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LANDSTAD DUALITY FOR C^*-COACTIONS

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Abstract.

We prove a dual version of Landstad's characterization of reduced C^*-crossed products: C^*-cocrossed products by coactions of a locally compact group G are characterized by the presence of an action of G and an equivariant representation of $C_0(G)$.

1. Introduction.

A decade and a half ago, Landstad [Lan 1] was able to characterize reduced C^*-(and W^*-) crossed products by a given group. Loosely speaking, he showed that a C^*-algebra A is a reduced C^*-crossed product by a locally compact group G if and only if there are a C^*-coaction (see the next section for the definition) of G on A and a unitary representation of G in $M(A)$ satisfying a certain equivariance condition relative to the coaction. Nonabelian C^*- (and W^*-) crossed product duality, in its infancy when this result was published, can be used to cast a different perspective on Landstad's characterization. Imai and Takai [IT] proved (in different terminology) that every reduced crossed product of A by G carries a "dual" coaction such that the corresponding cocrossed product (defined in the next section) is isomorphic to $A \otimes \mathscr{K}$, where $\mathscr{K}$ denotes the compact operators on $L^2(G)$ (see also [Val]). This generalized Takai duality [Tak] for crossed products by abelian groups, which employed actions of both G and its Pontryagin dual group. When G is nonabelian, coactions of G replace actions by the (nonexistent) dual group. Landstad's characterization, which we call Landstad duality, can be regarded as telling which coactions of G on A are dual, namely, those for which there is a suitable unitary representation of G in $M(A)$.

Half the intervening time elapsed before Katayama [Kat] proved a dual version of Imai-Takai duality: every (reduced) cocrossed product of A by G carries a "dual" action such that the corresponding reduced crossed product is isomorphic to (again) $A \otimes \mathscr{K}$. Nonabelian C^*-crossed product duality has not

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yet been placed in a self-dual category, as has W^*-crossed product duality [Eno], [ES], although the first steps have been taken [Ior], [Val], [BS1], [BS2]. In the dictionary of crossed product duality, coactions are dual to actions, cocrossed products are dual to crossed products, and representations of $C_0(G)$ are dual to unitary representations of G. Our main result (Theorem 3.3) is the dual version of Landstad duality: an action of G on A is dual if and only if there is an equivariant representation of $C_0(G)$ in $M(A)$. Here equivariance is with respect to translation on $C_0(G)$ and the given action on A (extended in the canonical way to $M(A)$). We remark that Landstad [Lan3, Theorem 1] proved the special case of this dual version for compact G.

In Section 2 we present a few tools designed to make C^*-coactions a little more manageable. The basic idea is to recognize that C^*-coactions usually arise as restrictions of W^*-coactions. One result in particular (Proposition 2.6), which improves [LPRS, Proposition 4.3], shows that when G is amenable things go about as smoothly as could be expected.

In Section 4 we discuss a dual analogue of another aspect of Landstad duality, namely "Landstad's conditions". Here again, things do not go as well as one might hope unless G is amenable.

We remark that it will be interesting to try to generalize Landstad duality to the twisted coactions of Phillips and Raeburn [PR].

2. Preliminaries on coactions.

We begin by establishing our notation for C^*-coactions. We use the "reduced" C^*-coactions of [Kat] and [LPRS] (as opposed to the "full" coactions of [Rae1], [Rae2]: a *C^*-coaction* of a locally compact group G on a C^*-algebra B is a nondegenerate monomorphism $\delta\colon B \to \tilde{M}(B \otimes C_r^*(G))$ (the multipliers of $B \otimes C_r^*(G)$ which multiply $\mathbf{C} \otimes C_r^*(G)$ into $B \otimes C_r^*(G)$) such that $(\delta \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes \delta^G) \circ \delta$ (the "coaction identity"), where δ^G itself is the canonical coaction of G on $C_r^*(G)$ defined by $\delta^G(\lambda(s)) = \lambda(s) \otimes \lambda(s)$, λ being the left regular representation of G. When δ is a C^*-coaction of G on B we call the triple (B, G, δ), or the pair (B, G), a *C^*-cosystem.* Tensor products of C^*-algebras are completions relative to the minimal C^*-tensor norm, except in the obvious (because of the notational signal) cases where the von Neumann algebra tensor product is used. Nondegeneracy here is in the sense of Banach representations, i.e., $\delta(B)(B \otimes C_r^*(G))$ is dense in $B \otimes C_r^*(G)$. This is equivalent to the usual condition that $\delta(\varepsilon_i) \to 1$ strictly in $M(B \otimes C_r^*(G))$ for some (hence every) bounded approximate identity $\{\varepsilon_i\}$ of B. Unfortunately, this use of the term nondegenerate conflicts with a more specialized connotation (see below), but the meaning is always clear from the context.

A coaction δ of G on B gives rise to a Banach representation of $B_r(G)$ $(= C_r^*(G)^*$, the reduced Fourier-Stieltjes algebra of $G)$ on B via $\delta_f(a) =$

$\langle \delta(a), \mathrm{id} \otimes f \rangle$, where id $\otimes f$ denotes the slice map determined by f. Most of the time we only need the restriction of this representation to $A(G)$ ($= \mathcal{M}(G)_*$, the Fourier algebra of G, where $\mathcal{M}(G)$ itself is the reduced von Neumann algebra of G). Two cosystems (B, G, δ) and (C, G, ε) are *conjugate* if there is an isomorphism $\theta: B \to C$ such that $\varepsilon \circ \theta = (\theta \otimes \mathrm{id}) \circ \delta$, or equivalently θ intertwines the $A(G)$-module structures. δ is called a *nondegenerate* coaction if B is a nondegenerate $A(G)$-module, i.e., $\delta_{A(G)}(B)$ is dense in B, or equivalently B is a nondegenerate $B_r(G)$-module. Is every C^*-coaction automatically nondegenerate? This is the automatic nondegeneracy problem, originally posed by Landstad [Lan1]. Automatic nondegeneracy has been proven when G is amenable [Lan1, Lemma 3.8], [Kat, Proposition 6] or discrete [BS1]. Landstad et al point out in [LPRS, Remark 2.2 (3)] that the proofs given in [Lan1, Lemma 3.7], [Kat, Proposition 6] of automatic nondegeneracy for discrete G seem to be incorrect. Another equivalent formulation of nondegeneracy is that $\delta(B)(\mathbb{C} \otimes C^*_r(G))$ should be dense in $B \otimes C^*_r(G)$ [Kat, Theorem 5] (here and in the sequel the juxtaposition of two subsets S and T of an algebra denotes the linear span of $\{ab \mid a \in S, b \in T\}$). We mention that Katayama's Theorem 5 depends upon his Lemma 4, whose proof we have been unable to understand. Specifically, one of his assertions seems to imply that $x \in \overline{\delta^G_{A(G)}(x)}$ for all $x \in C^*_r(G)$, which as far as we know is an open problem. The corresponding weak* approximation problem for $\mathcal{M}(G)$ is an old question of Eymard [Eym, (4.14)]. Actually, the difficulty with Katayama's proof of his Lemma 4 appears to be roughly the same as with his Proposition 6. Fortunately, we have been able to rearrange Katayama's proof of his Theorem 5 to our satisfaction without appealing to his Lemma 4.

We recall a bit of the theory of W^*-coactions from [NT] (see also [Lan2], [Nak], [SVZ]). A W^*-*coaction* of G on a von Neumann algebra $\mathcal{M}$ is a normal, unital monomorphism $\delta: \mathcal{M} \to \mathcal{M} \bar{\otimes} \mathcal{M}(G)$ (the von Neumann algebra tensor product) satisfying the coaction identity (which now is interpreted at the level of von Neumann algebras rather than multiplier algebras of C^*-algebras). We refer to $(\mathcal{M}, G, \delta)$ as a W^*-*cosystem.* A W^*-coaction of G on $\mathcal{M}$ gives rise to a Banach representation of $A(G)$ on $\mathcal{M}$ in the same way as for C^*-coactions. W^*-coactions usually are implemented by representations of $C_0(G)$ in the following way: let $\mu: C_0(G) \to \mathcal{L}(\mathcal{H})$ (the bounded operators on the Hilbert space $\mathcal{H}$) be a representation. Let W_G denote the left regular representation of G regarded as a unitary element of $M(C_0(G) \otimes C^*_r(G))$, and let $W = \mu \otimes \mathrm{id}(W_G)$. Then $W \in \mathcal{L}(\mathcal{H}) \bar{\otimes} \mathcal{M}(G)$, and for $f \in A(G)$ we have $\mu(f) = \langle W, \mathrm{id} \otimes f \rangle$. Moreover, W implements a W^*-coaction δ^μ of G on $\mathcal{L}(\mathcal{H})$ via $\delta^\mu(x) = \mathrm{Ad}\, W(x \otimes 1)$. A useful identity relating μ and δ^μ is:

$$\delta^\mu_{f * g^\vee}(x) = \int \mu(t \cdot f) x \mu(t \cdot g)\, dt, \qquad f, g \in C_c(G),\ x \in \mathcal{L}(\mathcal{H}),$$

where the integral is taken in the weak operator topology, $g^\vee(s) = g(s^{-1})$, and $t \cdot f(s) = f(st)$.

We will present several technical results, mainly concerning restrictions of W^*-coactions to C^*-coactions. To avoid any possibility of ambiguity, we first record the following completely elementary

LEMMA 2.1. *Let B be a C^*-subalgebra of a von Neumann algebra $\mathcal{M}$. Then $M(B)$ is faithfully represented as the idealizer of B in the weak* closure $\bar{B}$ of B in $\mathcal{M}$.*

PROOF. Realize $\mathcal{M}$ as a von Neumann algebra on a Hilbert space $\mathcal{H}$. Then B, hence $\bar{B}$, is nondegenerately and faithfully represented on a unique closed subspace $\mathcal{H}_0$ of $\mathcal{H}$. In this situation, $M(B)$ is faithfully represented on $\mathcal{H}_0$ as the idealizer of B in $\mathcal{L}(\mathcal{H}_0)$ [Bus, Theorem 3.9], which of course agrees with the idealizer of B in $\bar{B}$. Moreover, it is easy to see that this representation of $M(B)$ in $\bar{B}$ is independent of the choice of $\mathcal{H}$.

Now suppose we have a W^*-coaction δ of G on a von Neumann algebra $\mathcal{M}$, and suppose B is a C^*- subalgebra of $\mathcal{M}$. We want to make sense out of restricting δ to a C^*-coaction on B. Since $\delta(\mathcal{M}) \subset \mathcal{M} \bar{\otimes} \mathcal{M}(G)$ and $B \otimes C_r^*(G)$ is a C^*-subalgebra of $\mathcal{M} \bar{\otimes} \mathcal{M}(G)$, by Lemma 2.1 $M(B \otimes C_r^*(G))$ is faithfully represented as the idealizer of $B \otimes C_r^*(G)$ in the weak* closure $\bar{B} \bar{\otimes} \mathcal{M}(G)$ of $B \otimes C_r^*(G)$ in $\mathcal{M} \bar{\otimes} \mathcal{M}(G)$. Thus, the question of whether $\delta(B) \subset \tilde{M}(B \otimes C_r^*(G))$ is unambiguous.

LEMMA 2.2. *Let $(\mathcal{M}, G, \delta)$ be a W^*-cosystem, and let B be a C^*-subalgebra of $\mathcal{M}$.*

(1) *δ restricts to a C^*-coaction of G on B if and only if both*

$$\delta(B)(\mathbf{C} \otimes C_r^*(G)) \subset B \otimes C_r^*(G) \quad \textit{and} \tag{2.1}$$

$$\overline{\delta(B)(B \otimes C_r^*(G))} = B \otimes C_r^*(G). \tag{2.2}$$

(2) *δ restricts to a nondegenerate C^*-coaction of G on B if and only if*

$$\overline{\delta(B)(\mathbf{C} \otimes C_r^*(G))} = B \otimes C_r^*(G). \tag{2.3}$$

PROOF. (1) Only the sufficiency requires proof, so assume (2.1) and (2.2). Since B, $C_r^*(G)$, and $B \otimes C_r^*(G)$ are self-adjoint, we have

$$(\mathbf{C} \otimes C_r^*(G))\delta(B) \subset B \otimes C_r^*(G), \tag{2.4}$$

$$\overline{(B \otimes C_r^*(G))\delta(B)} = B \otimes C_r^*(G) \tag{2.5}$$

as well.

By the discussion preceding the current lemma, to show that $\delta(B) \subset M(B \otimes C_r^*(G))$, we must show that $\delta(B)$ is contained in $\bar{B} \bar{\otimes} \mathcal{M}(G)$ and idealizes $B \otimes C_r^*(G)$. For the first part, let $b \in B$, and let $\{e_i\}$ be a bounded approximate identity for $C_r^*(G)$. Then $1 \otimes e_i \to 1$ weak* in $\mathcal{M} \bar{\otimes} \mathcal{M}(G)$. By (2.1) we have

$$\delta(b)(1 \otimes e_i) \in B \otimes C_r^*(G) \subset \bar{B} \bar{\otimes} \mathcal{M}(G)$$

for all i, so

$$\delta(b) = \text{weak*} \lim_i \delta(b)(1 \otimes e_i) \in \bar{B} \bar{\otimes} \mathcal{M}(G).$$

For the other part, (2.2) and (2.5) trivially imply that $\delta(B)$ idealizes $B \otimes C_r^*(G)$.

It now follows from (2.1) and (2.4) that in fact

$$\delta(B) \subset \tilde{M}(B \otimes C_r^*(G)).$$

Nondegeneracy of δ as a homomorphism into $M(B \otimes C_r^*(G))$ follows at once from (2.2). Since the coaction identity holds on $\mathcal{M}$, it continues to hold on B. We conclude that δ restricts to a C^*-coaction of G on B.

(2) Again, only the sufficiency requires proof, so assume (2.3). As before, we also have

$$\overline{(\mathbf{C} \otimes C_r^*(G))\delta(B)} = B \otimes C_r^*(G). \tag{2.6}$$

The verification of $\delta(B) \subset \bar{B} \bar{\otimes} \mathcal{M}(G)$ goes exactly as in the proof of (1), except we appeal to (2.3) rather than (2.1).

The idealizer property follows from

$$\begin{aligned}\delta(B)(B \otimes C_r^*(G)) &= \delta(B)(\mathbf{C} \otimes C_r^*(G))(B \otimes \mathbf{C}) \\ &\subset (B \otimes C_r^*(G))(B \otimes \mathbf{C}) \quad \text{by (2.3)} \\ &= B \otimes C_r^*(G),\end{aligned}$$

and a similar computation for $(B \otimes C_r^*(G))\delta(B)$.

As in the proof of (1), it now follows from (2.3) and (2.6) that in fact

$$\delta(B) \subset \tilde{M}(B \otimes C_r^*(G)).$$

To see that δ is a nondegenerate homomorphism into $M(B \otimes C_r^*(G))$, let $z \in B \otimes C_r^*(G)$, and let $\{e_i\}$ be a bounded approximate identity for B. Then

$$\begin{aligned} z &\approx \sum_j \delta(b_j)(1 \otimes x_j) \qquad \text{for finitely many } b_j \in B,\ x_j \in C_r^*(G) \\ &= \lim_i \sum_j \delta(b_j)(1 \otimes x_j)(e_i \otimes 1) \\ &\in \overline{\delta(B)(B \otimes C_r^*(G))}.\end{aligned}$$

Again, the coaction identity holds by restriction, and we conclude that δ restricts to a C^*-coaction of G on B. We appeal to (2.3) one more time to see that this coaction is nondegenerate.

The next lemma records a couple of vital integral identities. In one form or another these were first proven in the weak* sense by the pioneers of nonabelian W^*-crossed product duality [Nak], [SVZ]. Specifically, (2.7) below is a C^*-version due to Katayama [Kat, Lemma 1] of a result which can be traced to Nakagami [Nak, Lemma 4.3 (ii)] (see also [NT, Lemma II.1.5]), and (2.8) below is a C^*-version of a result of Strătilă et al [SVZ, Lemma II.1.4]. We mention also that Katayama has proven an identity [Kat, proof of Lemma 3] which, although not as convenient, can be used in the same way as (2.8).

LEMMA 2.3 (Katayama, Nakagami, Strătilă, Voiculescu, and Zsidó). *If* $(\mathcal{M}, G, \delta)$ *is a* W^**-cosystem, then for* $f \in A_c(G)$, $c \in C_r^*(G)$, $x \in \delta_{A_c(G)}(\mathcal{M})$ *we have*

$$\delta(x)(1 \otimes \lambda(f^\vee)c) = \int \delta_{f \cdot s^{-1}}(x) \otimes \lambda(s)c\, ds, \tag{2.7}$$

$$x \otimes \lambda(f)c = \int \delta(\delta_{s \cdot f}(x))(1 \otimes \lambda(s)c)\, ds, \tag{2.8}$$

both integrals taken in the norm topology.

PROOF. (2.7) can be proven as in [Kat, Lemma 1], even though his context is different: he starts with a C^*-coaction, whereas we are starting with a W^*-coaction. However, it is actually easiest to prove this in the weak* sense without the c (see [Nak, Lemma 4.3 (ii)] or [NT, Lemma II.1.5]), then multiply by $1 \otimes c$.

(2.8) is proven in the weak* sense without the c in [SVZ, Lemma II.1.4], and again one can multiply by $1 \otimes c$.

The integrals exist in the norm topology because the integrands are norm continuous and have compact support.

COROLLARY 2.4. *Let* $(\mathcal{M}, G, \delta)$ *be a* W^**-cosystem, and let* B *be a* C^**-subalgebra and a nondegenerate* $A(G)$*-submodule of* $\mathcal{M}$. *Then* δ *restricts to a nondegenerate* C^**-coaction on* B.

PROOF. We show (2.3). Let $Z = \overline{\delta(B)(\mathbf{C} \otimes C_r^*(G))}$. (2.7) *shows that* $B \otimes C_r^*(G)$ contains

$$\delta(\delta_{A_c(G)}(B))(\mathbf{C} \otimes \lambda(A_c(G))C_r^*(G)),$$

hence contains Z by density.

Similarly, (2.8) shows that Z contains

$$\delta_{A_c(G)}(B) \otimes \lambda(A_c(G))C_r^*(G),$$

hence contains $B \otimes C_r^*(G)$ by density.

COROLLARY 2.5. *Let $(\mathcal{M}, G, \delta)$ be a W^*-cosystem with G amenable, and let B be a C^*-subalgebra and an $A(G)$-submodule of $\mathcal{M}$ such that*

$$\delta(B) \subset \tilde{M}(C \otimes C_r^*(G))$$

for some C^*-subalgebra C of $\mathcal{M}$. Then δ restricts to a (nondegenerate) C^*-coaction on B.

PROOF. First note that we have

$$\delta(B) \subset \tilde{M}(C \otimes C_r^*(G)) \subset \tilde{M}(\mathcal{M} \otimes C_r^*(G)),$$

and further that nondegeneracy will be automatic by amenability of G.

By the preceding corollary it suffices to show that $\delta_{A(G)}(B) = B$. Since G is amenable, the constant function 1 is in $B_r(G)$. We first show $\delta_1 = \mathrm{id}$ on B, and for this it is enough to show $\delta \circ \delta_1 = \delta$ on B, since δ is injective. But

$$\delta \circ \delta_1 = \mathrm{id} \otimes \delta_1^G \circ \delta,$$

and $\mathrm{id} \otimes \delta_1^G = \mathrm{id}$ on $\mathcal{M} \otimes C_r^*(G)$, hence on $\mathcal{M}(\mathcal{M} \otimes C_r^*(G))$ by strict continuity [LPRS, Lemma 1.5].

Now let ω be an element of the annihilator $\delta_{A(G)}(B)^\perp$ of $\delta_{A(G)}(B)$ in $\mathcal{M}^*$. We finish by showing $\omega \in \delta_{B_r(G)}(B)^\perp$. Let $b \in B$. Then for all $f \in A(G)$ we have

$$0 = \langle \delta_f(b), \omega \rangle = \langle \langle \delta(b), \omega \otimes \mathrm{id} \rangle, f \rangle.$$

Now, $\langle \delta(b), \omega \otimes \mathrm{id} \rangle \in M(C_r^*(G))$ since $\delta(b) \in M(\mathcal{M} \otimes C_r^*(G))$ [LPRS, Lemma 1.5], and $A(G)$ separates $M(C_r^*(G))$, so we must have $\langle \delta(b), \omega \otimes \mathrm{id} \rangle = 0$. But then we have

$$\langle b, \omega \rangle = \langle \delta_1(b), \omega \rangle = \langle \langle \delta(b), \omega \otimes \mathrm{id} \rangle, 1 \rangle = 0,$$

so indeed $\omega \in \delta_{B_r(G)}(B)^\perp$.

We recall some more terminology from [Kat], [LPRS], and adapt a little of the terminology and point of view of [Rae1], [Rae2], [Qui]. A *representation* of a C^*-cosystem (B, G, δ) is a pair (π, μ) of representations of B and $C_0(G)$ on the same Hilbert space $\mathcal{H}$ such that $(\pi \otimes \mathrm{id}) \circ \delta = \delta^\mu \circ \pi$, where δ^μ is the W^*-coaction of G on $\mathcal{L}(\mathcal{H})$ implemented by μ as above. In this situation $\overline{\pi(B)\mu(C_0(G))}$ turns out to be a C^*-algebra (by essentially the same argument as [LPRS, Lemma 2.5]), which we denote by $C^*(\pi, \mu)$, and we have $\pi(\mathrm{B}) \cup \mu(\mathrm{C_0(G)}) \subset \mathrm{M}(\mathrm{C}^*(\pi, \mu))$. It follows easily from Lemma 2.2 that when (π, μ) is a representation of (B, G, δ) the implemented coaction δ^μ restricts to a C^*-coaction on $\pi(B)$, and this coaction is nondegenerate if δ is [Qui, Proposition 2.4]. Moreover, if π is faithful, then it implements a conjugacy between the cosystems (B, G) and $(\pi(B), G)$. A representation (π, μ) is *weakly contained* in another representation (ρ, ν) if there is a representation θ of $C^*(\rho, \nu)$ such that $\theta \circ \rho = \pi$, $\theta \circ \nu = \mu$. Two representations are *weakly* *equivalent* if each weakly contains the other. A representation is *faithful* if it weakly contains every representation. If (π, μ) is a faithful representation, we call the triple $(C^*(\pi, \mu), \pi, \mu)$, or just the C^*-algebra $C^*(\pi, \mu)$, a *cocrossed product* of the cosystem (B, G, δ). Of course, all faithful representations are weakly equivalent, so all cocrossed products are isomorphic. When a particular cocrossed product is understood, we denote it by $(B \rtimes\!\!\!\!\times_\delta G, j_B, j_G)$, and we call j_B and j_G the *canonical imbeddings* of B and $C_0(G)$ in $M(B \rtimes\!\!\!\!\times_\delta G)$, respectively. When δ is understood, we denote the cocrossed product by $B \rtimes\!\!\!\!\times G$. We warn that this notation is not standard; the notation and terminology of crossed product duality have not yet stabilized, and we are trying to find a symbol which is "obviously" the dual of the symbol for crossed product, and $\rtimes$ has been used frequently for the latter. For a representation (π, μ) of a cosystem (B, G) the unique representation θ of $B \rtimes\!\!\!\!\times G$ such that $\theta \circ j_B = \pi, \theta \circ j_G = \mu$ is denoted by $\pi \times \mu$. If π is a representation of B on $\mathscr{H}$, then the representation of (B, G, δ) *induced from* B is the representation on $\mathscr{H} \otimes L^2(G)$ given by $\operatorname{Ind} \pi = ((\pi \otimes \mathrm{id}) \circ \delta, 1 \otimes M)$, where M denotes the canonical representation of $C_0(G)$ on $L^2(G)$ [GL], [Man]. If π is faithful, so is $\operatorname{Ind} \pi$ [LPRS, Theorem 3.7], [Qui, Proposition 2.8], [Rae2, Theorem 4.1], and $C^*(\operatorname{Ind} \pi)$ is the original definition of the cocrossed product by a coaction [Kat], [LPRS]. The *dual action* $\hat{\delta}$ of G on $B \rtimes\!\!\!\!\times_\delta G$ is determined by

$$s \cdot j_B(b) = j_B(b), \quad s \cdot j_G(f) = j_G(s \cdot f), \quad s \in G.$$

When we refer to the system $(B \rtimes\!\!\!\!\times G, G)$, we have in mind the dual action. Katayama duality [Kat, Theorem 8] states that if δ is nondegenerate then

$$B \rtimes\!\!\!\!\times_\delta G \rtimes_{\hat{\delta}, r} G \cong B \otimes \mathscr{K},$$

where the subscript r indicates the reduced crossed product (recall that $\mathscr{K}$ denotes the C^*-algebra of compact operators on $L^2(G)$).

We conclude this section with a sample application of Corollary 2.5, namely an improvement of [LPRS, Proposition 4.3].

PROPOSITION 2.6. *Let (B, G, δ) be a C^*-cosystem with G amenable, and let C be a C^*-subalgebra and an $A(G)$-submodule of B. Then δ restricts to a (nondegenerate) C^*-coaction on C.*

PROOF. Without loss of generality replace B by $j_B(B)$, so that δ extends naturally to a W^*-coaction on $(B \rtimes\!\!\!\!\times G)^{**}$. Since

$$\delta(C) \subset \delta(B) \subset \tilde{M}(B \otimes C_r^*(G)),$$

the result follows from Corollary 2.5.

3. Landstad duality.

We begin with a couple of results which characterize both the canonical image of B in $M(B \rtimes_\delta G)$ (for nondegenerate δ) and representations of (B, G, δ) induced from B.

PROPOSITION 3.1. *A representation (π, μ) of a C^*-cosystem (B, G) is weakly equivalent to a representation induced from B if and only if there is an action of G on $C^*(\pi, \mu)$ such that $\pi \times \mu$ is G-equivariant* (i.e., "*$C^*(\pi, \mu)$ carries an image of the dual action*"). *In this case, if π faithful then so is $\pi \times \mu$, so that the system $(C^*(\pi, \mu), G)$ is a realization of the dual system $(B \rtimes G, G)$.*

PROOF. Assume that (π, μ) is weakly equivalent to $\operatorname{Ind} \sigma$ for some representation σ of B. By the first part of [GL, Proposition 2.11] (which does not require nondegeneracy), $\operatorname{Ind} \sigma$ extends to a representation of the dual system $(B \rtimes G, G)$. Therefore, $C^*(\operatorname{Ind} \sigma)$, hence $C^*(\pi, \mu)$, carries a suitable action of G.

For the converse, let $\mathcal{H}$ be the Hilbert space of the representation (π, μ), and suppose that $C^*(\pi, \mu)$ carries an image of the dual action. We use a trick of Landstad [Lan3]: define a faithful representation τ of $C^*(\pi, \mu)$ on $\mathcal{H} \otimes L^2(G)$ by

$$(\tau(a)\xi)(s) = s \cdot a\xi(s),$$

and then define $\theta = \operatorname{Ad} W \circ \tau$, where $W = \mu \otimes \operatorname{id}(W_G)$. One readily verifies that

$$\theta \circ \pi = (\pi \otimes \operatorname{id}) \circ \delta,$$

$$\theta \circ \mu = 1 \otimes M$$

(recall that M denotes the canonical representation of $C_0(G)$ on $L^2(G)$). Hence, θ implements a weak equivalence between (π, μ) and $\operatorname{Ind} \pi$.

The last statement of Proposition 3.1 is now obvious.

The above result, which should be compared to [Rae2, Corollary 4.3], can be regarded as a "weak" version of the imprimitivity theorem, which says in this context that a representation of $B \rtimes G$ is (unitarily) equivalent to a representation induced from B if and only if it extends to the dual system $(B \rtimes G, G)$ [Qui, Corollary 2.11]. As we mention in [Qui], this imprimitivity theorem is actually equivalent to Katayama duality, and when G is amenable it is a special case of a result of Mansfield [Man1, Theorem 28] (see also [Man2, Theorem 3]). The amenability restriction here comes from the fact that Mansfield deals with a more general induction process, namely from a "restricted cosystem" $(B, G/N, \delta|)$ for a normal amenable subgroup N. The restricted coaction $\delta|$ is defined as $(\operatorname{id} \otimes q) \circ \delta$, where $q\colon C_r^*(G) \to C_r^*(G/N)$ is the natural quotient map (which makes sense since N is amenable). Presumably the amenability requirement in Mansfield's imprimitivity theorem could be removed through the use of the full coactions of Raeburn [Rae1], [Rae2], for then the appropriate quotient map would be $C^*(G) \to C^*(G/N)$, which always makes sense. However, as Raeburn points out [Rae2, Remark 2.7], it is not obvious how to adapt the definition of Mansfield's induction process to full coactions.

PROPOSITION 3.2. *Let (B, G) be a nondegenerate C^*-cosystem. Then $j_B(B)$ is the unique C^*-subalgebra C of $M(B \rtimes G)$ satisfying:*

$$\delta^{j_G} \text{ restricts to a nondegenerate } C^*\text{-coaction on } C, \tag{3.1}$$

$$\overline{C j_G(C_0(G))} = B \rtimes G. \tag{3.2}$$

$$s \cdot c = c, \qquad s \in G, c \in C, \tag{3.3}$$

PROOF. For ease of writing let $A = B \rtimes G$ and replace B by $j_B(B)$, so that $\delta = \delta^{j_G}$. Let $\mathscr{S}$ denote the collection of C^*-subalgebras C of $M(A)$ satisfying (3.1)–(3.3). Of course, $B \in \mathscr{S}$, and we must show $\mathscr{S} = \{B\}$. First we show $\mathscr{S}$ is directed by inclusion. Specifically, let $C_1, C_2 \in \mathscr{S}$, and let D be the C^*-subalgebra of $M(A)$ generated by $C_1 \cup C_2$. We show that $D \in \mathscr{S}$. To show (3.1) we need only verify (2.3). Define $Z = \overline{\delta(D)(\mathbf{C} \otimes C_r^*(G))}$. To see that $Z \subset D \otimes C_r^*(G)$, let $x \in C_r^*(G)$, and define

$$D' = \{d \in D \mid \delta(d)(1 \otimes x) \in D \otimes C_r^*(G)\}.$$

Then D' is a closed subspace of D containing $C_1 \cup C_2$, so to show $D' = D$ it suffices to show that D' is a subalgebra. For $d, e \in D'$ we have

$$\begin{aligned}
\delta(de)(1 \otimes x) &= \delta(d)\delta(e)(1 \otimes x) \\
&\approx \delta(d) \sum_i e_i \otimes x_i \qquad \text{for some } e_i \in D,\ x_i \in C_r^*(G) \\
&= \sum_i \delta(d)(1 \otimes x_i)(e_i \otimes 1) \\
&\approx \sum_{i,j} (d_{ij} \otimes y_{ij})(e_i \otimes 1) \qquad \text{for some } d_{ij} \in D,\ y_{ij} \in C_r^*(G) \\
&= \sum_{i,j} (d_{ij} e_i \otimes y_{ij}) \\
&\in D \otimes C_r^*(G).
\end{aligned}$$

This shows that $Z \subset D \otimes C_r^*(G)$.

We prepare for the verification of the opposite containment by showing that Z is a C^*-subalgebra of $D \otimes C_r^*(G)$. Clearly Z is a closed subspace. To show that Z is self-adjoint, it suffices to show that $(\mathbf{C} \otimes C_r^*(G))\delta(D) \subset Z$. Let $x \in C_r^*(G)$, and define

$$D' = \{d \in D \mid (1 \otimes x)\delta(d) \in Z\}.$$

As before, D' is a closed subspace of D containing $C_1 \cup C_2$, so to show $D' = D$ we need only show that D' is a subalgebra. For $d, e \in D'$ we have

$$\begin{aligned}(1 \otimes x)\delta(de) &= (1 \otimes x)\delta(d)\delta(e) \\ &\approx \sum_i \delta(d_i)(1 \otimes x_i)(e) \qquad \text{for some } d_i \in D,\ x_i \in C_r^*(G) \\ &\approx \sum_{i,j} \delta(d_i)\delta(e_{ij})(1 \otimes y_{ij}) \qquad \text{for some } e_{ij} \in D,\ y_{ij} \in C_r^*(G) \\ &\in Z.\end{aligned}$$

Hence, Z is self-adjoint. To show that Z is a subalgebra, it suffices to show that it contains all elements of the form $\delta(d)(1 \otimes x)\delta(e)(1 \otimes y)$ for $d, e \in D$, $x, y \in C_r^*(G)$. From the above we have

$$\begin{aligned}\delta(d)(1 \otimes x)\delta(e)(1 \otimes y) &\approx \delta(d)\sum_i \delta(e_i)(1 \otimes x_i)(1 \otimes y) \quad \text{for some } e_i \in D,\ x_i \in C_r^*(G) \\ &= \sum_i \delta(de_i)(1 \otimes x_i y) \\ &\in Z.\end{aligned}$$

Hence, Z is a subalgebra, and therefore is in fact a C^*-subalgebra of $D \otimes C_r^*(G)$.

We are now prepared to show that $Z \supset D \otimes C_r^*(G)$: merely observe that Z is a C^*-subalgebra containing $(C_1 \otimes C_r^*(G)) \cup (C_2 \otimes C_r^*(G))$.

We have verified (3.1) for D, and we now check (3.2). Certainly

$$\overline{Dj_G(C_0(G))} \supset \overline{C_1 j_G(C_0(G))} = A.$$

For the opposite inclusion, let $f \in C_0(G)$ and define

$$D' = \{d \in D \mid dj_G(f) \in A\}.$$

Clearly D' is a closed subspace of D containing $C_1 \cup C_2$, so to show $D' = D$ we only need show that D' is a subalgebra. For $d, e \in D'$ we have

$$dej_G(f) \in dA \subset A,$$

the latter inclusion following from $D \subset M(A)$.

We have now shown that D satisfies (3.1) and (3.2). Of course, (3.3) is clear since the G-fixed points of $M(A)$ form a C^*-algebra. Hence, $D \in \mathscr{S}$, so that $\mathscr{S}$ is directed under inclusion.

Next we observe that for every $C \in \mathscr{S}$ the system (A, G) is a realization of the dual system $(C \rtimes G, G)$. To see this, note that (id, j_G) is a representation of the cosystem (C, G), $C^*(\mathrm{id}, j_G) = A$ by (3.2), and A carries an image of the dual action by (3.3), so (id, j_G) is faithful by Proposition 3.1.

Finally, we are ready to show that $\mathscr{S}$ contains at most one element. Suppose $C, D \in \mathscr{S}$ with $C \neq D$. By the above we may assume that $C \subset D$. Then

$$C \rtimes_{\delta} G \rtimes_{\hat{\delta},r} G = A \rtimes_{\hat{\delta},r} G = D \rtimes_{\delta} G \rtimes_{\hat{\delta},r} G,$$

and Katayama's isomorphism [Kat, proof of Theorem 8] of $D \rtimes_{\delta} G \rtimes_{\hat{\delta},r} G$ with $D \otimes \mathscr{K}$ takes $C \rtimes_{\delta} G \rtimes_{\hat{\delta},r} G$ to $C \otimes \mathscr{K}$, so we must have $C \otimes \mathscr{K} = D \otimes \mathscr{K}$. We derive a contradiction from the existence of a nonzero functional of the form $\phi \otimes \psi$ with $\phi \in D^*$, $\psi \in \mathscr{K}^*$, and $\phi | C = 0$.

THEOREM 3.3. *Let (A, G) be a system. Then there is a cosystem (B, G) such that (A, G) is a realization of the dual system $(B \rtimes G, G)$ if and only if there is a nondegenerate G-equivariant homomorphism μ: $C_0(G) \to M(A)$.*

Moreover, the cosystem (B, G) may be chosen to be nondegenerate, and then it is uniquely determined up to conjugacy by the further requirement that $j_G = \mu$.

PROOF. When (A, G) is a dual system, we can take $\mu = j_G$.

Assume the existence of μ. We must show existence and uniqueness of an appropriate cosystem (B, G). First we show uniqueness. Note that if (B, G) is a cosystem such that $(A, G) = (B \rtimes G, G)$ with $j_G = \mu$, then so is $(j_B(B), G)$, and this latter cosystem is conjugate to (B, G). Therefore, uniqueness follows from Proposition 3.2.

It remains to show existence of a suitable cosystem (B, G). We pave the way with several definitions and lemmas. Dualizing Landstad's strategy [Lan1], we copy from Olesen and Pedersen [OP1, Section 2], [OP2, note added in proof] the construction of an "averaging operator" from a dense self-adjoint subalgebra of A into the G-fixed elements of $M(A)$.

DEFINITION 3.4. We define the following sets:

$$\mathfrak{p} = \left\{ a \in M(A)^+ \mid \text{there exists } b \in M(A)^+ \text{ with } \langle b, \phi \rangle = \int_G \langle s \cdot a, \phi \rangle \, ds \text{ for all } \phi \in A^{*+} \right\}$$

$$\mathfrak{n} = \{ a \in M(A) \mid a^* a \in \mathfrak{p} \}$$

$$\mathfrak{m} = \mathfrak{n}^* \mathfrak{n}.$$

In the definition of $\mathfrak{p}$, clearly the element b is uniquely determined by a; we denote it by Ea. Part (1) of the following lemma is proven in [OP2, note added in proof]. Parts (2)–(4) follow from (1) using standard arguments, e.g., [Ped, Lemma 5.1.2].

LEMMA 3.5. (1) $\mathfrak{p}$ *is a hereditary cone in* $M(A)^+$.

(2) $\mathfrak{n}$ *is a left ideal in* $M(A)$.

(3) $\mathfrak{m}$ *is a self-adjoint subalgebra of* $M(A)$ *which is equal to the span of its positive part* $\mathfrak{m}^+$.

(4) $\mathfrak{m}^+ = \mathfrak{p}$.

COROLLARY 3.6. (1) *E extends uniquely to a positive linear map from* $\mathfrak{m}$ *into* $M(A)$.

(2) $Ea = \int_G s \cdot a\, ds$ *for all* $a \in \mathfrak{m}$, *where the integral is taken in the weak* topology.*

(3) *For* $b, c \in \mathfrak{m}$ *the map* $a \mapsto E(bac)$ *is norm continuous on* $M(A)$.

PROOF. (1) and (2) follow immediately, and (3) may be proven using an argument similar to, e.g., [OP1, Lemma 2.5].

DEFINITION 3.7. $A_0 = \mu(C_c(G))A\mu(C_c(G))$.

LEMMA 3.8. (1) $\mu(C_c(G)) \subset \mathfrak{m}$.

(2) $A_0 \subset \mathfrak{m} \cap A$.

(3) A_0 *is a dense G-invariant self-adjoint subalgebra of A.*

PROOF. (1) This is very similar to [OP1, Lemma 2.6], and essentially follows from the fact that for $f \in C_c(G)$ the integral $\int s \cdot f\, ds$ is weak* convergent in $C_0(G)^{**}$.

(2) This follows immediately from (1) and Lemma 3.5.

(3) This follows from (2), the definition of A_0, and nondegeneracy of μ.

DEFINITION 3.9. B will denote the C^*-subalgebra of $M(A)$ generated by $E(A_0)$.

LEMMA 3.10 (1) *Every element of B is fixed under the action of G.*

(2) $\overline{B\mu(C_0(G))} = A$.

PROOF. (1) It is enough to show it for elements of $E(A_0)$, and this easy.

(2) Let C denote the left hand side. To see that $C \subset A$, it is enough to show that A contains all elements of the form $E(a\mu(f))\mu(g)$ for $a \in A_0$, $f, g \in C_c(G)$:

$$E(a\mu(f))\mu(g) = \int s \cdot (a\mu(f))\, ds\, \mu(g)$$

$$= \int s \cdot a\mu(s \cdot fg)\, ds.$$

Since $s \cdot fg$ vanishes for s outside a compact set, the integrand is in $C_c(G, A)$, so the integral exists in A.

To show $A \subset C$, it suffices by Lemma 3.8 (3) to show $A_0\mu(C_c(G)) \subset C$. Let $a \in A_0$ and $f \in C_c(G)$, and choose a net $\{g_i\}$ in $C_c(G)$ consisting of nonnegative symmetric functions of integral one with supports shrinking to the identity element of G. Then $\{g_i * g_i\}$ is a bounded approximate identity for $L^1(G)$. For each i define h_i: $G \times G \to A$ by

$$h_i(s,t) = t\cdot(a\mu(fg_i\cdot s^{-1}))\mu(g_i\cdot s^{-1}),$$

where $g\cdot s(t) = g(st)$. It is routine to verify that $h_i \in C_c(G \times G, A)$. On the one hand we have

$$\begin{aligned}\int h_i &= \int\int t\cdot(a\mu(fg_i\cdot s^{-1}))\mu(g_i\cdot s^{-1})\,dt\,ds\\ &= \int E(a\mu(fg_i\cdot s^{-1}))\mu(g_i\cdot s^{-1})\,ds,\end{aligned}$$

and the latter integrand is in $C_c(G, C)$ (continuity following from Corollary 3.6 (3)), so $\int h_i \in C$. On the other hand,

$$\begin{aligned}\int h_i &= \int\int t\cdot(a\mu(f))\mu(t\cdot g_i\cdot s^{-1}g_i\cdot s^{-1})\,ds\,dt\\ &= \int t\cdot(a\mu(f))g_i * g_i(t)\,dt\\ &= (g_i * g_i)\cdot(a\mu(f)),\end{aligned}$$

so that $\int h_i$ tends to $a\mu(f)$ in norm, forcing $a\mu(f) \in C$.

Let δ be the W^*-coaction of G on A^{**} implemented by μ as in Section 2.

LEMMA 3.11. (1) *For* $b \in B$, $f, g \in C_c(G)$ *we have*

$$\delta_{f*g^\vee}(b) = E(\mu(f)b\mu(g)).$$

(2) $E(\mu(C_c(G))B\mu(C_c(G)))$ *is dense in* B.

PROOF. (1) We compute:

$$\begin{aligned}\delta_{f*g^\vee}(b) &= \int \mu(t\cdot f)b\mu(t\cdot g)\,dt\\ &= \int t\cdot(\mu(f)b\mu(g))\,dt\\ &= E(\mu(f)b\mu(g)).\end{aligned}$$

(2) It suffices to show that $\overline{E(\mu(C_c(G))B\mu(C_c(G)))}$ contains all elements of the form $E(\mu(f)a\mu(g))$ for $f, g \in C_c(G)$, $a \in A$. By Lemma 3.10 (2) we can choose a sequence $\{a_n\}$ in $B\mu(C_c(G))$ converging in norm to a. By Corollary 3.6 (3) we have $E(\mu(f)a_n\mu(g)) \to E(\mu(f)a\mu(g))$. But $\mu(f)a_n\mu(g) \in \mu(C_c(G))B\mu(C_c(G))$.

CONCLUSION OF THE PROOF OF THEOREM 3.3. It follows from Lemma 3.11 that B is a nondegenerate $A(G)$-submodule of A^{**}. From Corollary 2.4 we deduce that δ restricts to a nondegenerate C^*-coaction on B. It now follows by definition that (id, μ) is a representation of the cosystem (B, G), and Lemma 3.10 (2) says that $A = C^*(\mathrm{id}, \mu)$. Moreover, it follows from Lemma 3.10 (1) and the equivariance of μ that the representation $\mathrm{id} \times \mu$ of $B \rtimes G$ is G-equivariant. By Proposition 3.1 we conclude that the system (A, G) is a realization of the dual system $(B \rtimes G, G)$.

4. Landstad's coconditions.

Let (A, G) be a system, let $\mu\colon C_0(G) \to M(A)$ be a nondegenerate equivariant homomorphism, and let δ be the W^*-coaction on A^{**} implemented by μ. By Theorem 3.3 there is a unique C^*-subalgebra B of $M(A)$ such that δ restricts to a nondegenerate C^*-coaction on B and (A, G) is a realization of the dual system $(B \rtimes G, G)$ with $j_G = \mu$. In the dual situation, Landstad was able to characterize the corresponding version of B as the set of elements of $M(A)$ satisfying three conditions [Lan1, (3.6)–(3.8)] which have since become known as *Landstad's conditions*. We propose the following as a dual analogue of these conditions in our context:

DEFINITION 4.1. We say an element b of $M(A)$ satisfies *Landstad's coconditions* if

(1) $s \cdot b = b$ for all $s \in G$;
(2) $b\mu(f)$, $\mu(f)b \in A$ for all $f \in C_c(G)$;
(3) $\delta(b) \in \tilde{M}(M(A) \otimes C_r^*(G))$.

We let L denote the set of elements of $M(A)$ satisfying Landstad's coconditions.

The formulation of Landstad's cocondition (3) deserves comment. L is clearly a C^*-subalgebra of $M(A)$, and the idea is that δ should restrict to a C^*-coaction on L. Unfortunately, we have been unable to show this. However, we at least have the following:

PROPOSITION 4.2. (1) $B \subset L$.
(2) *L is an $A(G)$-submodule of $M(A)$.*

PROOF. (1) We must show that every element of B satisfies Landstad's coconditions. Landstad's cocondition (1) is Lemma 3.10 (1).

The first half of Landstad's cocondition (2) was shown in the proof of Lemma 3.10 (2), and the other half can be shown similarly.

Finally, to verify Landstad's cocondition (3), let $b \in B$. Then

$$\delta(b) \in \tilde{M}(B \otimes C_r^*(G)) \subset \tilde{M}(M(A) \otimes C_r^*(G)).$$

(2) Note that $M(A)$ itself is an $A(G)$-submodule of A^{**}. This follows easily from

the following observations: $W = \mu \otimes \mathrm{id}(W_G) \in M(A \otimes C_r^*(G))$ (since $\mu \otimes \mathrm{id}$: $C_0(G) \otimes C_r^*(G) \to M(A \otimes C_r^*(G))$ is a nondegenerate homomorphism), and for $f \in A(G)$ the slice map $\mathrm{id} \otimes f$ takes $M(A \otimes C_r^*(G))$ into $M(A)$ [LPRS, Lemma 1.5]. Now let $b \in L$, $f \in A(G)$. We must show that $\delta_f(b)$ satisfies Landstad's coconditions.

For Landstad's cocondition (1), let $s \in G$, and compute:

$$\begin{aligned} s \cdot \delta_f(b) &= s \cdot \langle \delta(b), \mathrm{id} \otimes f \rangle \\ &= \langle s \cdot \delta(b), \mathrm{id} \otimes f \rangle, \end{aligned}$$

letting G act on the first factor of $A^{**} \bar{\otimes} \mathscr{M}(G)$,

$$\begin{aligned} &= \langle \mathrm{Ad}\, s \cdot W(s \cdot b \otimes 1), \mathrm{id} \otimes f \rangle \\ &= \langle \mathrm{Ad}\, \mu \otimes \mathrm{id}(s \cdot W_G)(b \otimes 1), \mathrm{id} \otimes f \rangle, \end{aligned}$$

letting G act on the first factor of $\mathscr{L}(L^2(G)) \bar{\otimes} \mathscr{M}(G)$.

$$\begin{aligned} &= \langle \mathrm{Ad}\, \mu \otimes \mathrm{id}(W_G(1 \otimes \lambda(s)))(b \otimes 1), \mathrm{id} \otimes f \rangle \\ &= \langle \mathrm{Ad}\, \mu \otimes \mathrm{id}(W_G)(b \otimes 1), \mathrm{id} \otimes f \rangle \\ &= \delta_f(b). \end{aligned}$$

It suffices to show Landstad's cocondition (2) for $f = g * h^{\vee}$ with $g, h \in C_c(G)$. The computation is similar to the first part of the proof of Lemma 3.10(2). Letting $k \in C_c(G)$, we have

$$\begin{aligned} \delta_f(b)\mu(k) &= \int \mu(s \cdot g) b \mu(s \cdot h)\, ds\, \mu(k) \\ &= \int \mu(s \cdot g) b \mu(s \cdot hk)\, ds. \end{aligned}$$

Since $s \cdot hk$ vanishes for s outside a compact set, the integrand is in $C_c(G, A)$ (since $b \in L$), so $\delta_f(b)\mu(k) \in A$. A similar computation works for $\mu(k)\delta_f(b)$.

Finally, we verify Landstad's cocondition (3). For $x \in C_r^*(G)$ we have

$$\begin{aligned} \delta(\delta_f(b))(1 \otimes x) &= (\delta_f \otimes \mathrm{id})(\delta(b))(1 \otimes x) \\ &= (\delta_f \otimes \mathrm{id})(\delta(b)(1 \otimes x)), \end{aligned}$$

which is in $M(A) \otimes C_r^*(G)$ since $\delta(b)(1 \otimes x)$ is, and similarly for $(1 \otimes x)\delta(\delta_f(b))$.

Ideally, it should turn out that not only does δ restrict to a C^*-coaction on L, but in fact $L = B$. This would imply, among other things, a positive resolution of the automatic nondegeneracy problem. To see this, suppose (C, G, δ) is a (possibly degenerate) cosystem, let $A = C \rtimes G$ and $\mu = j_G$, and without loss of generality replace C by $j_C(C)$. Then μ is covariant for the dual action on A, so by Proposition 3.2 there is a unique C^*-subalgebra B of $M(A)$ such that δ restricts to a nondegenerate C^*-coaction on B, $\overline{B\mu(C_0(G))} = A$, and every element of B is fixed under G. It is not hard to see that necessarily $B \subset C \subset L$, so if Landstad's coconditions actually characterize B then we must have $B = C$ and the original cosystem (C, G) is nondegenerate. It seems natural to ask whether Landstad's coconditions characterize B when G is amenable or discrete. We answer this in the affirmative when G is amenable.

THEOREM 4.3. *Let (B, G, δ) be a C^*-cosystem with G amenable. Then $j_B(B)$ is characterized in $M(B \rtimes G)$ by Landstad's coconditions.*

PROOF. Let $A = B \rtimes G$ and $\mu = j_G$, and replace B by $j_B(B)$. Then δ extends naturally to a W^*-coaction on A^{**}.

Let L denote the set of elements of $M(A)$ satisfying Landstad's coconditions. Then $B \subset L$, so by Landstad's cocondition (2) we have

$$A \supset \overline{L\mu(C_0(G))} \supset \overline{B\mu(C_0(G))} = A.$$

Moreover, Landstad's cocondition (3), Proposition 4.2 (2), and Corollary 2.5 (with $C = M(A)$) show that δ restricts to a nondegenerate C^*-coaction of L. Hence, by Proposition 3.2 we must have $B = L$.

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