^\mu\|_{S_2(\mu)}$ and $\|H_f^\nu\|_{S_2(\nu)}$ actually are equal. One step of that proof (Lemma 5.2) uses the same methods as this paper.

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2. Proofs.

PROOF OF THEOREM 1: Since $L^2(\mu) = L^2(\nu)$,

$$\mathscr{D}(H_f^\mu) = \{g \in A : fg \in L^2(\mu)\} = \mathscr{D}(H_f^\nu);$$

in particular, if one of the operators is densely defined, so is the other. Furthermore, P_μ and P_ν are two projections on the same subspace A of $L^2(\mu) = L^2(\nu)$. Hence $P_\nu P_\mu = P_\mu$, and if $g \in \mathscr{D}(H_f^\mu)$,

$$(2.1) \qquad (I - P_\nu)H_f^\mu(g) = (I - P_\nu)(I - P_\mu)(fg) = (I - P_\nu)(fg) = H_f^\nu(g).$$

Thus $H_f^\nu = (I - P_\nu)H_f^\mu$, which shows that H_f^ν is bounded, compact or in S_p when H_f^μ is, and that then

$$(2.2) \qquad \begin{aligned} \|H_f^\nu\|_{S_p(\nu)} \leqq \|H_f^\mu\|_{S_p(\nu)} &\leqq \|I\|_{B(L^2(\mu),L^2(\nu))}\|H_f^\mu\|_{S_p(\mu)}\|I\|_{B(L^2(\nu),L^2(\mu))} \\ &= C\|H_f^\mu\|_{S_p(\mu)}. \end{aligned}$$

The proof of Theorem 2 is a little bit more involved and we start with a couple of lemmas. We assume that Ω, μ, A are as in Theorem 2; in particular μ is finite on compact sets and (1.4) holds, with some $K_1, \psi \in L^2(K, \mu)$ and C, for every compact $K \subset \Omega$.

If E is a subset of Ω, we define $A(E, \mu)$ to be the closed subspace of $L^2(E, \mu)$ spanned by the restrictions of functions in A. Thus $A(\Omega, \mu) = A$, and if $E \subset E_1 \subset \Omega$, the restriction map extends to a contradiction $R: A(E_1, \Omega) \to A(E, \Omega)$.

LEMMA 1. *If K and K_1 are compact subsets of Ω satisfying (1.4) for some $C < \infty$ and $K \subset K_1$, then $R: A(K_1, \mu) \to A(K, \mu)$ is Hilbert-Schmidt.*

PROOF. By assumption, R can be regarded as a map of $A(K_1, \mu)$ into the Banach space of measurable functions on K bounded by a constant times ψ. If $f_1, \dots, f_n$ is an orthonormal set in $A(K_1, \mu)$, then for any $x \in K$ and complex $\lambda_1, \dots, \lambda_n$,

$$(2.3) \qquad |\sum_1^n \lambda_i Rf_i(x)|/\psi(x) \leqq C\|\sum_1^n \lambda_i f_i\|_{A(K_1,\mu)} = C\left(\sum_1^n |\lambda_i|^2\right)^{1/2}$$

and thus

$$\left(\sum_1^n |Rf_i(x)|^2\right)^{1/2} \leqq C\psi(x). \tag{2.4}$$

Hence

$$\sum_1^n \|Rf_i\|^2_{A(K,\mu)} = \int_K \sum_1^n |Rf_i(x)|^2 \, d\mu(x) \leqq C^2 \int_K |\psi|^2 \, d\mu. \tag{2.5}$$

Since the right hand side of (2.5) is independent of the orthonormal set (f_i), this shows that R is Hilbert-Schmidt.

LEMMA 2. *If K is a compact subset of Ω, then the restriction mapping $A \to A(K, \mu)$ belongs to S_p for every $p > 0$.*

PROOF. By repeated applications of and Lemma1, there exists a sequence $K = K_0 \subset K_1 \subset K_2 \subset \ldots$ such that each restriction mapping $R_n: A(K_n, \mu) \to A(K_{n-1}, \mu)$ belongs to S_2. By the Schatten-Hölder inequality, the mapping $R = R_1 R_2 \ldots R_n: A(K_n, \mu) \to A(K_0, \mu)$ belongs to $S_{2/n}$. Hence $R: A \to A(K, \mu)$ belongs to $S_{2/n}$ for every $n \geqq 1$.

PROOF OF THEOREM 2. By the assumptions, for some compacts K_0 and K_1 and some $\psi, C, C_1 < \infty$, and every $g \in A$,

$$\begin{aligned} \|g\|^2_{L^2(\nu)} &= \int_{K_0} |g|^2 \, d\nu + \int_{\Omega \setminus K_0} |g|^2 \, d\nu \\ &\leqq \int_{K_0} C^2 \int_{K_1 \setminus K_0} |g(y)|^2 \, d\mu(y) \psi(x)^2 \, d\nu(x) + C_1 \int_{\Omega \setminus K_0} |g|^2 \, d\mu \\ &\leqq (C^2 \|\psi\|^2_{L^2(\nu)} + C_1) \|g\|^2_{L^2(\mu)}. \end{aligned} \tag{2.6}$$

A similar argument gives an inequality in the opposite direction. Thus $\| \ \|_{L^2(\nu)}$ and $\| \ \|_{L^2(\mu)}$ are equivalent norms on A, and A is a closed subspace of $L^2(\nu)$ too.

We also observe that if $K \subset \Omega$ is compact, an application of (iii) to $K \cup K_0$ yields a compact K_1, a function $\psi \in L^2(K \cup K_0, \mu + \nu)$ and a $C < \infty$ such that

$$\begin{aligned} \sup_{x \in K} |g(x)|/\psi(x) \leqq \sup_{x \in K \cup K_0} |g(x)|/\psi(x) &\leqq C\left(\int_{K_1 \setminus (K \cup K_0)} |g(x)|^2 \, d\mu\right)^{1/2} \\ &\leqq C'\left(\int_{K_1 \setminus K} |g(x)|^2 \, d\nu\right)^{1/2}, \quad g \in A. \end{aligned} \tag{2.7}$$

Hence assumption (iii) holds for ν too, so the assumptions are really symmetric in μ and ν.

Define the operators R_0 and R'_0 by $R_0 h = \chi_{K_0} h$ and $R'_0 = (I - R_0)h = \chi_{\Omega \setminus K_0} h$, for any function h on Ω, and note that $R'_0 h \in L^2(\mu) \Leftrightarrow R'_0 h \in L^2(\nu)$.

Hence, if $g \in \mathscr{D}(H^\mu_f)$, then $fg \in L^2(\mu)$ and $R'_0(fg) \in L^2(\nu)$. Moreover, $g \in A \subset L^2(\nu)$

and $R_0 f$ is by assumption bounded, whence also $R_0(fg) = (R_0 f)g \in L^2(\nu)$, and thus $fg \in L^2(\nu)$. Thus $g \in \mathscr{D}(H_f^\nu)$, and it follows by symmetry that $\mathscr{D}(H_f^\mu) = \mathscr{D}(H_f^\nu)$.

Assume now that $H_f^\nu \in S_p(A_0, L^2(\nu))$ for some $p > 0$. (The cases of boundedness and compactness are proved by the same argument.)

Since R_0' is a bounded operator from $L^2(\nu)$ into $L^2(\mu)$,

$$R_0' H_f^\nu = R_0'(I - P_\nu)M_f \in S_p(A_0, L^2(\mu)). \tag{2.8}$$

Let $K_1 \supset K_0$ be a compact set such that (1.4) holds with $K = K_0$, and let R_1 denote the restriction to $K_1 : R_1 h = \chi_{k_1} h$. Then, by (2.8),

$$R_1 R_0'(I - P_\nu)M_f \in S_p(A_0, L^2(\mu)). \tag{2.9}$$

By Lemma 2, $R_1 \in S_p(A_0, L^2(\mu))$, and because $R_1 M_f g = (R_1 f)(R_1 g)$ and $R_1 f \in L^\infty$,

$$R_1 M_f \in S_p(A_0, L^2(\mu)). \tag{2.10}$$

Combining (2.9) and (2.10) we find

$$R_1 R_0' P_\nu M_f = R_0' R_1 M_f - R_1 R_0'(I - P_\nu)M_f \in S_p(A_0, L^2(\mu)). \tag{2.11}$$

Since (1.4) implies that $\|R_0 P_\nu M_f g\|_{L^2(\mu)} \leqq C' \|R_1 R_0' P_\nu M_f g\|_{L^2(\mu)}$ for every g in $(\mathscr{D}H_f^\nu)$, there is a bounded operator T on $L^2(\mu)$ such that $R_0 P_\nu M_f = T R_1 R_0' P_\nu M_f$. Hence (2.11) implies

$$R_0 P_\nu M_f \in S_p(A_0, L^2(\mu)). \tag{2.12}$$

Combining (2.8), (2.10) and (2.12) we get, using $R_0 = R_0 R_1$,

$$(I - P_\nu)M_f = R_0'(I - P_\nu)M_f + R_0 R_1 M_f - R_0 P_\nu M_f \in S_p(A_0, L^2(\mu)) \tag{2.13}$$

and finally, as in the proof of Theorem 1,

$$H_f^\mu = (I - P_\mu)M_f = (I - P_\mu)(I - P_\nu)M_f \in S_p(A_0, L^2(\mu)). \tag{2.14}$$

The converse follows by symmetry.

3. Examples.

EXAMPLE 1. Let Ω be the unit circle, μ the Lebesgue measure, $A = A_0$ the Hardy space H^2 and $\nu = \varphi\mu$ with $0 < \inf\varphi \leqq \sup\varphi < \infty$. Theorem 1 shows that the Hankel operator H_f^ν has the same properties as the classical Hankel operator H_f^μ.

EXAMPLE 2. Let Ω be an open set in $\mathbf{C}^n$ and $\mu = \varphi_1 m$, $\nu = \varphi_2 m$, where m is the Lebesgue measure on Ω and φ_1, φ_2 are continuous non-negative functions on Ω such that $\varphi_1^{-1}\{0\}$ and $\varphi_2^{-1}\{0\}$ are compact (possibly empty) subsets of Ω and $\varphi_1(x) \to 1$ as $x \to \delta\Omega$. Let $A = A_0$ be the set of all analytic functions in $L^2(\Omega, \mu)$. It follows easily from the maximum principle and the mean value property that A is a closed subspace of $L^2(\mu)$ and that condition (iii) in Theorem 2 holds with $\psi \equiv 1$.

The other conditions hold too and thus H_f^μ is bounded (compact, S_p) if and only if H_f^ν is, for every locally bounded f.

EXAMPLE 3. Let Ω be a bounded domain in $\mathbf{C}^n$ and $\Phi:\Omega\to\Omega'$ a conformal mapping of Ω onto another bounded domain such that $\Phi\in C^1(\bar\Omega)$ and $\Phi^{-1}\in C^1(\bar\Omega')$.

Let A and A' be the sets of analytic functions in $L^2(\Omega,m)$ and $L^2(\Omega',m)$, respectively, where m is the Lebesgue measure. If f is a measurable function on Ω', then $H_f: A'\to L^2(\Omega',m)$ is unitarily equivalent to $H^\nu_{f\circ\Phi}: A\to L^2(\Omega,\nu)$, where $d\nu/dm = J_\Phi$, the Jacobian of Φ. Theorem 1 thus yields that H_f is bounded (compact, S_p) in $L^2(\Omega',m)$ if and only if $H_{f\circ\Phi}$ is bounded (compact, S_p) in $L^2(\Omega,m)$.

EXAMPLE 4. Peng, Rochberg and Wu (1991), Section 6, consider the space

$$A=\left\{\sum_0^k(\log|z|)^j f_j(z)\in L^2(\Omega,\mu): f_j\in\mathscr{H}(\Omega)\right\},$$

where Ω is the unit disc and $\mu=(-\log|z|)^\alpha m$, and also the corresponding space for the measure $\nu=(1-|z|^2)^\alpha m$, where $k\geqq 0$ is a fixed integer and $\alpha>-1$. They define their Hankel operators on A_0, the analytic functions in A.

Condition (ii) in Theorem 2 holds with, for example, $K_0=\{z:|z|\leqq 1/2\}$. In order to show that (iii) holds, let K be a compact set in Ω, let $r_0=\sup_K|z|$, choose r_1 and r_2 with $r_0<r_1<r_2<1$ and define $K_1=\{z:r_1\leqq|z|\leqq r_2\}$. If $f=\sum_0^k(\log|z|)^j f_j(z)\in \mathrm{A}$, then, for some $c,c'>0$,

$$\begin{aligned}\int_{K_1}|f|^2\,d\mu &\geqq c\int_{r_1}^{r_2}\int_0^{2\pi}\left|\sum_n\sum_j(\log r)^j\hat f_j(n)r^n e^{in\theta}\right|^2 d\theta dr\\ &=2\pi c\int_{r_1}^{r_2}\sum_n\left|\sum_j(\log r)^j\hat f_j(n)r^n\right|^2 dr\\ &\geqq c\sum_n r_1^{2n}\int_{r_1}^{r_2}\left|\sum_{j=0}^k(\log r)^j\hat f_j(n)\right|^2 dr\\ &\geqq c'\sum_{n=0}^\infty r_1^{2n}\sum_{j=0}^k|\hat f_j(n)|^2,\end{aligned}$$

where the last inequality follows because $(\log r)^j, j=0,\dots,k$, are linearly independent in $L^2((r_1,r_2))$. In particular, $|\hat f_j(n)|\leqq Cr_1^{-n}(\int_{K_1}|f|^2d\mu)^{1/2}$, and (1.4) follows with $\psi(r)=(1+|\log r|)^k$. (We redefine all functions in A to be 0 at the origin, or use Remark 4.) These estimates also imply that A is a closed subspace of $L^2(\mu)$, and that A_0 is closed in A.

Theorem 2 now implies that the results of Peng, Rochberg and Wu for the measures $(-\log|z|)^\alpha m$ and $(1-|z|^2)^\alpha m$ are equivalent.

4. Small Hankel operators.

We will here show that the results above fail for the small Hankel operator defined by

$$\tilde{H}_f(g) = \bar{P}(fg), \tag{4.1}$$

where $\bar{P}$ is the orthogonal projection of $L^2(\mu)$ onto $\bar{A}$. We take A to be the set of analytic functions in $L^2(\Omega, \mu)$.

EXAMPLE 5. Let Ω be the unit disc and μ the normalized Lebesgue measure on Ω; we will find a measure ν on Ω such that $d\nu = \varphi \, d\mu$ with $1 \leqq \varphi \leqq 2$, and a function f such that $\tilde{H}_f^\mu$ and $\tilde{H}_f^\nu$ are densely defined, $\tilde{H}_f^\mu = 0$ (and thus $\tilde{H}_f^\mu$ is bounded, compact and in every S_p) but $\tilde{H}_f^\nu$ is unbounded.

We construct these as follows. Let $r_k = 1 - 2^{-k}$, $k = 1, 2, \ldots$, let D_k be the disc wth centre r_k and radius 2^{-k-2} and define $d\nu = (1 + \sum_{k=1}^{\infty} \chi_{D_k}) d\mu$. Furthermore, let $f(z) = z(1-z)^{-\alpha}$ for som α with $0 < \alpha < 1$. Then $f \in L^2(\mu) = L^2(\nu)$ and consequently $\tilde{H}_f^\mu$ and $\tilde{H}_f^\nu$ are both defined on the dense subset of bounded functions in A. If g belongs to $\mathscr{D}(\tilde{H}_f^\mu)$, i.e. $g \in A$ and $fg \in L^2(\mu)$, and $h \in \bar{A}$, then $fg\bar{h}$ is an analytic function in $L^1(\mu)$ and thus

$$\langle \tilde{H}_f^\mu g, h \rangle_\mu = \langle fg, h \rangle_\mu = \int fg\bar{h} \, d\mu = fg\bar{h}(0) = 0. \tag{4.2}$$

Consequently $\tilde{H}_f^\mu g = 0$ for every $g \in \mathscr{D}(\tilde{H}_f^\mu)$. Similarly, if $g \in A$, $fg \in L^2(\nu)$ and $h \in \bar{A}$, then

$$\begin{aligned} \langle \tilde{H}_f^\nu g, h \rangle_\nu &= \int fg\bar{h} \, d\nu = \int_\Omega fg\bar{h} \, d\mu + \sum_{k=1}^{\infty} \int_{D_k} fg\bar{h} \, d\mu \\ &= 0 + \sum_{k=1}^{\infty} \mu(D_k) fg\bar{h}(r_k). \end{aligned} \tag{4.3}$$

In particular, choosing $g_j = \bar{h}_j = (1 - r_j^2)(1 - r_j z)^{-2}$, $\|g_j\|_{L^2(\nu)} = \|h_j\|_{L^2(\nu)} \leqq \sqrt{2} \|g_j\|_{L^2(\mu)} = \sqrt{2}$, but

$$\begin{aligned} \langle \tilde{H}_j^\nu g_j, h_j \rangle_\nu &= \sum_{k=1}^{\infty} 2^{-2k-4} r_k (1 - r_k)^{-\alpha} (1 - r_j^2)^2 (1 - r_j r_k)^{-4} \\ &\geqq 2^{-2j-4} r_j (1 - r_j)^{-\alpha} (1 - r_j^2)^{-2} \\ &\geqq 2^{-2j-7} (1 - r_j)^{-\alpha-2} \\ &= 2^{\alpha j - 7}. \end{aligned} \tag{4.4}$$

Hence $\tilde{H}_f^\nu$ is unbounded.

The reader may well object that this example is cheating, because we employ an analytic symbol, whereas it is well-known that for small Hankel operators, the case of main interest is when the symbol is conjugate analytic. We accordingly also give an example with such a symbol.

EXAMPLE 6. Let Ω and μ be as in the previous example and let f be a conjugate analytic function in the disc such that $\bar{f}$ is unbounded but belongs to the Bloch space. Then $\tilde{H}_f^\mu$ is bounded, see for example Janson, Peetre and Rochberg (1987) or Zhu (1990). (We may further choose f such that $\bar{f}$ belongs to the Besov space B_p for any $p > 1$, for example $\bar{f}(z) = \log(1 - \log(1 - z))$, and then H_f^μ is compact and belongs to $S_p, p > 1$.)

Let, for $\varphi \in L^\infty(\Omega)$, T_φ be the bilinear form $T_\varphi(g,h) = \int fgh\varphi \, d\mu$, defined for bounded functions g and h in A.

Assume that T_φ extends to a bounded bilinear form on A for every φ with $0 \leqq \varphi \leqq 1$. By linearity, T_φ then is a bounded bilinear form for every $\varphi \in L^\infty$, and the mapping $\varphi \to T_\varphi$ is a linear operator mapping L^∞ into the space of bounded bilinear forms on A. By the closed graph theorem, $\|T_\varphi\| \leqq C\|\varphi\|_{L^\infty}$ for some $C < \infty$, i.e.

$$\left|\int fgh\varphi \, d\mu\right| = |T_\varphi(g,h)| \leqq C\|\varphi\|_{L^\infty} \|g\|_A \|h\|_A \tag{4.5}$$

for all $\varphi \in L^\infty$, $g \in H^\infty$, $h \in H^\infty$. Choosing $\bar{\varphi} = \operatorname{sign}(fgh)$, we find, for g and $h \in H^\infty$,

$$\int |f|\,|g|\,|h| \, d\mu \leqq C\|g\|_A \|h\|_A. \tag{4.6}$$

In particular, $g = h = (1 - |z_0|^2)(1 - \bar{z}_0 z)^{-2}$ gives

$$\int_\Omega |f(z)|(1 - |z_0|^2)^2 |1 - \bar{z}_0 z|^{-4} \, d\mu(z) \leqq C\|g\|_A^2 = C, \quad |z_0| < 1. \tag{4.7}$$

On the other hand, if D is the disc $\{z : |z - z_0| < \frac{1}{2}(1 - |z_0|)\}$,

$$\begin{aligned} \int_D |f(z)|(1 - |z_0|^2)^2 \, |1 - \bar{z}_0 z|^{-4} \, d\mu(z) &\geqq c(1 - |z_0|)^{-2} \int_D |f(z)| \, d\mu(z) \\ &\geqq \frac{c}{4} |f(z_0)|. \end{aligned} \tag{4.8}$$

Since z_0 is arbitrary and f was assumed to be unbounded, this is a contradiction. (Alternatively, we may argue that (4.6) implies that f belongs to the Banach algebra of pointwise multipliers on A. Since point evaluations are multiplicative linear functionals, they have norm 1 and this implies $f \in H^\infty$.)

Consequently, there exists φ, with $0 \leqq \varphi \leqq 1$, such that T_φ is unbounded. We let $dv = (1 + \varphi) \, d\mu$ and obtain

$$\langle \tilde{H}_f^{\nu} g, \bar{h} \rangle_{\nu} = \int fgh\, d\nu = \int fgh\, d\mu + \int fgh\varphi\, d\mu \tag{4.9}$$
$$= \langle \tilde{H}_f^{\mu} g, \bar{h} \rangle_{\mu} + T_{\varphi}(g, h),$$

whence $\tilde{H}_f^{\nu}$ is unbounded.

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