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MEAN VALUE PROPERTIES OF THE HURWITZ ZETA-FUNCTION

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Introduction.

The Lindelöf hypothesis for the Hurwitz zeta-function states that $\zeta(\frac{1}{2} + it, x) = O(t^{\varepsilon})$ for each fixed x and $\varepsilon > 0$. When $x = 1$ we have the usual hypothesis. The hypothesis is far from being proved but in 1952 Koksma and Lekkerkerker proved the estimate

$$\int_0^1 |\zeta^*(\tfrac{1}{2} + it, x)|^2 \, dx = O(\log t),$$

where $\zeta^*(s, x) = \zeta(s, x + 1)$ and since then the result has been sharpened. The result is very similar to Lindelöf's hypothesis, but it states that the mean square is small, not the function for fixed x. In this paper I will continue in Lekkerkerker's tradition and calculate some integrals of which

$$\int_0^1 |\zeta^*(\tfrac{1}{2} + it, x)|^2 \, dx = \log t + \gamma - \log 2\pi + O(t^{-\frac{47}{56}+\varepsilon}), \qquad \varepsilon > 0$$

is an improvement on former estimates. Especially I will obtain a better error-term depending on the Lindelöf hypothesis. Thus the hypothesis implies a better mean square formula.

Integrals involving the Hurwitz zeta-function.

From Hurwitz formula for the Hurwitz zeta-function

$$(1) \qquad \zeta(s, x) = \frac{2\Gamma(1 - s)}{(2\pi)^{1-s}} \left(\sin\frac{\pi s}{2} \sum_{k=1}^{\infty} \frac{\sin 2\pi kx}{k^{1-s}} + \cos\frac{\pi s}{2} \sum_{k=1}^{\infty} \frac{\cos 2\pi kx}{k^{1-s}} \right),$$

$$\mathscr{R}(s) < 0 \;\&\; x \in [0, 1]$$

we see directly with Parsevals identity (see Miklós Mikolás [2]–[4]):

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$$\int_0^1 \zeta(z,x)\zeta(w,x)\,dx = \tag{2}$$

$$2(2\pi)^{z+w-2}\Gamma(1-z)\Gamma(1-w)\cos\left(\frac{\pi}{2}(z-w)\right)\zeta(2-z-w),$$

$$\max(\mathscr{R}(z),\mathscr{R}(w),\mathscr{R}(z+w)) < 1$$

The formula holds initially for $\max(\mathscr{R}(z),\mathscr{R}(w)) < 0$, but I will show later that it holds for the extended region. We have that $\zeta(s,x) = \zeta^*(s,x) + x^{-s}$, and $\zeta^*(s,x)$ is continuous w.r.t. x for $x \in [0,1]$ and thus

$$\int_0^1 \zeta^*(z,x)\zeta^*(w,x)\,dx = \int_0^1 (\zeta(z,x) - x^{-z})(\zeta(w,x) - x^{-w})\,dx = \tag{3}$$

$$\int_0^1 (\zeta(z,x)\zeta(w,x) + x^{-(z+w)} - \zeta(z,x)x^{-w} - \zeta(w,x)x^{-z})\,dx =$$

$$\int_0^1 (\zeta(z,x)\zeta(w,x) - x^{-(z+w)} - x^{-w}\zeta^*(z,x) - x^{-z}\zeta^*(w,x))\,dx =$$

$$\int_0^1 (\zeta(z,x)\zeta(w,x) - x^{-(z+w)})\,dx - \int_0^1 (\zeta^*(z,x)x^{-w} + \zeta^*(w,x)x^{-z})\,dx$$

$$\max(\mathscr{R}(w),\mathscr{R}(z)) < 1$$

I will now deduce an expression for the last two integrals

$$\int_0^1 \zeta^*(z,x)x^{-w}\,dx = (\text{partial integration}) = \left(\text{we use } \frac{\partial\zeta^*}{\partial x}(s,x) = -s\zeta^*(s+1,x)\right)$$

$$\left[\zeta^*(z,x)\frac{x^{1-w}}{1-w}\right]_0^1 + \frac{z}{1-w}\int_0^1 \zeta^*(z+1,x)x^{1-w}\,dx$$

$$= \frac{\zeta(z)-1}{1-w} + \frac{z}{1-w}\int_0^1 \zeta^*(z+1,x)x^{1-w}\,dx$$

Consider

$$s_n(z,w) = \sum_{k=0}^{n} \frac{(z)_k}{(1-w)_{k+1}}(\zeta(z+k)-1),$$

we see that
$$\int_0^1 \zeta^*(z,x)x^{-w}\,dx = s_n(z,w) + R_n$$

where
$$R_n = \frac{(z)_{n+1}}{(1-w)_{n+1}}\int_0^1 \zeta^*(z+n+1,x)x^{n+1-w}\,dx$$

after $n+1$ partial integrations. We have $\int_0^1 \zeta^*(n+z)x^{n-w}\,dx = O(\frac{1}{2}^n)$, since $\zeta^*(z+n,x) \sim (x+1)^{-z-n}$ when $n \to \infty$. We also have

$\lim_{n\to\infty}\frac{(1-w)_n}{(z)_n}\frac{(z)_{n+1}}{(1-w)_{n+1}} = 1$, hence $R_n \to 0$, and $\{s_n(z,w)\}$ converges, when $n \to \infty$, and

(4)
$$\int_0^1 \zeta^*(z,x)x^{-w}\,dx = \sum_{k=0}^{\infty}\frac{(z)_k}{(1-w)_{k+1}}(\zeta(z+k)-1),$$
$$z \notin \mathbf{Z}^- \cup \{0,1\} \;\&\; \mathcal{R}(w) < 1$$

We combine formula (3) and (4) and get

(5)
$$\int_0^1 \zeta^*(z,x)\zeta^*(w,x)\,dz =$$
$$2(2\pi)^{z+w-2}\Gamma(1-z)\Gamma(1-w)\cos\left(\frac{\pi}{2}(z-w)\right)\zeta(2-z-w)+$$
$$\frac{1}{1-z-w}-\sum_{k=0}^{\infty}\left(\frac{(z)_k}{(1-w)_{k+1}}(\zeta(z+k)-1)+\frac{(w)_k}{(1-z)_{k+1}}(\zeta(w+k)-1)\right),$$
$$z \notin \mathbf{Z} \;\&\; w \notin \mathbf{Z} \;\&\; z+w \neq 1$$

I will now show (by analytic continuity) that the equality really holds in the region stated. We know that the integral in (5) is analytic w.r.t. z and w for all $z, w \notin \mathbf{Z}$ (Since the function under the integral-sign is analytic w.r.t. z and w and uniformly continuous, w.r.t. x for $z, w \neq 1$. I will now show that the righthand side of the equality (5) is analytic w.r.t. z and w. The first product is clearly analytic since its factors are analytic. The expression $\frac{1}{1-z-w}$ is also analytic when $z + w \neq 1$. We therefore only have to consider the last sum. First we notice that by symmetry it is enough to prove that

$$\sum_{k=0}^{\infty}\frac{(z)_k}{(1-w)_{k+1}}(\zeta(z+k)-1)$$

is analytic for w and z. We have

$$\sum_{k=0}^{\infty}\frac{(z)_k}{(1-w)_{k+1}}(\zeta(z+k)-1) = \sum_{k=0}^{M-1}\frac{(z)_k}{(1-w)_{k+1}}(\zeta(z+k)-1)+$$
$$\frac{(z)_M}{(1-w)_{M+1}}\sum_{k=0}^{\infty}\frac{(z+M)_k}{(2-w+M)_k}(\zeta(k+M+z)-1)$$

Clearly the first term in this expression is analytic since it is a finite sum of analytic functions (when $z, w \notin \mathbf{Z}$). We see that it is enough to prove that the last sum is analytic, since it is multiplied by an analytic function. Let $w \in \{w\colon |w - w_0| < 1\}$

and $z \in \{z: |z - z_0| < 1\}$. Choose $M > |\min(\mathscr{R}(1 - w_0), \mathscr{R}(z_0))| + 3$. We see that

$$\left|\sum_{k=0}^{\infty} \frac{(z + M)_k}{(2 - w + M)_k}(\zeta(k + M + z) - 1)\right| \leq \sum_{k=0}^{\infty} \frac{(|z + M| + 1)_k}{(|2 - w + M| - 1)_k}(\zeta(2 + k) - 1)$$

which converges. By Weierstrass M-test the sum converges uniformly with z and w in the given neighbourhoods and is thus analytic. We already have the equality (5) for $\max(\mathscr{R}(z), \mathscr{R}(w)) < 0$ and by uniqueness of analytic continuation we get (5) in the region stated.

From (5) we see when $z = \sigma + it$ and $w = \sigma - it$

$$(6) \quad \int_0^1 |\zeta^*(\sigma + it, x)|^2 \, dx = 2(2\pi)^{2\sigma - 2} |\Gamma(1 - \sigma - it)|^2 \cosh(\pi t)\zeta(2 - 2\sigma) +$$

$$\frac{1}{2\sigma - 1} - 2\mathscr{R}\left(\sum_{k=0}^{\infty} \frac{(\sigma + it)_k}{(1 - \sigma + it)_{k+1}}(\zeta(\sigma + it + k) - 1)\right), \quad \neg(\sigma \in \mathbf{Z} \ \& \ t = 0)$$

From (6) we see (with Stirling's formula)

$$(7) \quad \int_0^1 |\zeta^*(\sigma + it, x)|^2 \, dx = \frac{1}{2\sigma - 1} + (2\pi)^{2\sigma - 1}\zeta(2 - 2\sigma)t^{1 - 2\sigma} -$$

$$\frac{2}{t}\mathscr{I}(\zeta(\sigma + it)) + O\left(\frac{1}{t}\right), \qquad \sigma > 0$$

From (3), (4) and (5) we also see that

$$(8) \quad \int_0^1 (\zeta(z, x)\zeta(w, x) - x^{-z-w}) \, dx =$$

$$2(2\pi)^{(z + w - 2)}\Gamma(1 - z)\Gamma(1 - w)\cos\left(\frac{\pi}{2}(z - w)\right)\zeta(2 - z - w) - \frac{1}{1 - z - w}$$

$$\max(\mathscr{R}(z), \mathscr{R}(w)) < 1 \ \& \ z + w \neq 1$$

From (8) we see that the convergence region in (2) holds. If we put $w := 1 - s - \varepsilon$ and $z := s - \varepsilon$ we get

$$\int_0^1 (\zeta(s - \varepsilon, x)\zeta(1 - s - \varepsilon, x) - x^{2\varepsilon - 1}) \, dx = (\text{according to (3)}) =$$

$$2(2\pi)^{(-2\varepsilon - 1)}\Gamma(1 - s + \varepsilon)\Gamma(s + \varepsilon)\cos\left(\frac{\pi}{2}(1 - 2s)\right)\zeta(1 + 2\varepsilon) - \frac{1}{2\varepsilon} =$$

$$(\text{Laurent series development}) =$$

$$\frac{1}{\pi}((1 - 2\log(2\pi)\varepsilon + O(\varepsilon^2)))(\Gamma(1 - s) + \Gamma'(1 - s)\varepsilon + O(\varepsilon^2))$$

$$(\Gamma(s) + \Gamma'(s)\varepsilon + O(\varepsilon^2))\sin(\pi s)\left(\frac{1}{2\varepsilon} + \gamma + O(\varepsilon)\right) - \frac{1}{2\varepsilon} =$$

$$\left(\frac{1}{2\pi}\Gamma(s)\Gamma(1-s)\sin(\pi s) - \tfrac{1}{2}\right)\frac{1}{\varepsilon} +$$

$$\frac{\sin(\pi s)}{\pi}((\gamma - \log 2\pi)\Gamma(1-s)\Gamma(s) +$$

$$\tfrac{1}{2}(\Gamma(1-s)\Gamma'(s) + \Gamma'(1-s)\Gamma(s))) + O(\varepsilon) =$$

(according to the reflexion formula for the Gamma-function) =

$$\gamma - \log 2\pi + \tfrac{1}{2}\left(\frac{\Gamma'(s)}{\Gamma(s)} + \frac{\Gamma'(1-s)}{\Gamma(1-s)}\right) + O(\varepsilon) =$$

$$\gamma - \log 2\pi + \tfrac{1}{2}(\Psi(s) + \Psi(1-s)) + O(\varepsilon)$$

We let ε tend to zero and get

$$\int_0^1 (\zeta(s,x)\zeta(1-s,x) - x^{-1})\,dx = \gamma - \log 2\pi + \tfrac{1}{2}(\Psi(s) + \Psi(1-s)), \tag{9}$$

$$\mathscr{R}(s) \in (0,1)$$

We now consider a special case. First when $w + z = 1$. From formula (9), (3) and (4) we get

$$\int_0^1 \zeta^*(s,x)\zeta^*(1-s,x)\,dx = \gamma - \log 2\pi + \tfrac{1}{2}(\Psi(1-s) + \Psi(s)) - \tag{10}$$

$$\sum_{k=0}^{\infty}\left(\frac{\zeta(s+k)-1}{s+k} + \frac{\zeta(1-s+k)-1}{1-s+k}\right), \qquad s \notin \mathbf{Z}$$

First we only have the formula for $\mathscr{R}(s) \in (0,1)$, but by the same argument as in (5) we know that the formula holds in the region stated. (In fact the integral and the sum is just a special case of the sum and integral discussed in the proof of (5). We also need that $\Psi(s)$ is an analytic function and then by analytic continuity the equality is valid). When $s = \frac{1}{2} + it$ we get

$$\int_0^1 |\zeta^*(\tfrac{1}{2} + it, x)|^2\,dx = \tag{11}$$

$$\gamma - \log 2\pi + \mathscr{R}\left(\Psi(\tfrac{1}{2} + it) - 2\sum_{k=0}^{\infty}\frac{\zeta(\frac{1}{2} + it + k) - 1}{\frac{1}{2} + it + k}\right)$$

Directly from (11) we see (since $\Psi(s) = \log(s) + O\left(\left|\frac{1}{s}\right|\right)$ and $\zeta(\frac{1}{2} + it) = O(t^{\frac{9}{56}+\varepsilon})$ (see [7]))

$$\text{(12)} \quad \int_0^1 |\zeta^*(\tfrac{1}{2} + it, x)|^2 \, dx = \log t + \gamma - \log 2\pi - \frac{2}{t}\mathscr{I}\left(\zeta(\tfrac{1}{2} + it)\right) + O\left(\frac{1}{t}\right) =$$

$$\log t + \gamma - \log 2\pi + O(t^{-\frac{47}{56}+\varepsilon}), \qquad \varepsilon > 0$$

We see that the truth of the Lindelöf hypothesis would imply the error-term $O(t^{\varepsilon - 1})$. The estimate (12) is however far better than the previously best estimate:

$$\int_0^1 |\zeta^*(\tfrac{1}{2} + it, x)|^2 \, dx = \log t + O(1)$$

See [6] V.V. Rane, and also [1], and [5] for former estimates.

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